Des équations aux dérivées partielles au calcul scientifique

JULY 2nd 2007

Giovanni Alberti (PISA)

Structure of null sets in Euclidean space
results and open problems

joint work with

David Preiss (Warwick)

&

Marianna Csörnyei (UCL)

PLAN OF THE TALK

- Review of original motivations
- A covering theorem for null sets in the plane
- Applications → Differentiability of Lipschitz functions
  → Tangent field to null sets
  → Laczkovich problem
- Proof of the covering theorem
- Open problems (extension to higher dimensions).

REFERENCES

G. Alberti, M. Csörnyei, D. Preiss
Structure of null sets in the plane and applications
Proceedings of IV ECM (Stockholm 2004)
European Math. Soc., 2005

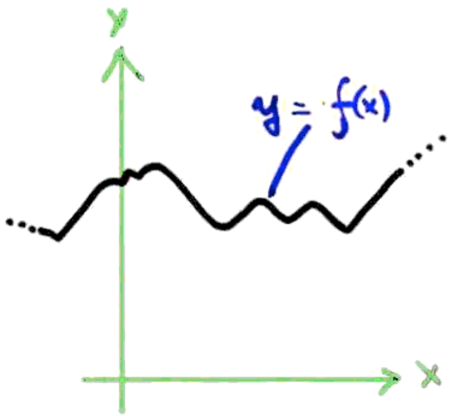
G. Alberti, M. Csörnyei, D. Preiss
Paper in preparation .....

1. MOTIVATIONS

- Differentiability of Lipschitz functions
- Rank-one property of BV functions (and structure of normal currents)
- Laczkovich problem

2. A COVERING THEOREM FOR NULL SETS

NOTATION



horizontal graph

i.e. graph of a function $y = f(x)$ with Lipschitz constant $\mathrm{Lip}(f) \leq 1$

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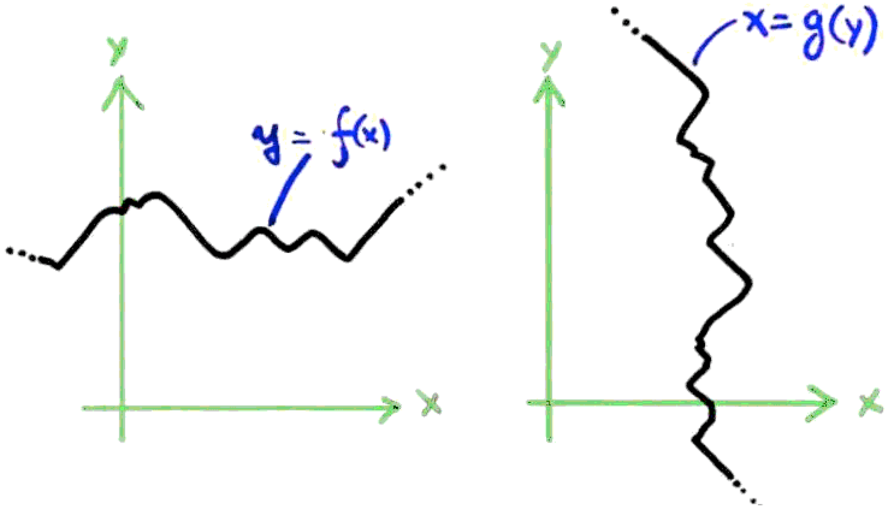

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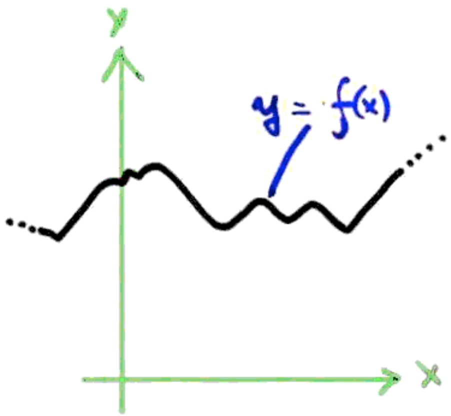

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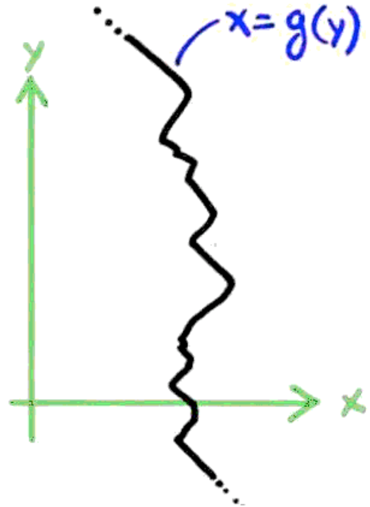

