

**ON THE STABILITY OF UNIFORMLY ROTATING VISCOUS
INCOMPRESSIBLE SELF-GRAVITATING LIQUID**

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We consider the problem of evolution of an isolated mass of a viscous incompressible self-gravitating liquid without taking into account the surface tension at the free boundary. In this problem unknown are a bounded domain $\Omega_t \subset R^3$, filled with the liquid, the vector field of velocities $\mathbf{v}(x, t) = (v_1, v_2, v_3)$ and a scalar pressure $p(x, t)$, $x \in \Omega_t$, satisfying the following relations:

$$\begin{aligned} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu \nabla^2 \mathbf{v} + \nabla p &= \kappa \nabla U(x, t), \\ \nabla \cdot \mathbf{v} &= 0, \quad x \in \Omega_t, \quad t > 0, \\ T(\mathbf{v}, p)\mathbf{n} &= 0, \quad V_n = \mathbf{v} \cdot \mathbf{n}, \quad x \in \Gamma_t \equiv \partial\Omega_t, \\ \mathbf{v}(x, 0) &= \mathbf{v}_0(x), \quad x \in \Omega_0. \end{aligned}$$

Here $\nu, \kappa = const > 0$,

$$U(x, t) = \int_{\Omega_t} \frac{dz}{|x - z|}$$

is the Newtonian potential depending on an unknown domain Ω_t , $T(\mathbf{v}, p) = -pI + \nu S(\mathbf{v})$ is the stress tensor, $S(\mathbf{v}) = \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial v_j} \right)_{j,k=1,2,3}$ is the doubled rate-of-strain tensor, \mathbf{n} is an exterior normal to Γ_t , V_n is the velocity of evolution of Γ_t in the normal direction. The density of a liquid is assumed to be equal to one. The domain Ω_0 is given.

By introducing a new pressure $p - \kappa U$ instead of p we can write the above problem in the form

$$\begin{aligned} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu \nabla^2 \mathbf{v} + \nabla p &= 0, \quad \nabla \cdot \mathbf{v} = 0, \quad x \in \Omega_t, \quad t > 0, \quad (1) \\ T(\mathbf{v}, p)\mathbf{n} &= \kappa U(x, t)\mathbf{n}, \quad V_n = \mathbf{v} \cdot \mathbf{n}, \quad x \in \Gamma_t \equiv \partial\Omega_t, \\ \mathbf{v}(x, 0) &= \mathbf{v}_0(x), \quad x \in \Omega_0. \end{aligned}$$

We are interested in the stability of a special solution of problem (1) corresponding to a rigid rotation of the liquid around a fixed axis (x_3 -axis). In this case the velocity and the pressure are given by

$$\mathbf{V}(x) = \omega(\mathbf{e}_3 \times \mathbf{x}) = \omega(-x_2, x_1, 0), \quad P(x) = \frac{\omega^2}{2}|x'|^2 + p_0 \quad (2)$$

where $x' = (x_1, x_2, 0)$, $p_0 = \text{const}$, \mathbf{e}_3 is a unit vector directed along the x_3 -axis and ω is the angular velocity of rotation. When we plug these functions into the boundary conditions, we obtain the equation that determines the domain \mathcal{F} filled with the rotating liquid; it is referred to as equilibrium figure. The equation reads

$$\frac{\omega^2}{2}|x'|^2 + \kappa\mathcal{U} + p_0 = 0, \quad x \in \mathcal{G} = \partial\mathcal{F}, \quad (3)$$

where

$$\mathcal{U}(x) = \int_{\mathcal{F}} \frac{dz}{|x-z|}.$$

Equilibrium figures of uniformly rotating self-gravitating liquid were studied by many great mathematicians, beginning with I. Newton (see for instance [1,2]). By analogy with classical mechanics the following criterion of stability was generally accepted: the figure \mathcal{F} is stable, if the second variation of the energy functional, i.e. the quadratic form

$$\begin{aligned} \delta^2 R[\rho] = & \int_{\mathcal{G}} b(x)\rho^2(x)dS + \frac{\omega^2}{\int_{\mathcal{F}} |z'|^2 dz} \left(\int_{\mathcal{G}} |y'|^2 \rho(y)dS \right)^2 \\ & - \kappa \int_{\mathcal{G}} \int_{\mathcal{G}} \frac{\rho(y)\rho(z)}{|y-z|} dS_y dS_z \end{aligned} \quad (4)$$

with

$$b(x) = -\omega^2 \mathbf{x}' \cdot \mathbf{N}(x) - \kappa \frac{\partial \mathcal{U}(x)}{\partial N} \geq b_0 > 0 \quad (5)$$

where \mathbf{N} is an exterior normal to \mathcal{G} is positive definite for arbitrary function ρ given on \mathcal{G} and satisfying the conditions

$$\int_{\mathcal{G}} \rho(x)dS = 0, \quad \int_{\mathcal{G}} \rho(x)x_i dS = 0, \quad i = 1, 2, 3, \quad (6)$$

and unstable, if δR can take negative values. But this criterion was never proved rigorously, and it is the aim of the present work to fill this gap. In the case when the surface tension is taken into account this has been done earlier [6-8].

Let us describe briefly our main result. We assume that the equilibrium figure defined by (3) is a given bounded domain with a smooth boundary, axially symmetric with respect to the x_3 -axis, whose barycenter is located at the origin. Hence, the relations

$$\int_{\mathcal{F}} x_i dx = 0, \quad \int_{\mathcal{F}} x_3 x_j dx = 0, \quad i = 1, 2, 3, \quad j = 1, 2$$

hold. Further we assume that the form (4) is positive definite, i.e.

$$c_i \|\rho\|_{L_2(\mathcal{G})}^2 \leq \delta^2 R[\rho] \leq c_2 \|\rho\|_{L_2(\mathcal{G})}^2 \quad (7)$$

for arbitrary ρ satisfying (6). Thus, the function $b(x)$ should be strictly positive. At the initial moment $t = 0$ we prescribe a small perturbation of \mathbf{V} , $\mathbf{v}_0(x)$, given in a domain Ω_0 sufficiently close to \mathcal{F} , so that the boundary Γ_0 of Ω_0 can be determined by the equation

$$x = y + \mathbf{N}(y)\rho_0(y), \quad y \in \mathcal{G} \quad (8)$$

with a small function $\rho_0(y)$. Moreover, we assume that

$$|\Omega_0| = |\mathcal{F}| \equiv \text{mes}\mathcal{F}, \quad \int_{\Omega_0} x_i dx = 0, \quad (9)$$

$$\int_{\Omega_0} \mathbf{v}_0(x) dx = \int_{\mathcal{F}} \mathbf{V}(x) dx = 0,$$

$$\int_{\Omega_0} \mathbf{v}_0(x) \cdot \boldsymbol{\eta}_i(x) dx = \int_{\mathcal{F}} \mathbf{V}(x) \cdot \boldsymbol{\eta}_i(x) dx = \omega \int_{\mathcal{F}} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx,$$

where $\boldsymbol{\eta}_i(x) = \mathbf{e}_i \times \mathbf{x}$, $\mathbf{e}_i = (\delta_{ij})_{j=1,2,3}$, $i = 1, 2, 3$, and that the compatibility conditions

$$\nabla \cdot \mathbf{v}_0 = 0, \quad S(\mathbf{v}_0)\mathbf{n}_0 - \mathbf{n}_0(\mathbf{n}_0 \cdot S(\mathbf{v}_0)\mathbf{n}_0)|_{\Gamma_0} = 0 \quad (10)$$

hold; by \mathbf{n}_0 we mean the exterior normal to Γ_0 . Finally, we assume that the smallness condition

$$\|\mathbf{v}_0 - \mathbf{V}\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} \leq \epsilon, \quad l \in (2, 5/2)$$

is satisfied (here W_2^l are Sobolev-Slobodetskii spaces defined below). Under these assumptions we prove that problem (1) is solvable in the infinite time interval $t \geq 0$, the surface Γ_t is representable in the form (8), i.e.

