CONTINUITY OF CALDERÓN-ZYGMUND OPERATORS

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En hommage respectueux à Luc Tartar

SUMMARY. Twenty years ago David and Journé discovered a criterion for the continuity on L^2 of Calderón-Zygmund operators defined by singular integrals. In their approach the distributional kernel of the given operator is locally Hölder continuous outside the diagonal. The aim of this paper is to prove a David-Journé theorem where this smoothness assumption is replaced by a weaker one. Our approach strongly relies on an algorithm developed by Beylkin, Coifman and Rokhlin.

1. The main theorem

Boundedness of Calderón-Zygmund operators under optimally weak regularity assumptions is an intriguing problem. This problem is far from being an academic question. Indeed standard Fourier methods do not work when the operator under study does not commute with translations. New tools are then needed. The first ingredient which will be used is the following fact : the paraproduct (defined below) between a function in BMO (which is defined below) and a function in L^2 still belongs to L^2. Since $\log|x|$ belongs to BMO, the Banach space BMO differs from L^∞ and the pointwise product between a function in BMO and a function in L^2 does not belong to L^2 in general. A variant of this action of BMO on L^2 already appears in one of the formulations of the div-curl lemma of Tartar and Murat (see Lemma 1 below). The second ingredient is related to some numerical schemes which have been proposed by Beylkin, Coifman and Rokhlin and which led to the Fast Multipole Algorithm. The ultimate goal is to replace the standard regularity assumptions on the distributional kernel of the operator (see (1.2)) by Hörmander condition (see (1.6)).

For the reader's convenience, the definitions of BMO and of the distributional kernel $K(x, y)$ of an operator T will be given now. Let T be a linear operator which is defined on the space $S(\mathbb{R}^n)$ of testing functions with values in the dual space $S'(\mathbb{R}^n)$ of tempered distributions. The distributional kernel $K(x, y)$ of $T : S(\mathbb{R}^n) \mapsto S'(\mathbb{R}^n)$ is a distribution in $2n$ variables which is defined by the following condition: for every pair (f, g) of two testing functions, we have $\langle T(f), g\rangle = \int\int K(x, y)g(x)f(y)dx\, dy$. In the standard theory of Calderón-Zygmund operators the distributional kernel $K(x, y)$ of T is locally Hölder continuous in the

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open set $\Omega = \{(x,y) \in \mathbb{R}^{2n};\ x \neq y\}$ and satisfies the following properties:

(1.1) $|K(x,y)| \leq \frac{C}{|x-y|^n}$ for every x and every $y \neq x$.

(1.2) There exists an exponent γ belonging to $(0,1)$ and a constant C such that $|K(x,y) - K(x,y')| + |K(y,x) - K(y',x)| \leq \frac{C|y-y'|^\gamma}{|x-y|^{n+\gamma}}$ when $|x-y| \geq 2|y-y'| > 0$.

These two properties are describing the restriction to $\Omega = \{(x,y) \in \mathbb{R}^{2n};\ x \neq y\}$ of the distributional kernel of T. This kernel is not locally integrable and the integral $Tf(x) = \int K(x,y)f(y)\,dy$ is a singular integral. That is why T is often referred to as a singular integral operator. In [3] David and Journé proved a theorem which immediately became famous under the name of the "$T(1)$ theorem". It says the following. Let T be an operator whose kernel satisfies (1.1) and (1.2). Then T is bounded on L^2 if and only if the following two conditions are satisfied:

(1.3) $|\iint K(x,y)f(x)g(y)dxdy| \leq C|Q|(||f||_\infty + |Q|^{\frac{1}{n}}||f'||_\infty)(||g||_\infty + |Q|^{\frac{1}{n}}||g'||_\infty)$ for every cube Q with volume $|Q|$ and every pair (f,g) of two continuously differentiable functions supported by Q