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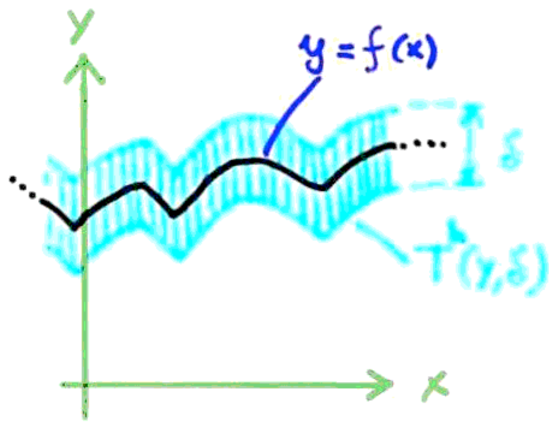

horizontal stripe with thickness δ

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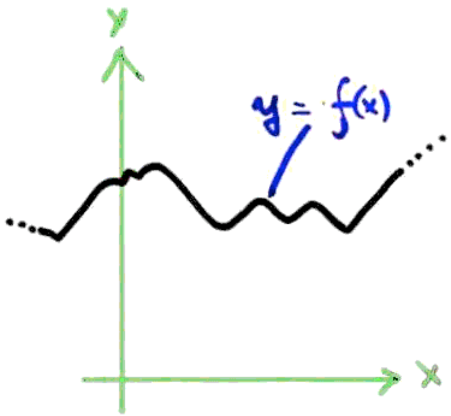

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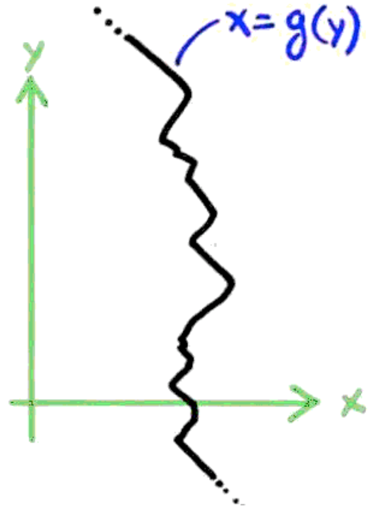

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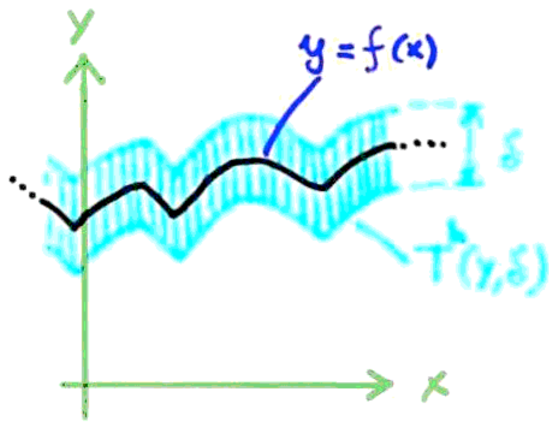

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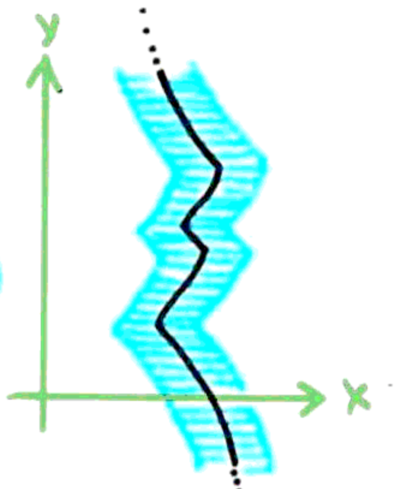

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STATEMENT

Let E be a null set in the plane (i.e. $\mathcal{L}^2(E)=0$)

Then $E = E^v \cup E^h$ so that:

i) $\forall \varepsilon>0$, E^h is covered by horizontal stripes T_i^h with thickness δ_i so that $\sum \delta_i \leq \varepsilon$

ii) $\forall \varepsilon>0$, E^v is covered by vertical stripes T_j^v with thickness η_j so that $\sum \eta_j \leq \varepsilon$

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This can be viewed as a refinement of Fubini.