$$x = z + \mathbf{N}(z)\rho(z, t), \quad z \in \mathcal{G} \quad (11)$$

and

$$\|\mathbf{v}(\cdot, t) - \mathbf{V}\|_{W_2^{l+1}(\Omega_t)} + \|\rho(\cdot, t)\|_{W_2^{l+1}(\mathcal{G})} \rightarrow 0$$

as $t \rightarrow \infty$, which means stability of the solution (2) of problem (1).

The proof is based on the analysis of the problem for perturbations

$$\mathbf{v}_r(x, t) = \mathbf{v}(x, t) - \mathbf{V}(x), \quad p_r(x, t) = p(x, t) - P(x)$$

written in the coordinate system rotating around the x_3 -axis with the angular velocity ω . It has the form

$$\mathbf{w}_t + (\mathbf{w} \cdot \nabla)\mathbf{w} + 2\omega(\mathbf{e}_3 \times \mathbf{w}) - \nu \nabla^2 \mathbf{w} + \nabla s = 0,$$

$$\begin{aligned}
\nabla \cdot \mathbf{w} &= 0, \quad y \in \Omega'_t, \quad t > 0, \\
T(\mathbf{w}, s)\mathbf{n}' &= \left(\frac{\omega^2}{2}|y'|^2 + \kappa U'(y, t) + p_0\right)\mathbf{n}', \\
V'_n &= \mathbf{w} \cdot \mathbf{n}', \quad y \in \Gamma'_t, \\
\mathbf{w}(y, 0) &= \mathbf{v}_0(y) - \mathbf{V}(y) \equiv \mathbf{w}_0(y), \quad y \in \Omega_0,
\end{aligned} \tag{12}$$

where

$$\mathbf{w}(y, t) = \mathcal{Z}^{-1}(\omega t)\mathbf{v}_r(\mathcal{Z}(\omega t)y, t), \quad q(y, t) = p_r(\mathcal{Z}(\omega t)y, t),$$

$$\mathcal{Z}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\Omega'_t = \mathcal{Z}^{-1}(\omega t)\Omega_t$, $\Gamma'_t = \partial\Omega'_t$, \mathbf{n}' is the exterior normal to Γ'_t , and

$$U' = \int_{\Omega'_t} |y - z|^{-1} dz.$$

Conditions (9) are transformed into

$$\begin{aligned}
\int_{\Omega_0} \mathbf{w}_0(x) dx &= 0, \\
\int_{\Omega_0} \mathbf{w}_0(x) \cdot \boldsymbol{\eta}_i(x) dx + \omega \int_{\Omega_0} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx &= \omega \int_{\mathcal{F}} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx,
\end{aligned}$$

and it is easily verified that they are satisfied for arbitrary $t \geq 0$:

$$\begin{aligned}
\int_{\Omega'_t} x_i dx &= 0, \quad \int_{\Omega'_t} \mathbf{w}(x, t) dx = 0, \\
\int_{\Omega'_t} \mathbf{w}(x, t) \cdot \boldsymbol{\eta}_i(x) dx + \omega \int_{\Omega'_t} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx &= \omega \int_{\mathcal{F}} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx,
\end{aligned}$$

$i = 1, 2, 3$.

As usual, we write free boundary problem (12) as a nonlinear problem in a fixed domain. We pass to the Lagrangian coordinates $\xi \in \Omega_0$ that are related to the Eulerian coordinates $x \in \Omega'_t$ by

$$x = \xi + \int_0^t \mathbf{u}(\xi, \tau) d\tau \equiv X(\xi, t), \tag{13}$$

where

$$\mathbf{u}(\xi, t) = \mathbf{w}(X(\xi, t), t).$$

Omitting primes, we can write (12) in an equivalent form

$$\begin{aligned}
\mathbf{u}_t + 2\omega(\mathbf{e}_3 \times \mathbf{u}) - \nu \nabla_u^2 \mathbf{u} + \nabla_u q &= 0, \\
\nabla_u \cdot \mathbf{u} &= 0, \quad \xi \in \Omega_0, \quad t > 0,
\end{aligned}$$

$$T_u(\mathbf{u}, q)\mathbf{n} = M(\xi, t)\mathbf{n}, \quad \xi \in \Gamma_0, \quad (14)$$

$$\mathbf{u}(\xi, 0) = \mathbf{w}_0(\xi), \quad \xi \in \Omega_0,$$

where $q(\xi, t) = s(X(\xi, t), t)$, ∇_u is a transformed gradient with respect to x , T_u is a transformed stress tensor, i.e.

$$\nabla_u = A\nabla, \quad T_u(\mathbf{u}, q) = -qI + \nu S_u, \quad S_u(\mathbf{u}) = (\nabla_u \mathbf{u}) + (\nabla_u \mathbf{u})^T,$$

$\nabla = \left(\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial \xi_3} \right) \equiv \nabla_\xi$, $A = (A)_{ij}$, A_{ij} is a co-factor of the element $a_{ij} = \delta_{ij} + \int_0^t \frac{\partial u_i(\xi, \tau)}{\partial \xi_j} d\tau$ of the Jacobi matrix of the transformation (13) (the determinant of this matrix equals one); $\mathbf{n} = \mathbf{n}(X) = A\mathbf{n}_0(\xi)/|A\mathbf{n}_0|$. finally,

$$M(\xi, t) = \kappa U(X(\xi, t)) + \frac{\omega^2}{2}|X'|^2 + p_0. \quad (15)$$

The function ρ in (11) can be expressed as a function of $\xi \in \Omega_0$:

$$\rho(z, t) = R(X(\xi, t)), \quad (16)$$

where

$$R(x) = \pm \text{dist}(x, \mathcal{G}),$$

the sign " - " corresponds to the case $x \in \mathcal{F}$ and the sign " + " to the case $x \in \mathbb{R}^3 \setminus \mathcal{F}$. The function $R(x)$ is smooth in a certain neighborhood of \mathcal{G} , and

$$\nabla R(x) = \mathbf{N}(\bar{x}),$$

where $\bar{x} = x - R(x)\nabla R(x)$ is the closest to x point of \mathcal{G} . Hence, $z = \bar{X}$ and

$$R_t(X) = \nabla_X R(X) \cdot X_t = \mathbf{N}(\bar{X}) \cdot \mathbf{u}(\xi, t). \quad (17)$$

By (3), we have

$$M = \kappa(U(X) - \mathcal{U}(\bar{X})) + \frac{\omega^2}{2}(|X'|^2 - |\bar{X}'|^2). \quad (18)$$

