ON LANDAU'S SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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1. Introduction. In this note we will study a special class of solutions of the three-dimensional steady-state Navier-Stokes equations

$$\begin{aligned} -\Delta u + u\nabla u + \nabla p &= 0\,, \\ \operatorname{div} u &= 0\,. \end{aligned} \tag{NSE}$$

The equations have a non-trivial scaling symmetry $u(x) \to \lambda u(\lambda x)$ and it is natural to try to find solutions which are invariant under this scaling. Explicit examples of such solutions were first calculated by L.D.Landau in 1944 ([L]) and can be found in standard textbooks. (See, for example, [LL], p. 82, or [B], p. 206.) The main idea of Landau's calculation is that if we impose an additional symmetry requirement, namely that the solutions are axi-symmetric, the system (NSE) reduces to a system of ODEs which, surprisingly, can be solved explicitly in terms of elementary functions. The solutions were also independently found by H.B.Squire in 1951 ([Sq]). More recently, the topic has been re-visited in [TX] and [CK], where issues concerning Landau's solutions are addressed from a slightly different viewpoint.

In this note we prove that even if we drop the requirement of axi-symmetry, Landau's solutions are still the only solutions of (NSE) which are invariant under the natural scaling. More precisely, we will prove the following:

Theorem 1. *Assume that $u\colon R^3 \setminus \{0\} \to R^3$ is a non-trivial smooth solution of (NSE) satisfying $\lambda u(\lambda x) = u(x)$ for each $\lambda > 0$. Then u is a Landau solution. In other words, u is axi-symmetric and, in a suitable coordinate frame, is described by formulae (E7) in Section 4.*

The proof of the theorem shows a connection between the scale-invariant solutions of (NSE) and the conformal geometry of the two-dimensional sphere. In fact, once the connection is understood, the formulae for Landau's solutions can be derived without much calculation, using just the geometrical properties of the two-dimensional sphere.
Some implications of Theorem 1 are considered in the next two sections.

2. Implications for regularity of very weak solutions. By a very weak solution of the steady-state Navier-Stokes system (NSE) in a domain $\Omega \subset R^3$ we mean a divergence-free vector field $u = (u_1, u_2, u_3) \in L^2_{\rm loc}(\Omega)$ which satisfies

$$\int_\Omega (u_i \Delta\varphi_i + u_i u_j \frac{\partial \varphi_i}{\partial x_j}) = 0$$

for each smooth, compactly supported, divergence-free vector field $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ in Ω.

It seems to be an open problem whether very weak solutions of (NSE) are regular. Standard regularity theory can be used to show that very weak solutions are regular under the additional requirement that $u \in L^3_{\rm loc}$. The equations (NSE) are usually considered with the assumption that $\nabla u \in L^2_{\rm loc}$, in which case regularity follows by a straightforward bootstrapping argument. The assumption $\nabla u \in L^2_{\rm loc}$ is of course very natural when considering solutions describing real physical flows. However, one can speculate that very weak solutions might arise from a blow-up procedure of the usual weak solutions of the time-dependent Navier-Stokes equations at a possible singularity (if a singularity exists). The time-dependent 3-dimensional Navier-Stokes equations are supercritical with respect to the natural energy estimates, and in a blow-up procedure the information about energy can be lost.

A natural first step in understanding the regularity of the very weak solutions above is to study the scale-invariant solutions in R^3 which are smooth in $R^3 \setminus 0$. A calculation (which can be found in [B], p. 209, and also in [T], and [CK]) shows that Landau's solutions are not very weak solutions of (NSE) across the origin. Therefore Theorem 1 implies the following.

Corollary. *Let u be a (-1)-homogeneous very weak solution of the Navier-Stokes equations in R^3, which is smooth away from the origin. Then $u \equiv 0$.*

This result rules out only the simplest conceivable singularity of a very weak solution. For example, the question if one can have a non-trivial very weak solution smooth away from the origin and satisfying $|u(x)| \le C|x|^{-1}$ in R^3 is not answered by Theorem 1 and – as far as I know – remains open.

Remark: The question about the existence of (-1)-homogeneous solutions of the system (NSE) in $R^n \setminus \{0\}$ (smooth away from the origin) can also be posed for $n \ge 4$. It turns out that for $n \ge 4$ there are no such solutions. This was proved independently by several authors ([St], [Sv], [T]) and it also follows from results in [FR].