3. Differentiability of Lipschitz maps

Let be given $f : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz

By Rademacher theorem, f is differentiable a.e.

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Question (weak version):

Given a singular measure μ on $\mathbb{R}^n$ ($\mu \perp \mathscr{L}^n$) is there a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$ which is <u>not</u> differentiable μ-a.e.

Remarks

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iii) What about the distance function

$$f(x) := \mathrm{dist}(x, E) \quad ?$$

This is not differentiable at $x \in E$ if and only if x is a porosity point of E : $\exists\, x_n \to x$, $r_n \to 0$ s.t. $B(x_n, r_n) \cap E = \emptyset$ and $|x_n - x| = O(r_n)$.

But there exist null sets with no porosity points....

Answers (so far...)

If $n=1$, $m=1$ the answer is positive (to both questions)

It is a classical construction...

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If $n>2$, nothing is known....

A construction for $n=1$

Assume E compact.

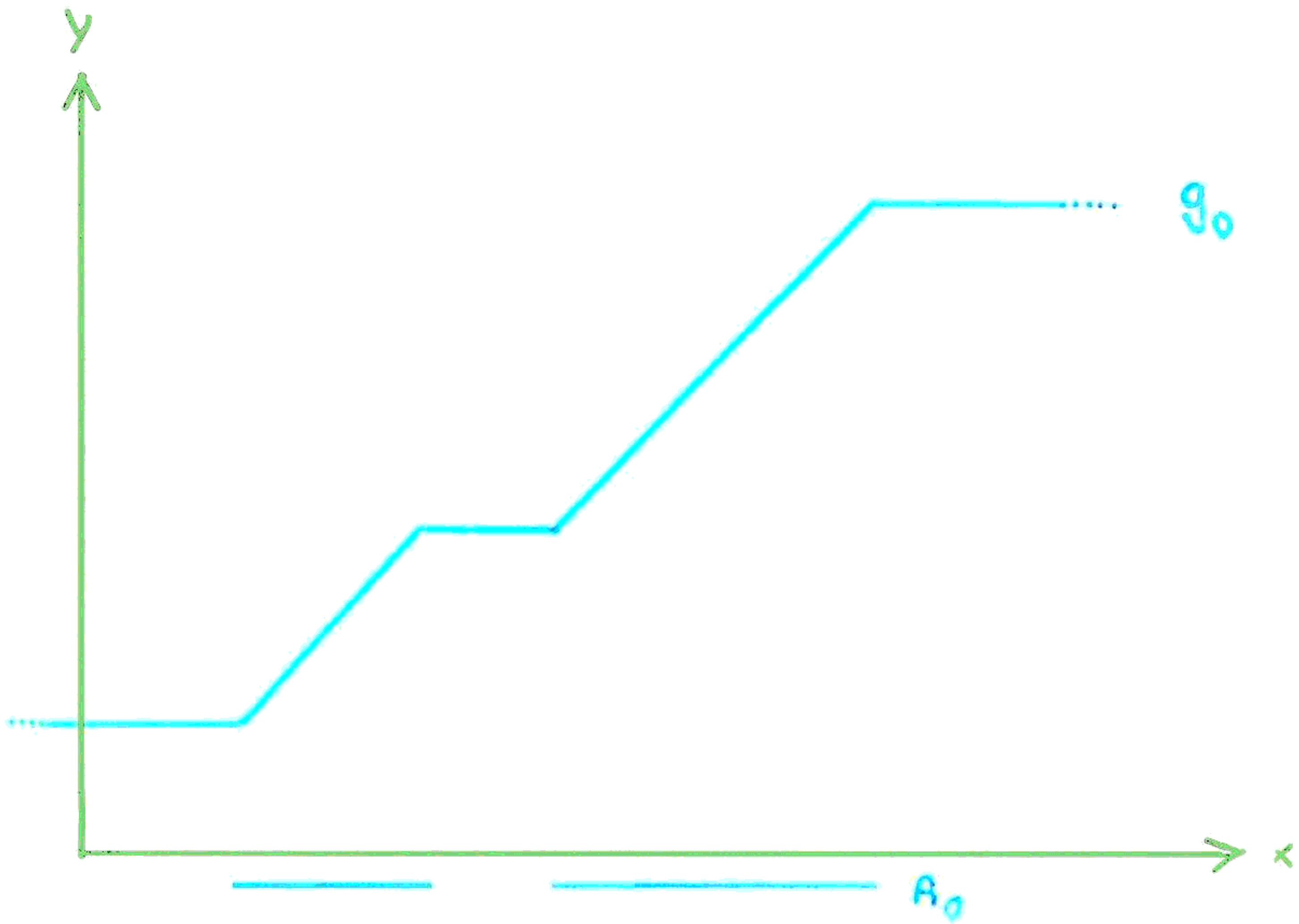
Since $\mathcal{L}^1(E)=0$, there exists open sets A_n s.t.