Let us make one more change of coordinates in (14). We have assumed that the surface Γ_0 is given by equation (8) (hence, $y = \xi$). We extend \mathbf{N} and ρ_0 inside \mathcal{F} and consider the mapping

$$\xi = y + \mathbf{N}(y)\rho_0(y) \equiv e_{\rho_0}(y) : \mathcal{F} \rightarrow \Omega_0.$$

For small ρ_0 , this mapping is invertible, and it transforms (14), (17) into

$$\begin{aligned} \mathbf{u}_t + 2\omega(\mathbf{e}_3 \times \mathbf{u}) - \nu \nabla^2 \mathbf{u} + \nabla q &= \mathbf{l}_1, \\ \nabla \cdot \mathbf{u} &= l_2 = \nabla \cdot \mathbf{L}, \quad y \in \mathcal{F}, \quad t > 0, \\ T(\mathbf{u}, q)\mathbf{N} - M\mathbf{N} &= \mathbf{l}_3, \\ r_t(y, t) - \mathbf{N}(y) \cdot \mathbf{u} &= l_4, \quad y \in \mathcal{G}, \end{aligned} \quad (19)$$

$$\mathbf{u}(y, 0) = \mathbf{w}_0(e_{\rho_0}(y)) \equiv \mathbf{u}_0(y), \quad y \in \mathcal{F}, \quad r(y, 0) = \rho_0(y), \quad y \in \mathcal{G},$$

where $r(y, t) = R(X(e_{\rho_0}(y), t))$ (we have conserved old notations for the transformed functions \mathbf{u} and q), and $\mathbf{l}_1, l_2, \mathbf{L}, \mathbf{l}_3, l_4$ are nonlinear functions of the solution whose explicit form we do not write here; M is defined in (18).

We consider problem (19) in weighted anisotropic Sobolev-Slobodetskii spaces. By $W_2^l(\Omega)$ where $l > 0$ and Ω is a domain in \mathbb{R}^n we mean the space with the norm

$$\|u\|_{W_2^l(\Omega)}^2 = \sum_{0 \leq |j| \leq l} \|D^j u\|_{L_2(\Omega)}^2 \equiv \sum_{0 \leq |j| \leq l} \int_{\Omega} |D^j u(x)|^2 dx,$$

if $l = [l]$, i.e. l is an integral number, and

$$\|u\|_{W_2^l(\Omega)}^2 = \|u\|_{W_2^{[l]}(\Omega)}^2 + \sum_{|j|=[l]} \int_{\Omega} \int_{\Omega} |D^j u(x) - D^j u(y)|^2 \frac{dx dy}{|x - y|^{n+2\lambda}},$$

if $l = [l] + \lambda$, $\lambda \in (0, 1)$. By $D^j u$ we mean, as usual, a partial derivative $\frac{\partial^{j_1} u}{\partial x_1^{j_1} \dots \partial x_n^{j_n}}$ where $j = (j_1, j_2, \dots, j_n)$ and $|j| = j_1 + \dots + j_n$. Anisotropic space $W_2^{l, l/2}(Q_T)$, $Q_T = \Omega \times (0, T)$ can be defined as

$$L_2((0, T), W_2^l(\Omega)) \cap W_2^{l/2}((0, T), L_2(\Omega))$$

and supplied with the norm

$$\|u\|_{W_2^{l, l/2}(Q_T)}^2 = \int_0^T \|u(\cdot, t)\|_{W_2^l(\Omega)}^2 dt + \int_{\Omega} \|u(x, \cdot)\|_{W_2^{l/2}(0, T)}^2 dx. \quad (20)$$

There exist many other equivalent norms in $W_2^{l, l/2}(Q_T)$. Sobolev spaces of functions given on smooth surfaces, in particular, on \mathcal{G} and on $\mathcal{G} \times (0, T)$, are introduced in a standard way, with the help of local maps and partition of unity. We also find it convenient to introduce the spaces

$$W_2^{l, 0}(Q_T) = L_2((0, T), W_2^l(\Omega)), \quad W_2^{0, l/2}(Q_T) = W_2^{l/2}((0, T), L_2(\Omega));$$

the squares of norms in these spaces coincide, respectively, with the first and the second term in (20). Finally, following Y.Hataya, whose paper [4] is devoted to the study of the viscous flow over a rigid bottom without surface tension, we introduce weighted spaces $\widetilde{W}_2^{l, l/2}(Q_T)$, $l > 1, T \leq \infty$, as the sets of functions $u \in W_2^{l, l/2}(Q_T)$ such that $tu \in W_2^{l-1, (l-1)/2}(Q_T)$, and we define the norm in $\widetilde{W}_2^{l, l/2}(Q_T)$ by

$$\|u\|_{\widetilde{W}_2^{l, l/2}(Q_T)}^2 = \|u\|_{W_2^{l, l/2}(Q_T)}^2 + \|tu\|_{W_2^{l-1, (l-1)/2}(Q_T)}^2.$$

In the same way spaces $\widetilde{W}_2^{l, l/2}(G_T)$ are introduced.

By interpolation inequalities,

$$\|(1+t)^\gamma u\|_{W_2^{l-\gamma, (l-\gamma)/2}(Q_T)} \leq c \|u\|_{\widetilde{W}_2^{l, l/2}(Q_T)}, \quad \forall \gamma \in [0, 1]$$

Our main result is as follows.

Theorem 1. *Let $\mathbf{v}_0 \in W_2^{l+1}(\Omega_0)$ with $l \in (2, 5/2)$ and let the surface Γ_0 be given by (8) with $\rho_0 \in W_2^{l+3/2}(\mathcal{G})$. Assume also that conditions (9), (10) hold and that*

$$\|\mathbf{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} \leq \epsilon$$

where ϵ is a sufficiently small positive number. Finally, let condition (7) be satisfied. Then problem (12)-(14)-(19) has a unique solution $\mathbf{u} \in \widetilde{W}_2^{l+2, l/2+1}(Q_\infty)$, $\nabla q \in \widetilde{W}_2^{l, l/2}(Q_\infty)$, $Q_\infty = \mathcal{F} \times \mathbb{R}_+$, defined in an infinite time interval $t \geq 0$, Γ_t is representable in the form (11), $r \in W_2^{l+1}(\mathcal{G})$, $\forall t \geq 0$, moreover, $r \in \widetilde{W}_2^{l+1/2, 0}(G_\infty)$, $G_\infty = \mathcal{G} \times \mathbb{R}_+$, and

$$\begin{aligned} & \|\mathbf{u}\|_{\widetilde{W}_2^{l+2, l/2+1}(Q_\infty)} + \|\nabla q\|_{\widetilde{W}_2^{l, l/2}(Q_\infty)} + \|q\|_{\widetilde{W}_2^{l+1, l/2+1/4}(G_\infty)} \\ & + \|r\|_{\widetilde{W}_2^{l+1/2, 0}(G_\infty)} + \sup_{t>0} \|r(\cdot, t)\|_{W_2^{l+1}(\mathcal{G})} + \sup_{t>0} t \|r(\cdot, t)\|_{W_2^l(\mathcal{G})} \\ & \leq c \left(\|\mathbf{w}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})} \right) \leq c \left(\|\mathbf{v}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})} \right). \end{aligned}$$

