Homogeneous hyperbolic Boundary Value Problems ; The variational case

Denis SERRE
UMPA, UMR 5669 CNRS
Ecole Normale Supérieure de Lyon

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En hommage à Luc TARTAR

Variational linear IBVP: the constant coefficient case

Set $\Omega = \mathbb{R}^{d-1} \times (0, +\infty) = \{x \in \mathbb{R}^d ; \, x_d > 0\}$.

Start with a quadratic Lagrangian ($u : \mathbb{R} \times \Omega \to \mathbb{R}^n$)

$$\mathcal{L}[u] := \int_{\mathbb{R}} \int_{\Omega} \left(\frac{1}{2} |\partial_t u|^2 - W(\nabla_x u) + f \cdot u \right) dx \, dt.$$

Write the Euler-Lagrange equations

$$\textbf{(Hyp)} \qquad \partial_t^2 u + Pu = f \qquad \text{in } (0,T) \times \Omega,$$

with

$$(Pu)_j = - \sum_{\alpha=1}^{d} \partial_\alpha \left(\frac{\partial W}{\partial F_{\alpha j}} (\nabla_x u) \right), \qquad j = 1, \dots, n,$$

The calculus of variations gives also a Neumann-type boundary condition (homogeneous !)

$$\textbf{(BC)} \qquad (Bu)_j := \sum_\alpha \nu_\alpha \frac{\partial W}{\partial F_{\alpha j}} (\nabla_x u) = 0, \qquad j = 1, \dots, n.$$

Examples

Notation: (internal or potential energy) $\mathcal{W}[u] = \int_\Omega W(\nabla_x u)dx$.

- The wave equation with Neumann boundary condition ($n = 1$)

$$\mathcal{W}[u] = \frac{c^2}{2}\int_\Omega |\nabla_x u|^2 dx.$$

Then

$$(\partial_t^2 - c^2\Delta_x)u = f, \qquad \frac{\partial u}{\partial \nu} = 0.$$

- The linear elasticity with zero normal stress ($n = d$). For instance

$$W(F) = \frac{\lambda}{4}|F + F^T|^2 + \frac{\mu}{2}(\mathrm{Tr}\, F)^2,$$

with λ, μ the constant Young moduli, such that

$$\lambda > 0, \qquad 2\lambda + \mu > 0$$

for rank-one convexity.

Hyperbolicity

Consider the Cauchy problem instead ($\Omega = \mathbb{R}^d$).

Proposition. (Classical) The Cauchy problem is strongly well-posed in some (and then in all) Sobolev space $H^s(\mathbb{R}^d)^n$ if, and only if, the operator $\partial_t^2 + P$ is strongly hyperbolic.

♡

Proposition. (Elementary) $\partial_t^2 + P$ is strongly hyperbolic if, and only if, $F \mapsto W(F)$ is strictly rank-one convex. That is $\exists c_0 > 0$ s. t.

$$W(\xi \otimes v) \geq c_0 |\xi|^2 |v|^2, \qquad \forall \xi \in \mathbb{R}^d, \forall v \in \mathbb{R}^n.$$

♠

Basic assumption

From now on, one always **assumes** hyperbolicity:

W is strictly rank-one convex.

Warning

- If $\Omega = \mathbb{R}^d$,
  - Strict rank-one convexity tells that $\frac{1}{C}\|\nabla u\|_{L^2}^2 \leq \mathcal{W}[u] \leq C\|\nabla u\|_{L^2}^2$.
  - The addition of a null-form to W does not modify $\mathcal{W}$.
- All **False** if $\Omega \neq \mathbb{R}^d$. The addition of a null form may change B.

The convex case (I)

... Back to the half-space.

Assume for a moment that $\mathcal{W} \geq C\|\nabla u\|_{L^2}^2$ ($C > 0$).