3. Asymptotic behavior of solutions in exterior domains. Theorem 1 has some relevance for the problem of long-range behavior of solutions of the Navier-Stokes equations in exterior domains. (See for example [G] for an overview of this topic.) For simplicity, we consider here only the following special case. Let f be a compactly supported vector field in R^3 and consider the equations

$$\begin{aligned} -\Delta u + u\nabla u + \nabla p = f\,, \\ \operatorname{div} u = 0 \end{aligned} \qquad \text{in} \quad R^3\,,$$

together with a "boundary condition" at ∞, which might take the form $u(x) \to 0$ at ∞ and $\int_{R^3} |\nabla u|^2 < \infty$, for example. The existence of such solution was proved in a classic paper by Leray ([Le]), but there are many open questions about the behavior of these solutions for large x, see [G]. Theorem 1 implies, roughly speaking, the following:

If a solution of the above exterior problem is asymptotically (-1)*-homogeneous, then the terms of order* $|x|^{-1}$ *must be given by a Landau solution.*

To give this a more precise meaning, let us consider the scaled functions u_λ and f_λ defined by $u_\lambda(x) = \lambda u(\lambda x)$ and $f_\lambda(x) = \lambda^3 f(\lambda x)$. The functions u_λ and f_λ satisfy the same equations as u and f. Moreover the functions f_λ converge to a distribution $\bar f$, given by $\bar f(x) = b\delta(x)$, where $b = \int_{R^3} f$ and δ is the Dirac function. *Assume* now that u_λ converges to a limit $\bar u$ in, say, $L^3_{\rm loc}(R^3 \setminus \{0\})$ as $\lambda \to \infty$. Our notion of "asymptotically (-1)- homogeneous" used above can be *defined* by requiring that this is really the case. (Of course, it is a difficult open problem to decide whether, for general large data, this might always be the case.) The limit functions $\bar u$ and $\bar f$ will again satisfy the same equations (in the sense of distributions). Under our assumptions the function $\bar u$ is smooth away from the origin, satisfies $\lambda \bar u(\lambda x) = \bar u(x)$ for each $\lambda > 0$, and, by Theorem 1, must therefore be a Landau solution or vanish identically. (The direction of the vector b will be the axis of symmetry of the solution.) For $b = 0$ we will have $\bar u = 0$, which means that, under the above assumptions, the solution u decays faster than $|x|^{-1}$.

4. Proof of Theorem 1. Let u be a (-1)-homogeneous vector field in R^3, smooth away from the origin. Clearly u is determined by its restriction to the unit sphere $S^2 \subset R^3$. For $x \in S^2$ we decompose $u(x)$ as $u(x) = v(x) + f(x)n(x)$, where $n(x) = x$ is the outer unit normal to S^2, and $v(x)$ is tangent to S^2 at x, i. e. $v(x) \cdot n(x) = 0$. We now write down the Navier-Stokes equations for u and as a system of PDEs on S^2. If u satisfies the Navier-Stokes equation in $R^3 \setminus \{0\}$ in the very weak sense defined above, it is easy to see that there exists a suitable pressure function p in $R^3 \setminus \{0\}$ which is (-2)-homogeneous and smooth away from the origin. The function p is also determined by its values on S^2, and the system (NSE) can be written down as a system of PDEs on S^2 for v, f and p. The differential operators in what follows will all be differential operators on S^2, defined by the usual conventions of Riemannian geometry. The differential forms on S^2 will be identified with vector fields and vice-versa, as is usual on Riemannian manifolds. The Hodge Laplacian $d\,d^* + d^*d$ on 1-forms will be denoted by $-\Delta_{\rm H}$. (The reason for writing it as $-\Delta_{\rm H}$, with the minus sign, is to keep the equations on S^2 in a form which resembles the standard euclidean form of the equations as much as possible.) The Navier-Stokes equations (NSE) for u written in terms of v, f and p as equations on S^2 are as follows:

$$
\begin{aligned}
-\Delta_{\rm H} v + v\nabla v + \nabla(p - 2f) &= 0\,, \\
-\Delta f + v\nabla f - f^2 - |v|^2 - 2p &= 0\,, \\
\operatorname{div} v + f &= 0\,.
\end{aligned}
\tag{S}
$$

A straightforward (although perhaps not the most illuminating) way to derive these equations is to write the system (NSE) in spherical coordinates (see, for example, [B], p. 601) and check that for (-1)-homogeneous vector fields it reduces to the system (S). We remark that the spherical coordinates version of (NSE) in the second edition of the book [LL] (p. 49) contains a misprint in the right-hand side of the first equation, where an incorrect expression $\sin^2\theta$ appears instead of the correct $\sin\theta$.