$$\alpha = T(1) \in BMO,\ \beta = T^*(1) \in BMO. \tag{1.4}$$

Let us comment on (1.3) and (1.4). Property (1.3) is usually referred as the "weak boundedness property". It is a scale invariant version of the hypothesis that T is defined on the space of compactly supported testing functions with values in the space of tempered distributions. The continuity on L^2 obviously implies (1.3). In (1.4) 1 stands for the constant function 1. The meaning of $T(1)$ is not clear. It is a tempered distribution modulo a constant function. Indeed if ψ denotes a test function with compact support and a vanishing integral then $\langle T(1), \psi\rangle = \langle 1, T^*(\psi)\rangle$ which is well defined since (1.2) implies that $T^*(\psi)$ is $O(|x|^{-n-\gamma})$ at infinity. The space BMO of functions with bounded mean oscillations was defined and studied by John and Nirenberg in [7]. This space consists of all functions $f(x) \in L^2_{loc}$ for which a constant C exists with the following property: for every ball $B \subset \mathbb{R}^n$ we have $(\frac{1}{|B|}\int_B |f(x) - m_B f|^2 dx)^{\frac{1}{2}} \leq C$ where $m_B f = \frac{1}{|B|}\int_B f dx$. The optimal C is the norm of f in BMO which implies that constant functions have a zero norm. Therefore a function in BMO is only defined modulo a constant. The space BMO contains L^∞ but enjoys some important properties which are not shared by L^∞. Calderón-Zygmund operators are not bounded on L^∞ but are bounded on BMO. More precisely Jack Peetre proved that any operator T which is bounded on L^2 and satisfies (1.2) maps L^∞ into the space BMO. Furthermore T maps BMO into itself if and only if $T(1) = 0$. The argument used by Peetre gives more as it will be told below. To conclude (1.3) and (1.4) are necessary to the continuity of T on L^2.

The converse implication in the "$T(1)$ theorem" is deeper and says that (1.2), (1.3) and (1.4) imply the continuity on L^2. For proving it David and Journé introduced two auxiliary operators T_α and T_β which are defined by the following conditions: $T_\alpha(f)$ is the paraproduct between f and $\alpha \in BMO$ and similarly the adjoint of T_β is the paraproduct between f and $\beta \in BMO$. We remind the reader that $T_\alpha(f) = \sum_{-\infty}^{\infty} \Delta_j(\alpha)S_{j-3}(f)$ where the notations refer to the classical Littlewood-Paley analysis. We then obviously have $T_\alpha(1) = \alpha$ and $T^*_\alpha(1) = 0$.

Nowadays the boundedness of these two operators on L^2 is an easy exercize using the famous characterization of BMO by Carleson measures. This construction is closely related to the div-curl lemma. Indeed this lemma says the following :

Lemma 1. *If $E = (E_1, ..., E_n)$ and $B = (B_1, ..., B_n)$ are two vector fields belonging to $L^2(\mathbb{R}^n) \times \cdots \times L^2(\mathbb{R}^n)$ and if both the divergence of E and the curl of B vanish identically, then $E \cdot B(x)$ belongs to the Hardy space H^1.*

For proving the div-curl lemma, one argues by duality and constructs a brand of paraproduct between a function in BMO and a function in L^2. During the conference Luc Tartar said that Lemma 1 should not be named the div-curl lemma since his formulation (with François Murat) of the div-curl lemma concerned weak limits of sequences. But our formulation also applies to this setting. Indeed H^1 is weakly sequencially complete. It means that every sequence which is bounded in H^1 contains a subsequence which is weakly convergent to a function in H^1. This is the Murat-Tartar div-curl lemma. The space L^1 is not weakly-sequencially complete.

It is now time to return to our Calderón-Zygmund operators. The continuity of T on L^2 is then reduced to the boundedness of $R = T - T_\alpha - T_\beta$. In other words (1.4) can be replaced by :

$$T(1) = T^*(1) = 0. \tag{1.5}$$

If (1.2), (1.3) and (1.5) are satisfied the continuity of T can be proved using a beautiful lemma devised by Mischa Cotlar and improved by Cotlar and Stein. We do not say more since Cotlar lemma does not apply to our framework and the reader is referred to [11].

This decomposition of T as a sum $T = T_\alpha + T_\beta + R$ can be used for proving estimates on most of the functional spaces. When T satisfies (1.1), (1.2), (1.3), and (1.5) one writes $T \in OpE_\gamma$. For $T \in OpE_\gamma$ and $0 \le s < \gamma$, P.G.Lemarié-Rieusset proved that T is continuous on the homogeneous Besov spaces $\dot{B}^{s,q}_p$ [9]. Frazier, Jawerth, Han and Weiss extended this theorem to the Triebel-Lizorkin spaces $\dot{F}^{s,q}_p$. Indeed they proved in [6] that an atom is mapped into a molecule by T.