Energy estimate:

$$\partial_t \left(\frac{1}{2}|u_t|^2 + W(\nabla u)\right) + \operatorname{div}\left(\frac{\partial W}{\partial F}(\nabla u) u_t\right) = f \cdot u_t.$$

Integrate over Ω, then use the boundary condition:

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \mathcal{W}[u]\right) = \int_\Omega f \cdot u_t dx - \int_{\partial\Omega} (Bu) \cdot u_t \, dy = \int_\Omega f \cdot u_t dx.$$

The convex case (II)

With Cauchy–Schwarz, one gets the so-called *maximal* estimate

$$
\begin{aligned}
e^{-2\gamma T}\|\nabla_{x,t}u(T)\|_{L^2}^2 \;&+\; \gamma\int_0^T e^{-2\gamma t}\|\nabla_{x,t}u(t)\|_{L^2}^2 dt \\
&\le\; C\left(\|(\nabla_{x,t}u)(0)\|_{L^2}^2+\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\|Lu(t)\|_{L^2}^2 dt\right).
\end{aligned} \tag{1}
$$

for every $\gamma > 0$ and every $T > 0$. Here $L := \partial_t^2 + P$.

Notice the invariance under the natural scaling $(x,t) \mapsto (\lambda x, \lambda t)$.

Questions

Assuming only strict rank-one convexity ...

- *Identify the quadratic energy densities W for which the IBVP in the half-space admits a maximal estimate,*
- *When does a maximal estimate imply the well-posedness of the IBVP ?*
- *What about more general domains $\Omega \subset \mathbb{R}^d$ and/or variable coefficients ($W = W(x,t;F)$) ?*

Comparison to Kreiss' theory (I)

- In H.-O. Kreiss' theory (1970, see also R. Sakamoto), the boundary condition is non-homogeneous:

$$Bu = g \qquad \text{(a data)}.$$

  Kreiss' maximal estimates contain additional terms:

$$\int_0^T \int_{\partial\Omega} |Bu|^2 dy\, dt \quad \text{in the input,}$$

  and

$$\int_0^T \int_{\partial\Omega} |\nabla_{x,t} u|^2 dy\, dt \quad \text{in the output.}$$

  Our boundary condition is *homogeneous*. No input $\longrightarrow$ we do not ask for an output.

- The pair $(\partial_t^2 + P, B)$ comes from a variational principle. Closer to physics ?

Comparison to Kreiss' theory (II)

- Same use of the Fourier transform in $y := (x_1, \dots, x_{d-1})$ and Laplace transform in t.

$$\nabla_y u \longrightarrow i\eta \otimes u \quad (\eta \in \mathbb{R}^{d-1}), \qquad \partial_t u \longrightarrow \tau u \quad (\tau \in \mathbb{C},\ \mathrm{Re}\,\tau > 0).$$

- Same characterization of the strong instability of Hadamard type: the Lopatinskii condition fails at some pair (τ, η) with $\mathrm{Re}\,\tau > 0$ and $\eta \in \mathbb{R}^{d-1}$. Necessary (but not sufficient!) condition for well-posedness.

- But the gap between maximal estimates (strong stability) and the Lopatinskii condition differs from that in Kreiss' theory.

Use of the Laplace–Fourier transform (I)

Notation. One needs only the restriction of W to matrices F in block form $F = (\eta \otimes v, z)$:

$$W(\eta \otimes v, z) = \frac{1}{2}\langle \Lambda z, z\rangle + \langle A_\eta z, v\rangle + \frac{1}{2}\langle \Sigma_\eta v, v\rangle,$$

with $\eta \mapsto A_\eta$ linear and $\eta \mapsto \Sigma_\eta$ quadratic.

Strict rank-one convexity tells that $\exists c_0 > 0$ s.t.

$$\forall \xi \in \mathbb{R}, \qquad \xi^2\Lambda + \xi(A_\eta + A_\eta^T) + \Sigma_\eta \geq c_0(\xi^2 + |\eta|^2)I_n.$$

Set

$$h(\eta) := \left\{ \min_{\xi \in \mathbb{R}} \lambda_1 \left(\xi^2\Lambda + \xi(A_\eta + A_\eta^T) + \Sigma_\eta\right)\right\}^{1/2} \quad (\geq \sqrt{c_0}|\eta|).$$

Use of the Laplace–Fourier transform (II)

In Fourier variables η, the maximal estimates decouple. Reduction to parametrized IBVP in $1+1$ dimension:

$$\begin{aligned} \partial_t^2 u - \Lambda u'' - i(A_\eta + A_\eta^T)u' + \Sigma_\eta u = f, &\qquad \text{in } x_d, t > 0, \\ \Lambda u' + iA_\eta^T u = 0, &\qquad \text{on } x_d = 0. \end{aligned}$$

The difficulty is that we need an estimate uniform in η.