We will denote by ω the function on S^2 given by $dv = \omega\Omega_0$, where Ω_0 is the canonical volume form of S^2. This corresponds to the formula $\omega = \operatorname{curl} v$ used in R^2.

By taking d of the first equation of the system (S) we obtain

$$-\Delta\omega + \operatorname{div}(v\omega) = 0. \tag{E1}$$

Lemma 1. With the notation introduced above, we have $\omega \equiv 0$.

Proof. Let L be the differential operator defined by $Lw = -\Delta w + \operatorname{div}(vw)$. The adjoint operator L^* is given by $L^*w = -\Delta w - v\nabla w$. The kernel of L^* consists of constant functions, as can be seen from the strong maximum principle. The kernel of L must therefore also be one dimensional. Let us denote by w_0 a non-trivial function in the kernel of L. If w_0 changed sign on S^2, we could find a strictly positive smooth function h on S^2 with $\int_{S^2} w_0 h = 0$. But this would mean that the equation $L^*w_1 = h$ has a solution. However, the last equation cannot be satisfied at points where w_1 attains its minimum. From this we see that the function ω cannot change sign. At the same time, the definition of ω immediately implies that $\int_{S^2}\omega = 0$, and we see that ω must vanish.

Remark. I assume the above argument is known in one form or another, but I was not able to find a good reference for it.

Once we know that $dv = 0$, the first equation of (S) simplifies. Indeed, when $dv = 0$ we have $-\Delta_{\mathrm{H}} v = -\nabla \operatorname{div} v = \nabla f$, and we also have $v\nabla v = \nabla |v|^2/2$. Using this, the first equation of (S) implies

$$\frac{1}{2}|v|^2 + p - f = c\,,$$

where c is a constant. The second equation of (S) now gives

$$-\Delta f - 2f + \operatorname{div}(fv) = 2c\,. \tag{E2}$$

Integrating (E2) over S^2 and using the third equation of (S) we see that $c = 0$. Since $dv = 0$ we can write $v = \nabla\varphi$ for a suitable smooth function φ on S^2. The equation (E2), together with the third equation of (S) and the fact that $c = 0$ now gives

$$\Delta^2\varphi + 2\Delta\varphi - \operatorname{div}(\Delta\varphi\nabla\varphi) = 0\,. \tag{E3}$$

Letting $w = 2 - \Delta\varphi$, the last equation can be re-written as

$$-\Delta w + \operatorname{div}(\nabla\varphi w) = 0\,.$$

The solutions of this equation are well-known: They are functions of the form $c_1 e^{\varphi}$, where c_1 is a constant. (An easy way to verify this is for example the following: Write w in the form $c_1(x)e^{\varphi(x)}$. We get an equation for c_1 for which the strong maximum principle implies that the solutions are exactly $c_1(x) \equiv \text{const.}$) Integrating w over the sphere we see that $c_1 > 0$. Hence we have

$$-\Delta\varphi + 2 = c_1 e^{\varphi}$$

for a constant $c_1 > 0$. Changing φ by a constant, if necessary, we can assume $c_1 = 2$ without loss of generality, and we end up with

$$-\Delta\varphi + 2 = 2e^{\varphi} \,. \tag{E4}$$

The interpretation of equation (E4) is well-known (see, for example, [CY]): Let $\bar{g}$ be the canonical metric on S^2 and let g be the metric on S^2 defined by $g = e^{\varphi}\bar{g}$. Equation (E4) says exactly that the Gauss curvature of the metric g is 1, i. e. the metric g is isometric to the metric $\bar{g}$. In other words, we have $g = h^*\bar{g}$ (pullback of $\bar{g}$ by h) for a suitable diffeomorphism h of S^2. From the definitions we also see that h has to be conformal or anti-conformal. Anti-conformal maps can be obtained from conformal maps by a composition with an isometry, and hence we can only consider the case when h is conformal. For a given conformal h, the function φ is given by