More difficult problems arise if one tries to replace (1.2) by a weaker condition.

Definition 1. *One writes $T \in OpH$ if* (1.3) *and* (1.5) *are satisfied together with the following Hörmander condition :*

(1.6) *There exists a constant C such that for every $y \in \mathbb{R}^n$ and every $y' \neq y$, we have* $\displaystyle\int_{|x-y| \ge 2|y-y'|} \{|K(x,y) - K(x,y')| + |K(y,x) - K(y',x)|\}dx \le C.$

The Hörmander condition is weaker than (1.2) and is needed to give a meaning to $T(1)$ and $T^*(1)$ in (1.5). At the present time we do not know whether any operator $T \in OpH$ is bounded on L^2 or not. More generally the continuity of $T \in OpH$ on Besov or Triebel-Lizorkin spaces is raising many interesting problems. Le us begin with two observations.

Lemma 2. *If $T \in OpH$, then T maps the homogeneous Besov space $\dot{B}^{0,1}_1$ into L^1.*

This is proved in [10] and it does require the boundedness of T on L^2. We obviously have $\dot{B}_1^{0,1} \subset H^1$ where H^1 denotes the Hardy space. Then a remarkable result by Jack Peetre says more.

Lemma 3. *Any $T \in OpH$ which is bounded on L^2 maps the Hardy space H^1 into L^1.*

Whe say that T is a convolution operator if $T(f) = f \star S$ for some generalized function S. If $T \in OpH$ is a convolution operator then T is bounded on L^2 and on the Hardy space H^1. Let us sketch the proof of these results. Let $\psi_j(x) = 2^{nj}\psi(2^j x)$ where ψ is a function in the Schwartz class with a vanishing integral. This sequence ψ_j is bounded in $\dot{B}_1^{0,1}$. Therefore $\|S \star \psi_j\|_1$ is bounded by Lemma 1. Moving to the Fourier transforms $\hat{S}\hat{\psi}_j$ is a bounded sequence in L^∞ and this implies $\hat{S} \in L^\infty$. Therefore T is bounded on L^2. We now treat the Hardy space case when $T \in OpH$ is a convolution operator. If R_j, $1 \le j \le n$, denote the Riesz transforms then $f \in H^1$ means that f and its n Riesz transforms $R_j(f)$ altogether belong to L^1. Moreover the Riesz transforms are bounded on H^1. These properties imply that every convolution operator T which maps H^1 into L^1 is mapping H^1 into H^1. Indeed we have $R_jT = TR_j$ which ends the proof. Arguing by duality it implies that T is bounded on the space BMO. One would like to extend this result to every $T \in OpH$. Theorem 1 below yields a partial answer. The kernel $K(x,y)$ is assumed to be locally integrable in $\Omega = \{(x,y) \in \mathbb{R}^{2n};\ x \neq y\}$ and new moduli of continuity are now defined as in [10] :

Definition 2. *For $(u,v,y) \in \mathbb{R}^{3n}$, for $r > 0$ and for each integer $R \in \mathbb{N}$, one considers the integrals $I(R,r,u,v,y) =$*
$$\int_{2^R r \le |x-y| < 2^{R+1} r} \{|K(x+u,y+v) - K(x,y)| + |K(y+u,x+v) - K(y,x)|\} dx.$$
One then defines $\epsilon(K,R) = \epsilon(R)$ as being the supremum of $I(R,r,u,v,y)$ over all $r > 0$, all $(u,v) \in \mathbb{R}^{2n}$ fulfilling $|u| + |v| \le r$, and all $y \in \mathbb{R}^n$.

For $a > 0$ and $u \in \mathbb{R}^n$, the kernels $K(x,y)$ and $\tilde{K}(x,y) = a^n K(ax+u, ay+u)$ are defining two operators T and $\tilde{T}$ which have the same operator norm acting on L^p. One immediately notices that $\epsilon(\tilde{K},R) = \epsilon(K,R)$.