Let us give a very general scheme of the proof of Theorem 1. It is based, in particular, on the estimates of solutions of non-homogeneous linear problems

$$\begin{aligned} \mathbf{v}_t + 2\omega(\mathbf{e}_3 \times \mathbf{v}) - \nu \nabla^2 \mathbf{v} + \nabla p &= \mathbf{f}(y, t), \\ \nabla \cdot \mathbf{v}(y, t) &= f(y, t), \quad y \in \mathcal{F}, \quad t > 0, \\ T(\mathbf{v}, p)\mathbf{N} + \mathbf{N}B_0(y)r &= \mathbf{d}(y, t), \\ r_t - \mathbf{N}(y) \cdot \mathbf{v}(y, t) &= g(y, t), \quad y \in \mathcal{G}, \\ \mathbf{v}(y, 0) &= \mathbf{v}_0(y), \quad y \in \mathcal{F}, \quad r(y, 0) = r_0(y), \quad y \in \mathcal{G}. \end{aligned} \tag{21}$$

and

$$\begin{aligned} \mathbf{v}_t - \nu \nabla^2 \mathbf{v} + \nabla p &= \mathbf{f}(x, t), \quad \nabla \cdot \mathbf{v} = f(x, t) \quad x \in \mathcal{F}, \\ T(\mathbf{v}, p)\mathbf{N} &= \mathbf{d}(x, t), \\ \mathbf{v}(x, 0) &= \mathbf{v}_0(x), \quad x \in \mathcal{F}. \end{aligned} \tag{22}$$

The expression

$$-B_0(y)r(y) = -b(y)r(y) + \kappa \int_{\mathcal{G}} \frac{r(z)dS_z}{|y-z|}$$

with $b(y)$ defined in (5) is a linear part (the first variation) of $M = \kappa(U(X) - \mathcal{U}(\bar{X})) + \frac{\omega^2}{2}(|X'|^2 - |\bar{X}'|^2)$ with respect to r .

Theorem 2. *Let $l \in (1/2, 5/2)$, $Q_T = \mathcal{F} \times (0, T)$, $T < \infty$, $G_T = \mathcal{G} \times (0, T)$ and let the data of problem (22) possess the following regularity properties: $\mathbf{f} \in$*

$W_2^{l, l/2}(Q_T)$, $f \in W_2^{1+l, 0}(Q_T)$, $f = \nabla \cdot \mathbf{F}$, $\mathbf{F}_t \in W_2^{0, l/2}(Q_T)$, $\mathbf{v}_0 \in W_2^{1+l}(\mathcal{F})$, $\mathbf{d} \in W_2^{l+1/2, l/2+1/4}(G_T)$; moreover, let the compatibility conditions

$$\nabla \cdot \mathbf{v}_0 = f(x, 0), \quad x \in \mathcal{F}, \quad \Pi_{\mathcal{G}} S(\mathbf{v}) \mathbf{N} = \Pi_{\mathcal{G}} \mathbf{d}(x, 0), \quad x \in \mathcal{G}$$

be satisfied where

$$\Pi_{\mathcal{G}} \mathbf{a} = \mathbf{a} - \mathbf{N}(\mathbf{a} \cdot \mathbf{N})$$

is a projection of the vector field \mathbf{a} given on \mathcal{G} on the tangential plane to \mathcal{G} . Then problem (22) has a unique solution $\mathbf{v} \in W_2^{2+l, 1+l/2}(Q_T)$, $\nabla p \in W_2^{l, l/2}(Q_T)$ with $p|_{G_T} \in W_2^{l+1/2, l/2+1/4}(G_T)$, and the solution satisfies the inequality

$$\begin{aligned} & \|\mathbf{v}\|_{W_2^{2+l, 1+l/2}(Q_T)} + \|\nabla p\|_{W_2^{l, l/2}(Q_T)} + \|p\|_{W_2^{l+1/2, l/2+1/4}(G_T)} \\ & \leq c(T) \left(\|\mathbf{f}\|_{W_2^{l, l/2}(Q_T)} + \|f\|_{W_2^{l+1, 0}(Q_T)} + \|\mathbf{F}_t\|_{W_2^{0, l/2}(Q_T)} \right. \\ & \quad \left. + \|\mathbf{v}_0\|_{W_2^{1+l}(\mathcal{F})} + \|\mathbf{d}\|_{W_2^{l+1/2, l/2+1/4}(G_T)} \right). \end{aligned} \quad (23)$$

Theorem 3. Let $\mathbf{f}, f, \mathbf{v}_0, \mathbf{d}$ satisfy the assumptions of Theorem 2 and $g \in W_2^{l+3/2, l/2+3/4}(G_T)$ with $l \in (1/2, 5/2)$. Then problem (21) has a unique solution $\mathbf{v} \in W_2^{2+l, 1+l/2}(Q_T)$, $\nabla p \in W_2^{l, l/2}(Q_T)$, $r \in W_2^{l+1/2, 0}(\mathcal{G})$, such that $p|_{G_T} \in W_2^{l+1/2, l/2+1/4}(G_T)$, $r(\cdot, t) \in W_2^{l+1}(\mathcal{G})$ for arbitrary $t \in (0, T)$, and

$$\begin{aligned} Y(T) & \equiv \|\mathbf{v}\|_{W_2^{2+l, 1+l/2}(Q_T)} + \|\nabla p\|_{W_2^{l, l/2}(Q_T)} + \|p\|_{W_2^{l+1/2, l/2+1/4}(G_T)} + \|r\|_{W_2^{l+1/2, 0}(G_T)} \\ & + \sup_{t < T} \|r(\cdot, t)\|_{W_2^{l+1}(\mathcal{G})} \leq c \left(\mathcal{N} + \left(\int_0^T (\|\mathbf{v}\|_{L_2(\mathcal{F})}^2 + \|r\|_{W_2^{-1/2}(\mathcal{G})}^2) dt \right)^{1/2} \right), \end{aligned} \quad (24)$$

where

$$\begin{aligned} \mathcal{N} & = \|\mathbf{f}\|_{W_2^{l, l/2}(Q_T)} + \|f\|_{W_2^{l+1, 0}(Q_T)} + \|\mathbf{F}\|_{W_2^{0, 1+l/2}(Q_T)} \\ & + \|\mathbf{v}_0\|_{W_2^{1+l}(\mathcal{F})} + \|\mathbf{d}\|_{W_2^{l+1/2, l/2+1/4}(G_T)} + \|g\|_{W_2^{l+3/2, l/2+3/4}(G_T)}. \end{aligned}$$