Necessary condition for the maximal estimate:

- Drop the data f. The only input that remains is the initial data.
- Test on solutions of the form $u = e^{\tau t} v(x_d)$, with $v \in H^1(0, +\infty)$, $\mathrm{Re}\,\tau > 0$.

Use of the Laplace–Fourier transform (III)

Test $\longrightarrow$ reduction to an ODE, parametrized by (τ, η):

$$-\Lambda v'' - i(A_\eta + A_\eta^T)v' + \Sigma_\eta v + \tau^2 v = 0. \tag{2}$$

Definition. *Call v a stable solution of (2) if $v(+\infty) = 0$.*

Proposition 1 (adapted from Hersh). Assume

$$\eta \in \mathbb{R}^{d-1} \quad \text{and} \quad \tau^2 \in \mathbb{C} \setminus (-\infty, -h(\eta)^2].$$

Then the central subspace of (2) is trivial. The stable and unstable subspaces have equal dimensions:

$$\dim S(\tau, \eta) = \dim U(\tau, \eta) = n.$$

♣

The Lopatinskii condition

Necessary condition for the maximal estimates: The Lopatinskii condition holds at every pair (τ, η) with $\mathrm{Re}\, \tau > 0$, and also at every pair $(0, \eta)$ with $\eta \neq 0$.

> [*Lopatinskii condition* at (τ, η)] The only solution of $(2)_{(\tau,\eta)}$, satisfying the boundary condition $\Lambda v'(0) + iA_\eta^T v(0) = 0$, is $v \equiv 0$.

Failure of the Lopatinskii yields:

- Hadamard instability if $\mathrm{Re}\, \tau > 0$,
- Loss of regularity if $\tau = 0$ and $\eta \neq 0$.

Special parameters (I)

Lemma 1. Assume that $z \in (-h(\eta)^2, +\infty)$. Then the algebraic Ricatti equation

$$-\Lambda Q^2 - i(A_\eta + A_\eta^T)Q + \Sigma_\eta + zI_n = 0_n$$

admits a unique stable solution $Q(z,\eta)$. In addition, $Q(z,\eta)$ has the properties

1. $H(z,\eta) := \Lambda Q(z,\eta) + iA_\eta^T$ is Hermitian,

2. $z \mapsto H(z,\eta)$ is decreasing for the order in $\mathbb{H}_n$,

3. $H(z,\eta) \sim -\sqrt{z}\,\Lambda^{1/2}$ as $z \to +\infty$ (thus $H(z,\eta)$ becomes $< 0_n$),

4. $Q(z,\eta)$ is a $(\Lambda, \Sigma_\eta + zI_n)$-isometry:

$$Q^*\Lambda Q = \Sigma_\eta + zI_n.$$

Special parameters (II)

Lemma 2. Let (τ, η) be given with either $\tau \in [0, \infty)$ or $\pm i\tau \in (0, h(\eta))$. Then $S(\tau, \eta)$ is the solution set of a first-order ODE

$$v' = Q(\tau^2, \eta)v.$$

♢

Corollary. For parameters (τ, η) with $\tau \in [0, +\infty)$, the Lopatinskii condition writes

$$\det H(\tau^2, \eta) \neq 0.$$

♡

Monotonicity of $z \mapsto H(z, \eta)$ gives:

Corollary. Given $\eta \neq 0$, the Lopatinskii condition holds true for every parameter τ with $\tau \in [0, +\infty)$ if, and only if, $H(0, \eta)$ is negative definite.

♣

Summary: Each of the following statement implies the next one

S1. The stored energy $\mathcal{W}$ is coercive over $\dot{H}^1(\Omega)$.

S2. The IBVP with the homogeneous Neumann boundary condition admits a maximal estimate.