$$\varphi(x) = \log|h'(x)|^2 \,, \tag{E5}$$

where $h'(x)$ denotes the (complex) derivative of h at x. It is well-known (see e. g. [DFN]) that all conformal diffeomorphisms of S^2 can be produced as follows. Let $P\colon S^2 \to \mathbf{C}$ be the standard stereographic projection, and let $M_\lambda\colon \mathbf{C} \to \mathbf{C}$ be defined by $z \to \lambda z$. Let $h_\lambda = P^{-1} \circ M_\lambda \circ P$. Then any conformal diffeomorphism of S^2 can be produced by composing a suitable h_λ (with $\lambda > 0$) with isometries of S^2. If φ is given by (E5) and we compose h with and isometry, then the function φ either does not change or only changes by being shifted by the isometry. Therefore in a suitable coordinate frame all solutions φ of (E4) look like the solutions generated by the special h_λ above. We now consider the standard spherical coordinates (θ, ψ) on S^2, given by

$$\begin{aligned} x_1 &= \sin\theta\cos\psi \,, \\ x_2 &= \sin\theta\sin\psi \,, \\ x_3 &= \cos\theta \,. \end{aligned} \tag{SC}$$

We will use the usual notation $e_\theta = \frac{\partial x}{\partial\theta}$ for the tangent vector field on S^2 corresponding to $\frac{\partial}{\partial\theta}$. Letting $\lambda = e^{-\kappa}$, calculating the maps h_λ above in these coordinates, and using the formula (E5), we obtain

$$\varphi(x) = -2\log\left(\cosh\kappa - \sinh\kappa\,\cos\theta\right) . \tag{E6}$$

This gives

$$\begin{aligned} v &= \frac{\partial\varphi}{\partial\theta}\,e_\theta = \frac{-2\sin\theta}{\coth\kappa - \cos\theta}\,e_\theta \,, \\ f &= -\,\Delta\varphi = 2e^{\varphi} - 2 = \frac{2}{(\cosh\kappa - \sinh\kappa\,\cos\theta)^2} - 2 \,, \end{aligned} \tag{E7}$$

which agrees with the formulae in [B], p. 207 if we set $\coth\kappa = 1 + c$ and with the formulae in [LL], p. 82, if we set $\coth\kappa = A$. The proof of Theorem 1 is finished.

Remarks:

1. As we already mentioned in Section 1, the Landau solutions (given by (E7)) do not satisfy the Navier-Stokes equations (NSE) across the origin. A calculation in [B], p. 209, shows that for Landau's solutions we have, in distributions,

$$-\Delta u + \operatorname{div}(u \otimes u) + \nabla p = b\delta\,,$$

where δ is the Dirac function and $b = b(\kappa)$ is a non-zero vector in R^3 depending in a non-trivial way on the parameter κ which parametrizes the solutions in the above coordinate frame. The exact formula for b can be found in [B], p. 209, and was also calculated in [CK].

2. If $h\colon S^2 \to S^2$ is a non-trivial holomorphic map (which is not necessarily a diffeomorphisms) the formula (E5) gives a function φ which is regular away from a finite set $a_1, \dots, a_m \in S^2$ where h' vanishes. The function φ will generate a (-1)-homogeneous solution of the Navier-Stokes equations in the region $R^3 \setminus (\cup_{j=1}^{j=m} R_+ \cdot a_j)$, where $R_+ = [0, \infty)$. However, the vector field will not be locally square integrable in R^3, except for the case of Landau's solutions, when h' does not vanish at any point.

5. Open problems. An interesting problem is to try to repeat the above analysis when R^3 is replaced by the half-space $R^3_+ = \{x \in R^3,\, x_3 > 0\}$ and the boundary condition $u = 0$ is imposed on $\partial R^3_+ \setminus \{0\}$. A simple calculation shows that there are no non-trivial axi-symmetric (-1)-homogeneous solutions in that case. However, it seems to be an open problem if this conclusion is still true without assuming the rotational symmetry. We refer the reader to the very interesting paper [Se], where a related situation is studied in a different context.

Another interesting question is the following: *Among smooth vector fields in $R^3 \setminus \{0\}$ satisfying $|u(x)| \le C|x|^{-1}$ for some $C > 0$, are the Landau solutions the only ones which satisfy the Navier-Stokes equations (NSE) in $R^3 \setminus \{0\}$?* Such questions are relevant for the problem of asymptotic behavior of steady-state solutions in exterior domains mentioned in Section 3. A first natural step in addressing this question is to look at possible infinitesimal deformations of Landau solutions in the above class. This leads to linear equations which can be reduced to ODEs by classical methods of separation of variables, due to the symmetries of Landau's solutions. Based on numerical experiments with these ODEs, the author conjectures that the Landau solutions are rigid with respect to infinitesimal deformations, i. e. it seems that there are no new solutions bifurcating from Landau's solutions.

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