- $A_n \downarrow E$
- $\mathcal{L}^1(A_n) \leq 2^{-n} \mathcal{L}^1(I)$
  $\forall I$ connected component of A_{n-1}

For every n, take g_n s.t. $g_n' = 1_{A_n}$

Set

$$f(x) := \sum_{n=0}^{\infty} (-1)^n g_n(x)$$


y
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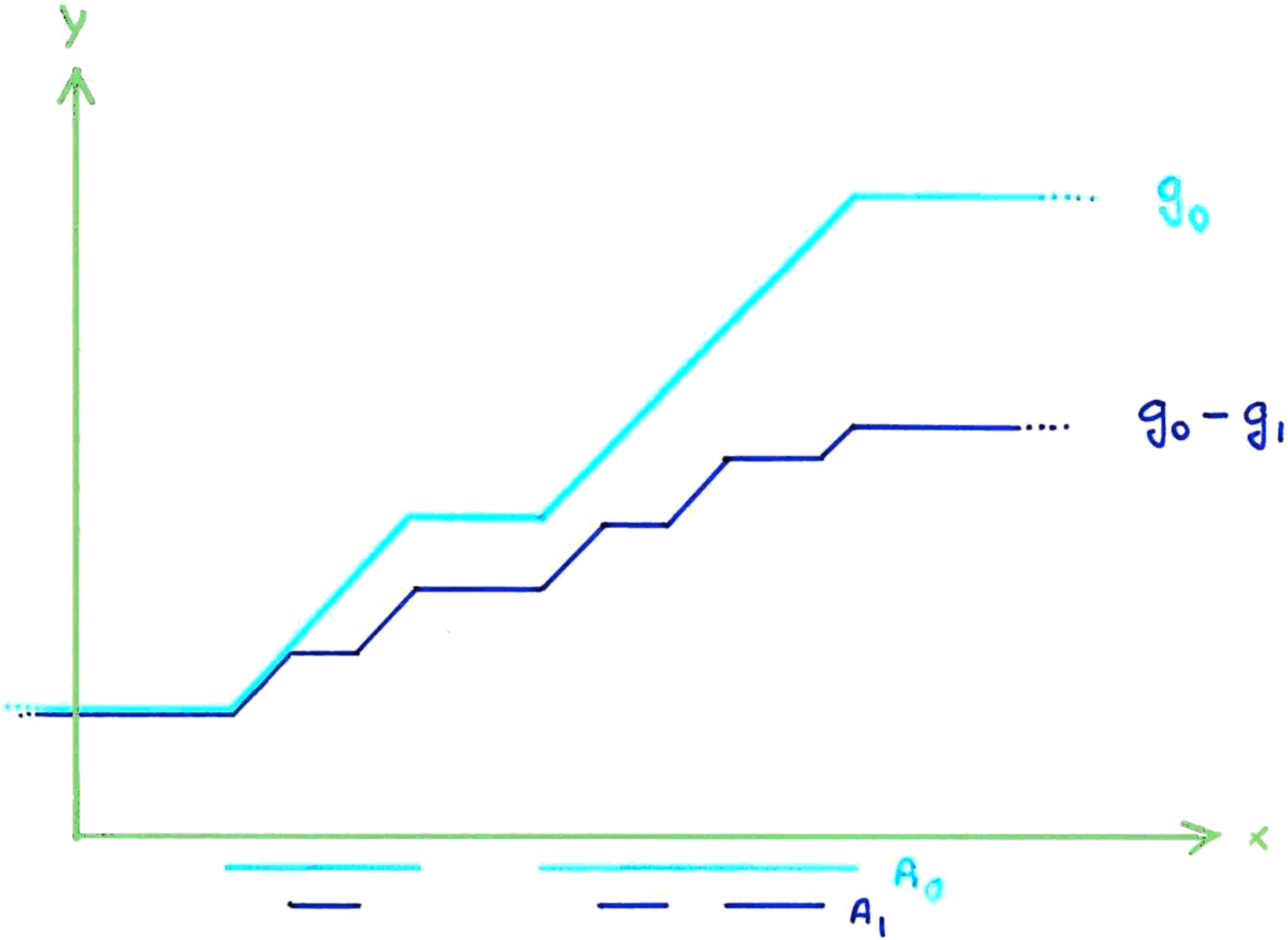
y
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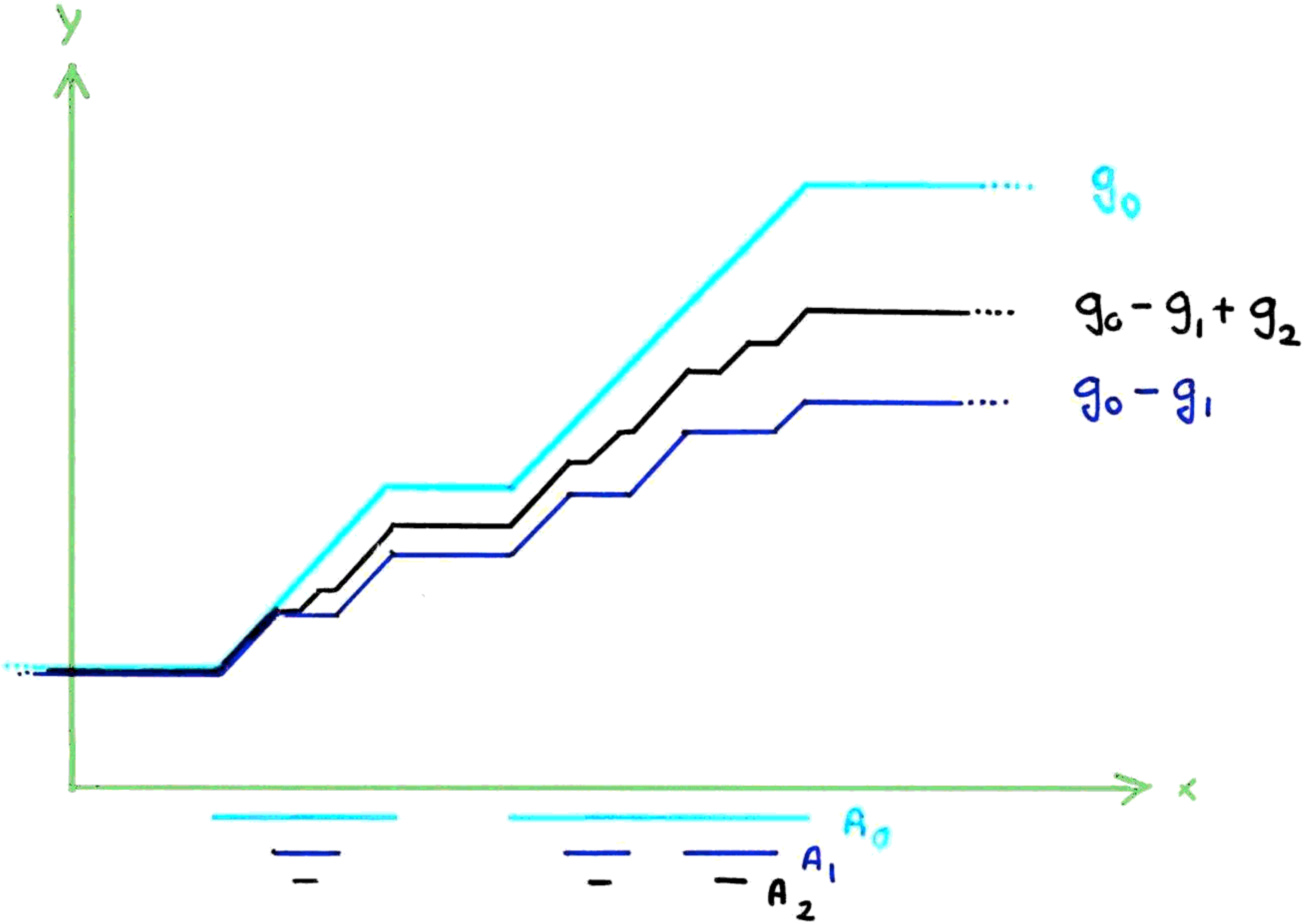

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- for each stripe T_i^h (T_j^v) we consider as building block the function $g_i^h(x,y)$ $(g_j^v(x,y))$ s.t.

$$\frac{\partial g_i^h}{\partial y} = 1_{T_i^h} \quad \left(\frac{\partial g_j^v}{\partial x} = 1_{T_j^v} \right)$$

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- and then....