Moreover, if $\mathbf{f} \in \widetilde{W}_2^{l, l/2}(Q_T)$, $\mathbf{d} \in \widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)$, $g \in \widetilde{W}_2^{l+3/2, l/2+3/4}(G_T)$, $f \in \widetilde{W}_2^{l+1, 0}(Q_T)$, $\mathbf{F} \in \widetilde{W}_2^{0, 1+l/2}(Q_T)$ (this means that $f \in W_2^{l+1, 0}(Q_T)$, $tf \in W_2^{l, 0}(Q_T)$), $\mathbf{F} \in W_2^{0, 1+l/2}(Q_T)$, $t\mathbf{F} \in W_2^{0, (l+1)/2}(Q_T)$) with $l \in (3/2, 5/2)$, then

$$\begin{aligned} \widetilde{Y}(T) & \equiv \|\mathbf{v}\|_{\widetilde{W}_2^{2+l, 1+l/2}(Q_T)} + \|\nabla p\|_{\widetilde{W}_2^{l, l/2}(Q_T)} + \|p\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)} + \|r\|_{\widetilde{W}_2^{l+1/2, 0}(G_T)} \\ & + \sup_{t < T} \|r(\cdot, t)\|_{W_2^{l+1}(\mathcal{G})} + \sup_{t < T} t \|r(\cdot, t)\|_{W_2^l(\mathcal{G})} \\ & \leq c \left(\widetilde{\mathcal{N}} + \left(\int_0^T (1+t^2) (\|\mathbf{v}\|_{L_2(\mathcal{F})}^2 + \|r\|_{W_2^{-1/2}(\mathcal{G})}^2) dt \right)^{1/2} \right), \end{aligned} \quad (25)$$

where

$$\widetilde{\mathcal{N}} = \|\mathbf{f}\|_{\widetilde{W}_2^{l, l/2}(Q_T)} + \|f\|_{\widetilde{W}_2^{l+1, 0}(Q_T)} + \|\mathbf{F}\|_{\widetilde{W}_2^{0, 1+l/2}(Q_T)}$$

$$+\|\mathbf{v}_0\|_{W_2^{1+l}(\mathcal{G})} + \|\mathbf{d}\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)} + \|g\|_{\widetilde{W}_2^{l+3/2, l/2+3/4}(G_T)}.$$

The constants in (24) and (25) are independent of T .

Inequalities (24), (25) hold without additional terms containing $\|\mathbf{v}\|_{L_2(\mathcal{F})}^2 + \|r\|_{W_2^{-1/2}(\mathcal{G})}^2$ in the right hand side but then $c = c(T)$.

The norm $\|r\|_{W_2^{-1/2}(\mathcal{G})}$ is defined in a standard way as

$$\|r\|_{W_2^{-1/2}(\mathcal{G})} = \sup_{\varphi \in W_2^{1/2}(\mathcal{G})} \frac{\left| \int_{\mathcal{G}} r(x)\varphi(x)dx \right|}{\|\varphi\|_{W_2^{1/2}(\mathcal{G})}}.$$

Since we are going to apply inequalities (24) and (25) to the solution of problem (19), we need estimates of the nonlinear terms in (19) and of the norms $\|\mathbf{u}\|_{L_2(\mathcal{F})}$ and $\|r\|_{W_2^{-1/2}(\mathcal{G})}$.

Proposition 1. *Assume that $\mathbf{u} \in \widetilde{W}_2^{2+l, 1+l/2}(Q_T)$, $\nabla q \in \widetilde{W}_2^{l, l/2}(Q_T)$, $r \in \widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)$ and that the smallness conditions*

$$\|\mathbf{u}\|_{\widetilde{W}_2^{2+l, 1+l/2}(Q_T)} + \|r\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)} \leq \delta_1, \quad \|\rho_0\|_{W_2^{l+3/2}(\Omega_0)} \leq \delta_1 \quad (26)$$

are satisfied with a certain small $\delta_1 > 0$. Moreover, let $\rho_0 \in W_2^{l+3/2}(\mathcal{G})$ be extended into \mathcal{F} so that

$$\|\rho_0\|_{W_2^{l+1}(\mathcal{F})} \leq c\|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})}. \quad (27)$$

Then

$$\begin{aligned} & \|\mathbf{l}_1\|_{\widetilde{W}_2^{l, l/2}(Q_T)} + \|l_2\|_{\widetilde{W}_2^{1+l, 0}(Q_T)} + \|\mathbf{L}\|_{\widetilde{W}_2^{0, (l+1)/2}(Q_T)} \\ & + \|\mathbf{l}_3\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)} + \|l_4\|_{\widetilde{W}_2^{l+3/2, l/2+3/4}(G_T)} \\ & \leq c\delta_1 \left(\|\mathbf{u}\|_{\widetilde{W}_2^{2+l, 1+l/2}(Q_T)} + \|\nabla q\|_{\widetilde{W}_2^{l, l/2}(Q_T)} + \|r\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)} \right) \end{aligned} \quad (28)$$

with the constant c independent of $T \geq 1$.

Moreover, if \mathbf{u} , \mathbf{u}' , r , r' satisfy (26), then

$$\begin{aligned} & \|\mathbf{l}_1 - \mathbf{l}'_1\|_{\widetilde{W}_2^{l, l/2}(Q_T)} + \|l_2 - l'_2\|_{\widetilde{W}_2^{1+l, 0}(Q_T)} + \|\mathbf{L} - \mathbf{L}'\|_{\widetilde{W}_2^{0, (l+1)/2}(Q_T)} \\ & + \|\mathbf{l}_3 - \mathbf{l}'_3\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)} + \|l_4 - l'_4\|_{\widetilde{W}_2^{l+3/2, l/2+3/4}(G_T)} \\ & \leq c\delta_1 \left(\|\mathbf{u} - \mathbf{u}'\|_{\widetilde{W}_2^{2+l, 1+l/2}(Q_T)} + \|\nabla(q - q')\|_{\widetilde{W}_2^{l, l/2}(Q_T)} + \|r - r'\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)} \right), \end{aligned} \quad (29)$$

where $\mathbf{l}'_1 = \mathbf{l}_1(\mathbf{u}', q')$ etc.

The norms $\|\mathbf{u}\|_{L_2(\mathcal{F})}$ and $\|r\|_{W_2^{-1/2}(\mathcal{G})}$ are estimated due to the following propositions.

Proposition 2. Assume that the solution of problem (12) is defined for $t \in (0, T)$, is square integrable together with the derivatives entering in (12), and Γ_t is representable in the form (11) with $\rho(z, t)$ satisfying

$$\sup_{t < T} |\rho(\cdot, t)|_{C^1(\mathcal{G})} \leq \delta_2 \ll 1.$$

Then there exist two positive functions, $E(t)$ and $E_1(t)$, such that

$$\frac{dE(t)}{dt} + E_1(t) = 0, \quad (30)$$

$$c_1 \left(\|\mathbf{w}(\cdot, t)\|_{L_2(\Omega_t)}^2 + \|\rho(\cdot, t)\|_{L_2(\mathcal{G})}^2 \right) \leq E(t) \leq c_2 \left(\|\mathbf{w}(\cdot, t)\|_{L_2(\Omega_t)}^2 + \|\rho(\cdot, t)\|_{L_2(\mathcal{G})}^2 \right), \quad (31)$$

and

$$E_1(t) \geq c_3 \left(\|S(\mathbf{w}(\cdot, t))\|_{L_2(\Omega_t)}^2 + \|\rho(\cdot, t)\|_{W_2^{-1/2}(\mathcal{G})}^2 \right),$$

with the constants independent of \mathbf{w} , ρ , T .

Proposition 3. If the solution of (19) is defined in the time interval $t \in (0, T)$ and \mathbf{u} satisfies (26), then

$$\|\mathbf{u}(\cdot, t)\|_{L_2(\mathcal{F})} \leq c \|\mathbf{w}(\cdot, t)\|_{L_2(\Omega_t)} \leq c \|\mathbf{u}(\cdot, t)\|_{L_2(\mathcal{F})}, \quad (32)$$

$$\|r(\cdot, t)\|_{W_2^{-1/2}(\mathcal{G})} \leq c \|\rho(\cdot, t)\|_{W_2^{-1/2}(\mathcal{G})} \leq c \|r(\cdot, t)\|_{W_2^{-1/2}(\mathcal{G})}.$$

The idea of construction of the function E satisfying (30) and (31) is due to Professor M.Padula [5].