One could also consider the double integral $J(R,r,u,v,z) =$
$$r^{-n} \int_{\{|y-z|\le r\}} \int_{\{2^R r \le |x-y| < 2^{R+1} r\}} \{|K(x+u,y+v) - K(x,y)| + |K(y+u,x+v) - K(y,x)|\}\, dxdy$$
and define $\eta(R)$ as being the supremum of $J(R,r,u,v,z)$ over (r,u,v,z) as above. Theorem 1 will remain valid if $\varepsilon(R)$ is being replaced by $\eta(R)$.

Let $\mu \ge 0$ be an exponent. A new class of Calderón-Zygmund operators is now defined :

Definition 3. *One writes $T \in OpM_\mu$ if (1.3), (1.5) are satisfied together with*

$$(1.7) \quad \sum_{R \ge 1} R^\mu \varepsilon(R) < +\infty.$$

We have $OpM_\mu \subset OpM_\nu$ if $\nu \le \mu$. If $T \in OpM_\mu$ and $\mu \ge 0$, then $T \in OpH$. The property $T \in OpM_0$ is slightly more precise that Hörmander's condition (1.6). If $T \in OpM_\mu$ so does its adjoint T^*. It is not difficult to prove that $T \in OpE_\gamma$ implies

$\varepsilon(R) < C2^{-R\gamma}$. But $\varepsilon(R) \le CR^{-\beta}$ and $\beta > 1+\mu$ imply $T \in OpM_\mu$. Obviously $OpE_\gamma \subset OpM_\mu$.

The first author proved the following: If $T \in OpM_1$ then T and its adjoint T^* are bounded on $\dot{B}_1^{0,1}$. Therefore T is bounded on L^2. The hypothesis $T \in OpM_1$ was at that time the weakest regularity assumption implying L^2 estimates. The proof uses the action of T on the so-called weak molecules [10]. Afterwards Han and Hofmann proved that if T belongs to a space which is slightly different from OpM_1 then for $1 \le p, q \le \infty$, T is bounded on $\dot{B}_p^{0,q}$. Moreover for $1 < p, q < \infty$, T is bounded on $\dot{F}_p^{0,q}$ [8]. But these methods do not extend to the continuity on $BMO = (\dot{F}_1^{0,2})^*$. Deng, Yan et Yang constructed an operator $T \in OpH$ which is bounded on $\dot{B}_1^{0,1}$ but is not bounded on $\dot{F}_1^{0,2}$ [5]. This operator is not a convolution operator. This construction raises the following problem. What is the smallest exponent μ such that every $T \in OpM_\mu$ is bounded on BMO? The answer given by Theorem 1 says that this exponent belongs to the interval [1, 3/2]. Concerning the continuity on L^2 the author proved that the minimal exponent ν with the property that every $T \in OpM_\nu$ is bounded on L^2 belongs to $[0, 1/2]$. This is striking since it says that for $T \in OpH$ the continuity on L^2 does not imply the continuity on BMO. This sharply contrasts with the case of a convolution operator.

Our main result says that there exists an operator T which belongs to OpM_μ for every $\mu \in (0,1)$ and which is not bounded on BMO. In contrast every $T \in OpM_{3/2}$ is bounded on BMO.

Theorem 1. *(i) Every $T \in OpM_{3/2}$ is bounded on BMO.*

(ii) There exists an operator T with the following properties:

(a) *For all $\mu \in (0,1)$ we have $T \in OpM_\mu$*

(b) *For all $1 \le p < \infty, 1 \le q \le \infty, T$ is bounded on $\dot{F}_p^{0,q}$ and on $\dot{B}_p^{0,q}$*

(c) *However T is not bounded on BMO.*

The proofs will appear in Rev. Mat. Complut. Let us mention some related work. Donggao Deng and Yongsheng Han are writing a remarkable book entitled “Harmonic Analysis on Spaces of Homogeneous Type” where Calderón-Zygmund theory is being developed in a general setting. A space of homogeneous type is a set X equipped with a distance $d(x,y)$ and a positive measure μ such that for any ball $B = B(x,R)$, we have $\mu(\tilde{B}) \le C\mu(B)$ where $\tilde{B}$ is the ball centered at x with radius $2R$. Working on spaces of homogeneous type is a challenge since one cannot rely on the Fourier transformation. It implies that some adapted spectral theory should be built which is what Donggao Deng and Yongsheng Han are doing.

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