4. Tangent field of a null set

Definition

Let E be a set in the plane.

Let τ assign to each $x \in E$ a line ($\tau(x) \in G(2,1)$)

Then τ is a weak tangent field of E if for every curve C of class C^1 there holds

$$\tau(x) = \mathrm{Tan}(C, x) \quad \text{for a.e. } x \in E \cap C$$

with respect to length, or $\mathcal{H}^1$

REMARKS

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THEOREM

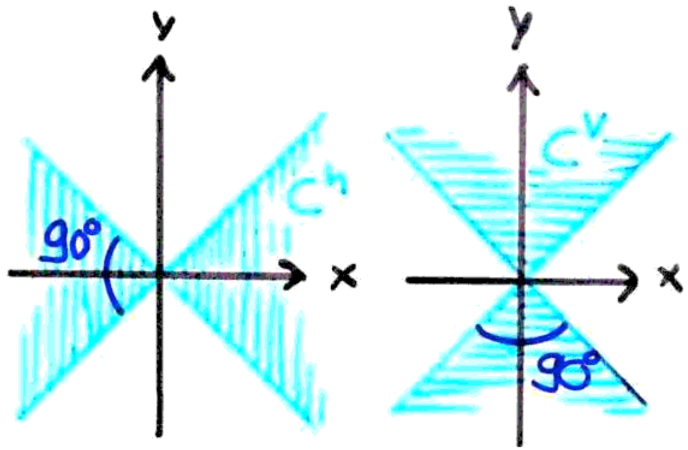
Every null set E in the plane admits a tangent field τ_E

PROOF (for E compact)

STEP 1. Write $E = E^h \cup E^v$

and set

$$\mathscr{C}(x) := \begin{cases} C^h & \text{if } x \in E^h \\ C^v & \text{if } x \in E^v \setminus E^h \end{cases}$$



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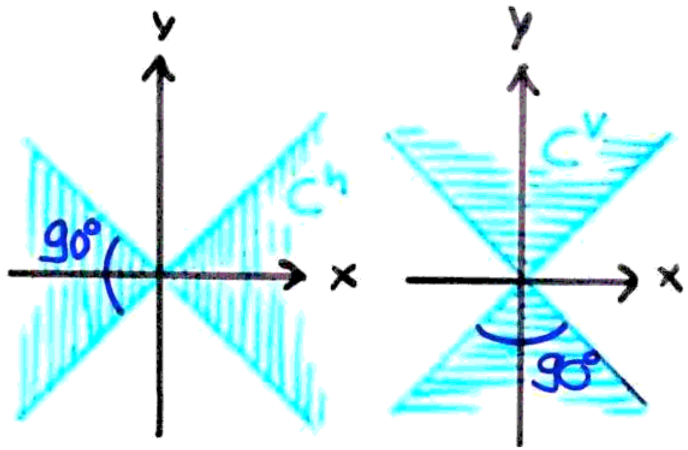

Then the cone-field $\mathcal{C}(x)$ is tangent to E in the sense that for every curve C of class $\mathcal{C}^1$

$$\mathrm{Tan}(C, x) \subset \mathcal{C}(x) \quad \text{for a.e. } x \in E \cap C .$$

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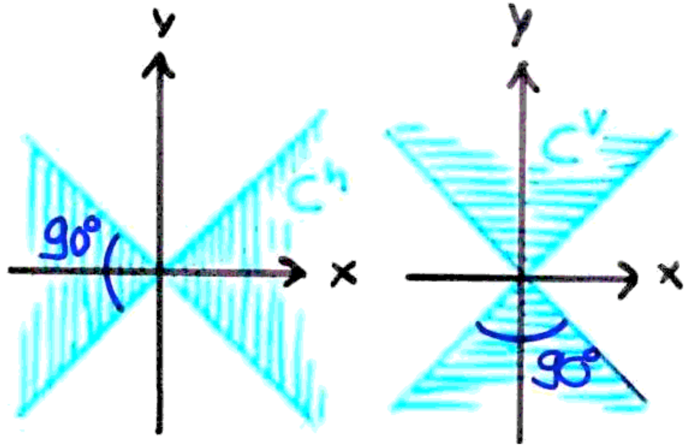