Theorem 3 and Propositions 1-3 allow us to obtain the following uniform estimate of the solution of Problem (19).

Theorem 4. Let the solution of (19) be defined for $t \in (0, T)$ and

$$\tilde{Y}(T) \leq \delta_1 \quad (33)$$

where δ_1 is sufficiently small positive number ($\tilde{Y}(T)$ is defined in (25)). Then

$$\tilde{Y}(T) \leq cN_0, \quad (34)$$

where

$$N_0 = \|\mathbf{u}_0\|_{W_2^{1+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{1+1}(\mathcal{G})}.$$

Proof. It follows from (30) and (32) that

$$\int_0^T (\|\mathbf{u}\|_{L_2(\mathcal{F})}^2 + \|r\|_{W_2^{-1/2}(\mathcal{G})}^2) dt \leq c \int_0^T E_1(t) dt \leq cE(0) \leq c(\|\mathbf{u}_0\|_{L_2(\mathcal{F})}^2 + \|\rho_0\|_{L_2(\mathcal{G})}^2).$$

Hence, making use of (24) and (28) we obtain

$$Y(T) \leq c(\delta_1 \tilde{Y}(T) + N_0). \quad (35)$$

Now, we multiply (30) by t^2 which leads to

$$\frac{dt^2 E(t)}{dt} + t^2 E_1(t) = 2tE(t),$$

and, as a consequence, to

$$\begin{aligned} t^2 E(t) + \int_0^t \tau^2 E_1(\tau) d\tau &= 2 \int_0^t \tau E(\tau) d\tau \leq 2 \sqrt{\int_0^t \tau^2 E(\tau) d\tau} \sqrt{\int_0^t E(\tau) d\tau} \\ &\leq c(N_0 \tilde{Y}(T) + \delta_1 \tilde{Y}^2(T)). \end{aligned}$$

By (25), (35), we have

$$\tilde{Y}^2(T) \leq c_1 \delta_1 \tilde{Y}^2(T) + c_2 N_0 \tilde{Y}(T),$$

and if

$$\delta_1 \leq \frac{1}{2c_1},$$

then

$$\tilde{Y}(T) \leq 2c_2 N_0,$$

q.e.d.

Let us turn to the question of solvability of problem (19), at first in a finite time interval $t \in (0, 1)$. We assume that ρ_0 is extended in \mathcal{F} as indicated in Proposition 1. Then it is easily verified that the norms $\|u\|_{W_2^{l_1}(\Omega_0)}$ and $\|u(e_{\rho_0}(\cdot))\|_{W_2^{l_1}(\mathcal{F})}$ are equivalent to each other for arbitrary $l_1 \in [0, 2 + l]$, i.e.

$$c_3 \|u\|_{W_2^{l_1}(\Omega_0)} \leq \|u(e_{\rho_0}(\cdot))\|_{W_2^{l_1}(\mathcal{F})} \leq c_4 \|u\|_{W_2^{l_1}(\Omega_0)}.$$

Therefore we can consider problem (14) that is equivalent to (19).

We write the function M (15) in the form

$$M = \kappa(U(X) - \mathcal{U}(\bar{\xi})) + \frac{\omega^2}{2}(|X'|^2 - |\bar{\xi}'|^2) = M_1(\mathbf{u}) + M_2(\rho_0)$$

where

$$M_1 = \kappa(U(X) - U_0(\xi)) + \frac{\omega^2}{2}(|X'|^2 - |\xi'|^2), \quad (36)$$

$$M_2 = \kappa(U_0(\xi) - \mathcal{U}(\bar{\xi})) + \frac{\omega^2}{2}(|\xi'|^2 - |\bar{\xi}'|^2),$$

$$U_0(\xi) = \int_{\Omega_0} \frac{d\eta}{|\xi - \eta|}.$$

and we represent the difference of the volume potentials in (36) as

$$U(X) - U_0(\xi) = \int_{\Omega_0} \frac{d\eta}{|X(\xi, t) - X(\eta, t)|} - \int_{\Omega_0} \frac{d\eta}{|\xi - \eta|} = \int_0^1 \frac{\partial U_s}{\partial s} ds$$

$$= \frac{\partial U_s}{\partial s} \Big|_{s=0} + \int_0^1 (1-s) \frac{\partial^2 U_s}{\partial s^2} ds$$

where

$$U_s(\xi, t) = \int_{\Omega_0} \frac{D_s(\eta, t) d\eta}{|X_s(\xi, t) - X_s(\eta, t)|},$$

$X_s(\xi, t) = \xi + s \int_0^t \mathbf{u}(\xi, \tau) d\tau$, D_s is the Jacobian of the transformation X_s (we note that $D_s = 1$ for $s = 1$ and $s = 0$.) A simple calculation shows that

$$\frac{\partial U_s}{\partial s} \Big|_{s=0} = - \int_{\Omega_0} \frac{\xi - \eta}{|\xi - \eta|^3} d\eta \cdot \int_0^t \mathbf{u}(\xi, \tau) d\tau + \int_{\Gamma_0} \mathbf{n}_0(\eta) \cdot \int_0^t \mathbf{u}(\eta, \tau) d\tau \frac{dS}{|\xi - \eta|}$$

is the first variation of $U(X) - U_0(\xi)$ with respect to \mathbf{u} , and $\int_0^1 (1-s) \frac{\partial^2 U_s}{\partial s^2} ds$ is a nonlinear remainder. Problem (14) can be written in the form

$$\begin{aligned} \mathbf{u}_t + 2\omega(\mathbf{e}_3 \times \mathbf{u}) - \nu \nabla^2 \mathbf{u} + \nabla q &= \mathbf{m}_1(\mathbf{u}, q), \\ \nabla \cdot \mathbf{u} &= m_2(\mathbf{u}), \quad \xi \in \Omega_0, \quad t > 0, \\ T(\mathbf{u}, q) \mathbf{n}_0 - \mathbf{n}_0 \lambda_1(\mathbf{u}) &= \mathbf{m}_3(\mathbf{u}) + \mathbf{n}_0 M_2(\rho_0), \\ \mathbf{u}(\xi, 0) &= \mathbf{w}_0(\xi), \quad \xi \in \Omega_0, \end{aligned} \tag{37}$$

where

$$\lambda_1(\mathbf{u}) = \frac{\partial U_s}{\partial s} \Big|_{s=0} + \omega^2 \xi' \cdot \int_0^t \mathbf{u}(\xi, \tau) d\tau$$

and m_i are expressions of the same type as l_i in (19).

In comparison with the Neumann problem (21), (37) contains extra terms $2\omega(\mathbf{e}_3 \times \mathbf{u})$ and $\lambda_1(\mathbf{u})$. Since they are of a lower order, Theorem 2 still can be applied. Nonlinear terms satisfy inequalities of the type (28), (29), and

$$\|M_2(\rho_0)\|_{W_2^{t+1/2}(\mathcal{G})} \leq c \|\rho_0\|_{W_2^{t+1/2}(\mathcal{G})}.$$

Making use of these inequalities, one can prove the following theorem.