S3. The Lopatinskii condition holds true at every pair (τ, η) with either $\operatorname{Re}\tau > 0$, or $\tau = 0$ (and then $\eta \neq 0$).

S4. The Lopatinskii condition holds true at every pair (τ, η) with $\tau \in \mathbb{R}^+$ ($\eta \neq 0$ if $\tau = 0$).

S5. For every $\eta \neq 0$ in $\mathbb{R}^{d-1}$, the Hermitian matrix $-H(0, \eta)$ is positive definite.

Questions

At each step, we dropped some information. It seems that condition (**S5**) is definitely weaker than (**S1**).

- How far is this true ?
- For which W is the homogeneous IBVP well-posed in Sobolev spaces ?

Answers

Theorem 1. Actually, $(\mathbf{S1})$ implies $(\mathbf{S5})$!! Therefore

$$(\mathbf{S1}) \iff (\mathbf{S2}) \iff (\mathbf{S3}) \iff (\mathbf{S4}) \iff (\mathbf{S5}).$$

The homogeneous IBVP is well-posed if, and only if, $\mathcal{W}$ is coercive over $H^1(\Omega)^n$.

$\diamondsuit$

The theorem provides a test for either the coerciveness of $\mathcal{W}$, or the well-posedness of the IBVP:

- Solve an algebraic Ricatti equation,
- Then check whether some Hermitian matrix is positive definite.

(I) Sketch of the proof of $(\mathrm{S1}) \iff (\mathrm{S5})$

Fix $\eta \in \mathbb{R}^{d-1}$ with $\eta \neq 0$.

By assumption, $\exists\, a < 0$ such that $K := -H(a,\eta)$ be positive definite. Define

$$S := \begin{pmatrix} Q(a,\eta)^* \\ -I_n \end{pmatrix} \Lambda \begin{pmatrix} Q(a,\eta), & -I_n \end{pmatrix} - a \begin{pmatrix} I_n & 0_n \\ 0_n & 0_n \end{pmatrix}.$$

By construction, S is positive definite.

By Lemma 1, we have

$$\begin{pmatrix} \Sigma & -iA_\eta \\ iA_\eta^T & \Lambda \end{pmatrix} = S + \begin{pmatrix} 0_n & H(a,\eta) \\ H(a,\eta) & 0_n \end{pmatrix}.$$

(II) Sketch of the proof of $(\mathrm{S1}) \iff (\mathrm{S5})$

There follows

$$2\int_0^{+\infty} W(i\eta\otimes v, v')\,dx_d \;=\; \int_0^{+\infty}(v,v')S\begin{pmatrix} v \\ v' \end{pmatrix}dx_d \\ -\langle H(a,\eta)v(0), v(0)\rangle.$$

The first term in the right-hand side is coercive on $H^1(\mathbb{R}^+)$. The second one is non-negative.

Uniformity. By regularity of $(z,\eta)\mapsto H(z,\eta)$ and homogeneity, one may choose $a = -\mathrm{cst}|\eta|^2$. Then

$$(v,v')S\begin{pmatrix} v \\ v' \end{pmatrix} \geq \mathrm{cst}\left(|\eta|^2|v|^2+|v'|^2\right).$$

(III) Sketch of the proof of $(\mathrm{S1}) \iff (\mathrm{S5})$

By Plancherel, this gives coerciveness:

$$\begin{aligned}\mathcal{W}[u] \;&=\; \int_\Omega W(\nabla u)\,dx = \int_{\mathbb{R}^{d-1}}\int_0^{+\infty} W(i\eta\otimes\hat{u},\hat{u}')\,dx_d dy \\ &\geq \text{cst}\int_{\mathbb{R}^{d-1}}\int_0^{+\infty}\left(|\eta|^2|\hat{u}|^2+|\hat{u}'|^2\right)\,dx_d dy = \text{cst}\int_\Omega |\nabla u|^2 dx.\end{aligned}$$

♢

Proof that $(\mathrm{S5})$ implies well-posedness

Hille–Yosida Theorem for a maximal monotone operator over a Hilbert space.

♡

Hyperbolic IBVP *vs* elliptic BVP

Compare the hyperbolic IBVP

$$(\partial_t^2 + P)u = f, \qquad u(0) = u^0,\ u_t(0) = u^1, \qquad Bu = 0$$

with the elliptic BVP

$$Pu = f \quad \text{in } \Omega, \qquad Bu = g \quad \text{on } \partial\Omega.$$

The former is well-posed iff

$$H(0,\eta) < 0_n, \qquad \forall \eta \in \mathbb{S}^{d-2},$$

while the latter is well-posed (Agmon, Douglis & Nirenberg, Lopatinskii) iff

$$\det H(0,\eta) \neq 0_n, \qquad \forall \eta \in \mathbb{S}^{d-2}.$$

Conclusion: Hyperbolic well-posedness $\Longrightarrow$ elliptic well-posedness.

(But $\not\Leftarrow$ ...)

Application

Recall the isotropic elasticity:

$$W(F) = \frac{\lambda}{4}|F + F^T|^2 + \frac{\mu}{2}(\mathrm{Tr}\, F)^2,$$

with (rank-one convexity)

$$\lambda > 0, \qquad 2\lambda + \mu > 0.$$

Define the Pressure/Shear velocities

$$c_P := \sqrt{2\lambda + \mu}, \qquad c_S := \sqrt{\lambda}.$$

Proposition 2. The IBVP is well-posed if, and only if, $\lambda + \mu > 0$. That is when $c_P > c_S$.

♠

When $\lambda + \mu \leq 0 < 2\lambda + \mu$, the Cauchy problem is well-posed, but the IBVP is not.

Surface waves (I)

What about the parameters $\tau \in (-ih(\eta), ih(\eta))$? Recall

Lemma 1. For $z \in (-h(\eta)^2, +\infty)$, the algebraic Ricatti equation

$$-\Lambda Q^2 - i(A_\eta + A_\eta^T)Q + \Sigma_\eta + zI_n = 0_n$$

admits a unique stable solution $Q(z,\eta)$. It has the properties

1. $H(z,\eta) := \Lambda Q(z,\eta) + iA_\eta^T$ is Hermitian, with $z \mapsto H(z,\eta)$ is decreasing for the order in $\mathbb{H}_n$,

2. $H(z,\eta) \sim -\sqrt{z}\,\Lambda^{1/2}$ as $z \to +\infty$ (thus $H(z,\eta)$ becomes $< 0_n$),

3. $Q(z,\eta)$ is a $(\Lambda, \Sigma_\eta + zI_n)$-isometry:

$$Q^*\Lambda Q = \Sigma_\eta + zI_n.$$

♢

Surface waves (II)

From 1 and 3, the limit $H(-h(\eta)^2,\eta)$ exists. Whence the limit $Q(-h(\eta)^2,\eta)$.

Given η, there exists a ξ s.t. $\lambda_1(\xi^2\Lambda+\xi(A_\eta+A_\eta^T)+\Sigma_\eta)=0$.

Lemma 3. $i\xi$ is an eigenvalue of $Q(-h(\eta)^2,\eta)$: this matrix is only semi-stable. Given an associated eigenvector X_η, one has

$$X_\eta^* H(-h(\eta)^2,\eta)X_\eta = 0.$$

♠

Thus $H(-h(\eta)^2,\eta)$ is not negative definite:

Corollary. Given $\eta\neq 0$, the determinant of $H(z,\eta)$ vanishes somewhere in $[-h(\eta)^2,+\infty)$.

♡

Surface waves (III)

Whence the alternative:

- Either the IBVP is ill-posed (H vanishes for some $\eta \neq 0$ and some $z \geq 0$),
- Or (vanishing at some $-h(\eta)^2 \leq z < 0$) there are surface waves

$$u^\eta(x,t) = e^{i(\rho(\eta)t+\eta\cdot y)}v(x_d),$$

  - where $\det H(-\rho^2,\eta) = 0$,
  - and $v \in S(-\rho^2,\eta)$ with $H(-\rho^2,\eta)v(0) = 0$.

Surface waves (IV)

- If $\rho \in (0, h(\eta))$, v decays exponentially at $+\infty$: Modulation yields finite energy surface waves:

$$u^{\phi}(x,t) := \int_{\mathbb{R}^d} u^{\eta}(x,t)\phi(\eta)\, d\eta, \qquad \phi \in \mathcal{D}(\mathbb{R}^d).$$

  Example: Rayleigh waves in elasticity.