Theorem 5. *Let $\Omega_T^0 = \Omega_0 \times (0, T)$, $G_T^0 = \Gamma_0 \times (0, T)$. There exists such $\epsilon > 0$, that if*

$$\|\mathbf{w}_0\|_{W_2^{t+1}(\Omega_0)} + \|\rho_0\|_{W_2^{t+1}(\mathcal{G})} \leq \epsilon,$$

problem (37) has a unique solution $\mathbf{u} \in W_2^{2+t, 1+t/2}(Q_1^0)$, $\nabla q \in W_2^{t, t/2}(Q_1^0)$, and

$$\begin{aligned} &\|\mathbf{u}\|_{W_2^{2+t, 1+t/2}(Q_1^0)} + \|\nabla q\|_{W_2^{t, t/2}(Q_1^0)} + \|q\|_{W_2^{t+1, (t+1)/2}(G_1^0)} \\ &\leq c \left(\|\mathbf{w}_0\|_{W_2^{t+1}(\Omega_0)} + \|\rho_0\|_{W_2^{t+1}(\mathcal{G})} \right). \end{aligned} \tag{38}$$

We should also consider the problem of extension of the solution of (19) given in the time interval $[0, T]$ of arbitrary length T into a larger interval $[T, T+1]$.

This can be done again by solving problem (14). This time we represent the function M in a different way, namely,

$$M = \kappa(U(X(\xi, t)) - U(\bar{X}(\xi, T))) + \frac{\omega^2}{2} (|X'(\xi, t)|^2 - |\bar{X}'(\xi, T)|^2) = M_3(\mathbf{u}) + M_4(\rho_1)$$

where $\rho_1(z) = \rho(z, T)$,

$$M_3 = \kappa(U(X(\xi, t)) - U_1(X(\xi, T))) + \frac{\omega^2}{2} (|X'(\xi, t)|^2 - |X'(\xi, T)|^2),$$

$$M_4 = \kappa(U_1(X(\xi, T)) - U(\bar{X}(\xi, T))) + \frac{\omega^2}{2} (|X'(\xi, T)|^2 - |\bar{X}'(\xi, T)|^2),$$

$$U_1(\sigma) = \int_{\Omega_T} \frac{d\mu}{|\sigma - \mu|}.$$

If the norm $\|\mathbf{u}\|_{\widetilde{W}_2^{2+l, 1+l/2}(Q_T^0)}$ is small, then the transformation $X(\xi, T) = \sigma$ establishes one-to one correspondence between Ω_0 and Ω_T , and we have

$$X(\xi, t) = X(\xi, T) + \int_T^t \mathbf{u}(\xi, \tau) d\tau = \sigma + \int_T^t \mathbf{u}_1(\sigma, \tau) d\tau \equiv Y(\sigma, t),$$

where $t > T$ and

$$\mathbf{u}_1(X(\xi, T), t) = \mathbf{u}(\xi, t).$$

Hence,

$$\begin{aligned} U(X(\xi, t)) - U_1(X(\xi, T)) &= \int_{\Omega_T} \frac{dS_\mu}{|Y(\sigma, t) - Y(\mu, t)|} - \int_{\Omega_T} \frac{dS_\mu}{|\sigma - \mu|} = \int_0^1 \frac{\partial V_s}{\partial s} ds \\ &= \frac{\partial V_s}{\partial s} \Big|_{s=0} + \int_0^1 (1-s) \frac{\partial^2 V_s}{\partial s^2} ds \end{aligned}$$

where

$$V_s(\xi, t) = \int_{\Omega_T} \frac{D_s(\mu, t) d\mu}{|Y_s(\sigma, t) - Y_s(\mu, t)|},$$

$Y_s(\sigma, t) = \sigma + s \int_T^t \mathbf{u}_1(\sigma, \tau) d\tau$, D_s is the Jacobian of the transformation Y_s and

$$\begin{aligned} \frac{\partial V_s}{\partial s} \Big|_{s=0}(\mathbf{u}) &= - \int_{\Omega_T} \frac{\sigma - \mu}{|\sigma - \mu|^3} d\mu \cdot \int_T^t \mathbf{u}_1(\sigma, \tau) d\tau \\ &\quad + \int_{\Gamma_T} \mathbf{n}_1(\mu) \cdot \int_T^t \mathbf{u}_1(\mu, \tau) d\tau \frac{dS_\mu}{|\sigma - \mu|} \end{aligned}$$

(\mathbf{n}_1 is the normal to Γ_T).

We introduce the functions \mathbf{u}_0 and q_0 that coincide with \mathbf{u} and q for $t < T$ and are extended to $t \in [T, T+1]$ with the preservation of class, and we set $\mathbf{v} = \mathbf{u} - \mathbf{u}_0$, $p = q - q_0$. Problem (14) can be written as

$$\begin{aligned}
\mathbf{v}_t + 2\omega(\mathbf{e}_3 \times \mathbf{v}) - \nu \nabla^2 \mathbf{v} + \nabla p &= \mathbf{m}'_1 + \mathbf{f}(\xi, t), \\
\nabla \cdot \mathbf{v} &= m'_2 + f(\xi, t), \quad \xi \in \Omega_0, \quad t > T, \\
T(\mathbf{v}, p)\mathbf{n}_0 - \mathbf{n}_0 \lambda_2(\mathbf{v}) &= \mathbf{m}'_3 + \mathbf{d}(\xi, t), \quad \xi \in \Gamma_0, \\
\mathbf{v}(\xi, T) &= 0, \quad \xi \in \Omega_0,
\end{aligned} \tag{39}$$

where

$$\lambda_2(\mathbf{v}) = \frac{\partial V_s}{\partial s} \Big|_{s=0}(\mathbf{v}) + \omega^2 \sigma' \cdot \int_T^t \mathbf{v}_1(\sigma, \tau) d\tau,$$

m'_i are again nonlinear terms and $\mathbf{f}, f, \mathbf{d}$ are functions depending on \mathbf{u}_0, q_0 .

Theorem 6. *Assume that the solution of problem (14) is given for $t \in (0, T)$ and that inequality (33) holds with a small δ_1 . Then problem (39) is uniquely solvable in the interval $(T, T+1)$, and*

$$\begin{aligned}
&\|\mathbf{v}\|_{W_2^{2+l, 1+l/2}(Q_{T, T+1}^0)} + \|\nabla p\|_{W_2^{l, l/2}(Q_{T, T+1}^0)} + \|p\|_{W_2^{l+1/2, l/2+1/4}(G_{T, T+1}^0)} \\
&+ T \left(\|\mathbf{v}\|_{W_2^{1+l, 1/2+l/2}(Q_{T, T+1}^0)} + \|\nabla p\|_{W_2^{l-1, l/2-1/2}(Q_{T, T+1}^0)} + \|p\|_{W_2^{l-1/2, l/2-1/4}(G_{T, T+1}^0)} \right) \\
&\leq c \left(\|\mathbf{u}\|_{\widetilde{W}_2^{2+l, 1+l/2}(Q_T)} + \|\nabla q\|_{\widetilde{W}_2^{l, l/2}(Q_T)} \right. \\
&\quad \left. + \|\rho(\cdot, T)\|_{W_2^{l+1}(\mathcal{G})} + T \|\rho(\cdot, T)\|_{W_2^l(\mathcal{G})} \right),
\end{aligned}$$

where $Q_{T, T+1}^0 = \Omega_0 \times (T, T+1)$, $G_{T, T+1}^0 = \Gamma_0 \times (T, T+1)$.