- If $\rho = h(\eta)$, v is only bounded. Surface wave in a generalized sense

  Example:

$$\partial_t^2 u = c^2 \Delta u + f, \qquad \frac{\partial u}{\partial \nu} = 0.$$

Surface waves *vs* Kreiss' theory

Existence of surface waves

$$\Longleftrightarrow$$

Failure of the Lopatinskii condition at some frequency $\tau = i\rho \in i\mathbb{R}$ ($\mathrm{Re}\,\tau = 0$ instead of $\mathrm{Re}\,\tau > 0$).

$$\Longrightarrow$$

The Lopatinskii condition, if satisfied for all (τ, η) with $\mathrm{Re}\,\tau > 0$, is *not uniformly* satisfied.

Consequence. The non-homogeneous ($Bu = g$) IBVP is not strongly well-posed in Kreiss' sense. Its estimates suffer from a loss of regularity.

♣

Surface waves (VI): An alternate approach

Define

$$I[v] := \int_0^{+\infty} W(i\eta \otimes v, v')\, dx_d, \qquad E[v] := \frac{1}{2}\int_0^{+\infty} |v|^2 dx_d$$

and then

$$\beta(\eta) := \inf \left\{ \frac{I[v]}{E[v]} \,;\, v \in H^1(\mathbb{R}^+),\, v \not\equiv 0 \right\}.$$

Testing on fields $v = e^{-\omega x_d} V$, with $\mathrm{Re}\,\omega > 0$ and $V \in \mathbb{C}^n$, gives

$$\beta(\eta) \leq \lambda_1(\Theta(\omega)),$$

with

$$\Theta(\omega) := |\omega|^2 \Lambda + i(\omega A_\eta - \bar{\omega} A_\eta^T) + \Sigma_\eta.$$

Whence

$$\beta(\eta) \leq \inf_{\mathrm{Re}\,\omega>0} \lambda_1(\Theta(\omega)) \leq \inf_{\mathrm{Re}\,\omega=0} \lambda_1(\Theta(\omega))$$
$$= \min_{\xi\in\mathbb{R}} \lambda_1(\Theta(i\xi)) = h(\eta)^2.$$

Proposition 3. Assume the strict inequality

$$\beta(\eta) < h(\eta)^2.$$

Then there exists a non-zero $v_0 \in H^1(\mathbb{R})^n$ such that $I[v_0] = \beta(\eta)E[v_0]$.

◊

Then

$$e^{i(\rho t+\eta\cdot y)}v_0(x_d), \qquad \rho := \sqrt{\beta(\eta)}$$

is a surface wave.

Example: 3-D wave-like systems

Given $V, Y, Z \in \mathbb{R}^3$, set

$$\begin{aligned} W(\nabla_x u) \;\; := \;\; & \frac{1}{2}|\nabla_x u|^2 + \partial_3 u_1 V \cdot \nabla_y u_2 - \partial_3 u_2 V \cdot \nabla_y u_1 + \partial_3 u_2 Y \cdot \nabla_y u_3 \\ & -\partial_3 u_3 Y \cdot \nabla_y u_2 + \partial_3 u_3 Z \cdot \nabla_y u_1 - \partial_3 u_1 Z \cdot \nabla_y u_3 . \end{aligned}$$

This is the Dirichlet energy + a null form.

Then $P = -\Delta_x \otimes I_3$ and

$$Bu = \partial_3 u + \begin{pmatrix} V \cdot \nabla_y u_2 - Z \cdot \nabla_y u_3 \\ Y \cdot \nabla_y u_3 - V \cdot \nabla_y u_1 \\ Z \cdot \nabla_y u_1 - Y \cdot \nabla_y u_2 \end{pmatrix} .$$

Example (continuation)

Then

$$\det H(0,\eta) = \omega(q(\eta) - \omega^2), \qquad \omega := \sqrt{\tau^2 + |\eta|^2}$$

where

$$q(\eta) := (V \cdot \eta)^2 + (Y \cdot \eta)^2 + (Z \cdot \eta)^2.$$

Proposition 4.

1. The hyperbolic IBVP is well-posed iff $\lambda_+(q) < 1$.

2. The elliptic BVP is well-posed if either $\lambda_+(q) < 1$ or $\lambda_-(q) > 1$.

♣

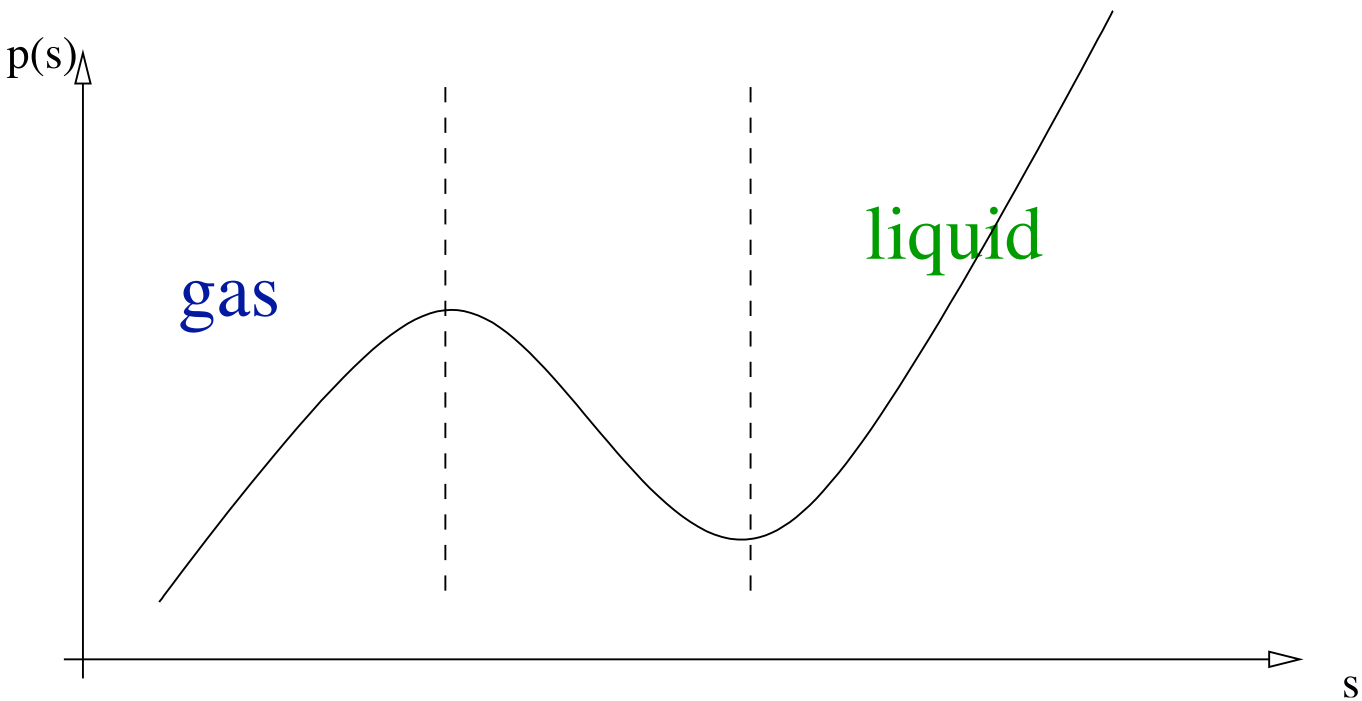
A non-linear illustration (I)

The Euler equations of a compressible, inviscid fluid:

$$\begin{aligned} \partial_t \rho + \mathrm{div}(\rho v) &= 0, \\ \partial_t(\rho v) + \mathrm{Div}(\rho v \otimes v) + \nabla_x p &= 0 \end{aligned}$$

have a variational origin.

Assume a van der Waals-like equation of state $\rho \mapsto p(\rho)$.



A non-linear illustration (II)

Then there are liquid-vapour phase transitions. These are subsonic discontinuities; an extra jump condition is needed besides Rankine–Hugoniot. Several choices (Slemrod, Truskinovski,...) can be made.