Theorems 4-6 permit us to construct the solution of (19) in an infinite time interval. By Theorem 5, the solution \mathbf{u}, q exists for $t \in [0, 1]$ and satisfies (38). By (17), the function

$$R(X) = R(\xi) + \int_0^t \mathbf{N}(\bar{X}) \cdot \mathbf{u}(\xi, \tau) d\tau$$

satisfies the estimate

$$\begin{aligned}
\|R(X)\|_{W_2^{l+1}(\Gamma_0)} &\leq \|R(\cdot)\|_{W_2^{l+1}(\Gamma_0)} + c \int_0^t \|\mathbf{u}(\cdot, \tau)\|_{W_2^{l+1}(\Gamma_0)} d\tau \\
&\leq c \left(\|\mathbf{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})} \right).
\end{aligned}$$

This implies the solvability of Problem (19) and the estimate (34) for $T = 1$. If ϵ is sufficiently small, then condition (33) is satisfied, and we can extend the solution to the interval $t \in [1, 2]$. Assume that the solution is defined for $t \in [0, T]$ and that inequality (34) holds. Then, by virtue of Theorem 6,

$$\begin{aligned}
&\|\mathbf{u}\|_{\widetilde{W}_2^{2+l, 1+l/2}(Q_{T+1}^0)} + \|\nabla q\|_{\widetilde{W}_2^{l, l/2}(Q_{T+1}^0)} + \|q\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(G_{T+1}^0)} \\
&\leq \|\mathbf{u}_0\|_{\widetilde{W}_2^{2+l, 1+l/2}(Q_{T+1}^0)} + \|\nabla q_0\|_{\widetilde{W}_2^{l, l/2}(Q_{T+1}^0)} + \|q_0\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(G_{T+1}^0)}
\end{aligned}$$

$$\begin{aligned}
& + \|\mathbf{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_{T+1}^0)} + \|\nabla v\|_{\widetilde{W}_2^{l,l/2}(Q_{T+1}^0)} + \|v\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_{T+1}^0)} \\
& \leq c \left(\|\mathbf{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T^0)} + \|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q_T^0)} + \|q\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T^0)} \right. \\
& \quad \left. + \|\rho(\cdot, T)\|_{W_2^{l+1}(\mathcal{G})} + T\|\rho(\cdot, T)\|_{W_2^l(\mathcal{G})} \right).
\end{aligned}$$

It can be shown that

$$\begin{aligned}
& \|\rho(\cdot, T)\|_{W_2^{l+1}(\mathcal{G})} + T\|\rho(\cdot, T)\|_{W_2^l(\mathcal{G})} \\
& \leq c \left(\|R(X(\cdot, T))\|_{W_2^{l+1}(\Gamma_0)} + T\|R(X(\cdot, T))\|_{W_2^l(\Gamma_0)} \right) \\
& \leq c \left(\|r(\cdot, T)\|_{W_2^{l+1}(\mathcal{G})} + T\|r(\cdot, T)\|_{W_2^l(\mathcal{G})} \right).
\end{aligned}$$

It follows that the solution \mathbf{u}, q, r of (19) is defined for $t \in [0, T+1]$ and satisfies in this time interval inequality (34), maybe with another greater constant c_1 than c_0 in Theorem 4. But if we impose on ϵ one more last restriction $c_1\epsilon \leq \delta_1$, then, according to Theorem 4, it will satisfy (34) with the constant c_0 . So we can extend the solution step by step to the infinite time interval and obtain estimate (34) for $T = \infty$. This completes the proof of Theorem 1.

Let us say a few words about the case when the equilibrium figure \mathcal{F} is not axially symmetric. In this case equation (3) defines a one-parameter family of equilibrium figures, \mathcal{F}_θ , obtained by rotation of one of them, \mathcal{F}_0 , about the x_3 -axis through the angle θ . Functions (2) given in $\mathcal{F}_{\omega t + \varphi_0}$ represent a periodic solution of problem (1). Criterion of stability (7) is conserved but it should be verified in the set of functions ρ satisfying (6) and one more orthogonality condition

$$\int_{\mathcal{G}} \rho(y) h(y) dS = 0$$

where $h(y) = \mathbf{N}(y) \cdot (\mathbf{e}_3 \times \mathbf{x})$ (for axially symmetric \mathcal{F} , $h(y) = 0$). It can be proved that $\delta^2 R[h] = 0$.

In this case problem (1) has a unique solution belonging to the weighted Sobolev space and as $t \rightarrow \infty$, it tends to the solution (2), whereas Γ_t tends to a certain \mathcal{G}_ψ .

Finally, if $\delta^2 R$ can take negative values for some "admissible" ρ , then the solution (2) of problem (1) is unstable. The proof relies on the analysis of a linear spectral problem

$$\begin{aligned}
\lambda \mathbf{w} + 2\omega(\mathbf{e}_3 \times \mathbf{w}) - \nu \nabla^2 \mathbf{w} + \nabla p &= 0, \\
\nabla \cdot \mathbf{w}(y, t) &= 0, \quad y \in \mathcal{F}, \\
T(\mathbf{w}, p) \mathbf{N} + \mathbf{N} B_0(y) \rho &= 0, \\
\lambda \rho - \mathbf{N}(y) \cdot \mathbf{w}(y, t) &= 0 \quad y \in \mathcal{G},
\end{aligned} \tag{40}$$

considered in the space of complex-valued functions satisfying the orthogonality conditions (6) and

$$\int_{\mathcal{F}} \mathbf{w} dy = 0,$$

$$\int_{\mathcal{F}} \mathbf{w} \cdot \boldsymbol{\eta}_i dy + \omega \int_{\mathcal{G}} \rho \boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_i dS = 0, \quad i = 1, 2, 3.$$

The equilibrium figure is assumed to satisfy the condition

$$\min_{\theta \in [0, 2\pi]} \int_{\mathcal{F}} ((x \cos \theta + x_2 \sin \theta)^2 - x_3^2) dS > 0.$$

One can show that the spectrum of problem (40) consists of a countable number of eigenvalues of a finite algebraic multiplicity with the accumulation points infinity and zero. There are no non-zero purely imaginary eigenvalues. If $\delta^2 R$ can take negative values, then problem (40) has a finite number of eigenvalues with positive real parts. This fact can be deduced from the Pontriagin-Krein-Langer-Azizov theorem about the spectrum of dissipative operators in the Hilbert space with indefinite metrics [3]. With the problem (40) one can associate operators \mathcal{A} and \mathcal{J} defined in a Hilbert space H whose elements are vector-valued functions $V = (\mathbf{v}(y), r(y))$ (\mathbf{v} is given in \mathcal{F} and ρ on \mathcal{G}), possessing the following properties:

1. \mathcal{J} is a bounded self-adjoint operator, and the quadratic form $(\mathcal{J}V, V)$ is positive in the subspace H_+ and negative in the complementary subspace $H \ominus H_+ \equiv H_-$, of a finite dimension m .

2. \mathcal{A} is densely defined in H , it satisfies the condition

$$Im(\mathcal{J}AV, V) \geq 0.$$

and has a discrete spectrum.

Then, according to the theorem cited above, it has a finite dimensional invariant subspace, and its eigenvalues in this subspace have the property $Im\lambda \leq 0$. For problem (40) this implies the existence of finitely many eigenvalues with $Re\lambda \geq 0$. But since they can not be purely imaginary, their real parts are positive.

With this information, the proof of instability of the solution (2) can be carried out by standard methods.

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