Theorem (S. Benzoni-Gavage 1998). Choose the conservation of energy across phase boundaries. Then the phase boundaries admit finite energy surface waves.

♡

The conservation of energy is precisely the jump condition that comes from the variational formulation.

Mind that this problem, besides nonlinear, is first-order!

General domains and variable coefficients (I)

Assumptions:

Ω an open subset of $\mathbb{R}^d$, with smooth boundary $\partial\Omega$. Outer normal $\nu(x)$.

Energy density $W(x; \nabla u)$ s.t.

- $x \mapsto W(x; \cdot)$ is $\mathcal{C}^2$ from $\overline{\Omega}$ into the quadratic forms over $\mathbf{M}_{n\times d}(\mathbb{R})$,
- For every fixed $x_0 \in \Omega$, $F \mapsto W(x_0; F)$ is strictly rank-one convex,
- For every $x_0 \in \partial\Omega$,

$$u \mapsto \mathcal{W}_{x_0}[u] := \int_{\Pi(x_0)} W(x_0; \nabla u)\, dx$$

  is coercive over $\dot{H}^1(\Pi(x_0))^n$, where

$$\Pi(x_0) := \{x \in \mathbb{R}^d;\ (x - x_0)\cdot \nu(x_0) < 0\}.$$

General domains and variable coefficients (II)

Theorem 2 (Gårding-type inequality). Under the above assumptions, there exist $\epsilon > 0$ and $C < \infty$ such that

$$\mathcal{W}[u] \geq \epsilon \|\nabla u\|_{L^2}^2 - C\|u\|_{L^2}^2, \qquad \forall u \in H^1(\Omega)^n.$$

♡

Corollary. The corresponding IBVP is well-posed in $H^1(\Omega)$.

♣

Whence the (computer assisted) strategy for checking the well-posedness of a “variational’ IBVP:

C. A. Strategy

- Freeze $x \in \overline{\Omega}$, check that $W(x, \cdot)$ is strictly rank-one convex,

- Freeze $x \in \partial\Omega$,

  - For $\eta \neq 0$, solve the algebraic Ricatti equation ($z = 0$)

    $$-\Lambda Q^2 - i(A_\eta + A_\eta^T)Q + \Sigma_\eta = 0_n, \qquad Q \text{ stable.}$$

    In general, solved iteratively ...

  - Then check whether $\Lambda Q + iA_\eta^T$ is negative definite.

If n is small enough or if the symmetry group is large enough, calculations can be made explicitly.

Proof of Lemma 1 and 2

1. For solutions of (2), $x_d \mapsto \langle \Lambda v', v' \rangle - \langle (\Sigma_\eta + zI_n)v, v \rangle$ is constant.

2. Thus stable solutions (and unstable too) satisfy

$$\langle \Lambda v', v' \rangle = \langle (\Sigma_\eta + zI_n)v, v \rangle.$$

3. Thus $S(\tau, \eta)$ is transversal to $\{0\} \times \mathbb{C}^n$: there exists $Q(z, \eta) \in \mathbf{M}_n(\mathbb{C})$ such that stable solutions satisfy $v' = Qv$.

4. This Q is a stable solution of the Ricatti equation.

5. A stable solution of this algebraic equation defines a stable subspace of the ODE, thus equals $Q(\tau, \eta)$. Whence uniqueness.

6. From 1, one has $Q^* \Lambda Q = \Sigma_\eta + zI_n$.

7. For solutions of (2), the derivative

$$\frac{d}{dx_d} \langle \Lambda v' + iA_\eta^T v, v \rangle$$

is real.

8. Thus stable solutions (and unstable too) satisfy

$$\langle \Lambda v' + iA_\eta^T v, v \rangle \in \mathbb{R}.$$

This means $H(z,\eta) := \Lambda Q + iA_\eta^T \in \mathbb{H}_n$.

9. From 6, one has the Lyapunov equation

$$Q^* \frac{dH}{dz} + \frac{dH}{dz} Q = I_n.$$

10. Since Q is stable and $I_n > 0_n$, this tells that

$$\frac{dH}{dz} < 0_n.$$