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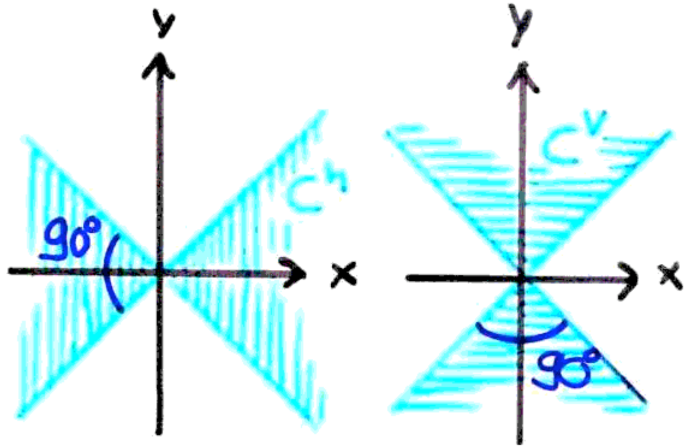

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STEP 3. Let

$$\tau_E(x) := \bigcap_{\theta \in \mathbb{Q}} \mathcal{C}_\theta(x)$$

COROLLARY

distributional gradient

Let $u \in BV(\mathbb{R}^2)$. Thus $Du \in \mathcal{M}(\mathbb{R}^2; \mathbb{R}^2)$

Let μ be a singular measure on $\mathbb{R}^2$ supported on the null set E.

Then

$$\frac{d(Du)}{d\mu}(x) \perp \tau_E(x) \quad \text{for } \mu\text{-a.e. } x$$

Radon-Nikodym derivative of Du w.r.t. μ

tangent field of E

Proof. Just the coarea formula....

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COROLLARY (Rank-one property of BV functions)

Let $u \in BV(\mathbb{R}^2, \mathbb{R}^n)$. Then $Du = M \cdot |Du|$ and $M(x)$ is a rank-one matrix for a.e. x.

Proof. Straightforward....

5. Laczkovich problem

Question:

Let A be a set with positive measure in $\mathbb{R}^n$.
Is there $f : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz s.t. $f(A)$ has non-empty interior?

Answers (so far...):

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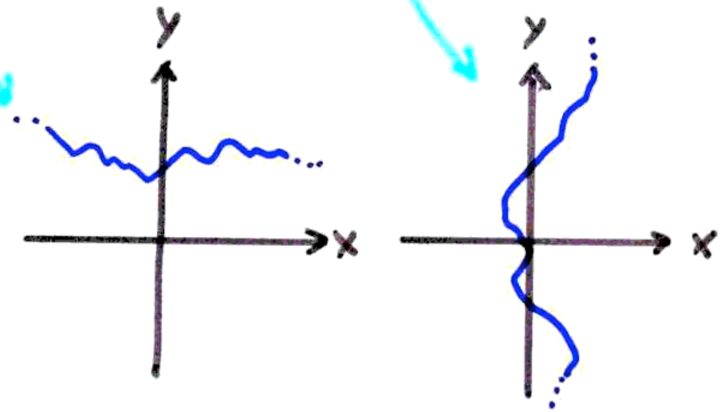
If $n \geq 3$ nothing is known.

6. PROOF OF THE COVERING THEOREM

A covering resul for discrete sets (Dillworth's lemma)

Let E be a finite set in the plane. Set $n := \#E$.

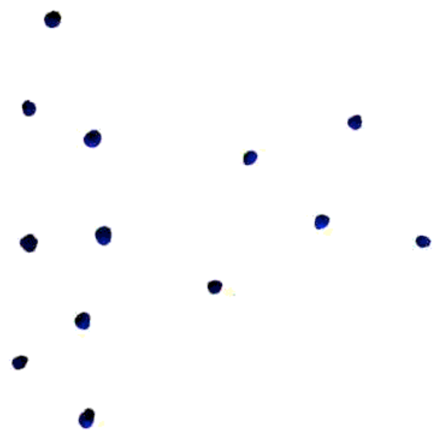
Then E can be covered by $\sqrt{n}$ vertical graphs and $\sqrt{n}$ horizontal graphs.



Remark.
This is actually a geometric version of Erdős-Szekeres theorem on monotone subsequences.

Proof.

E



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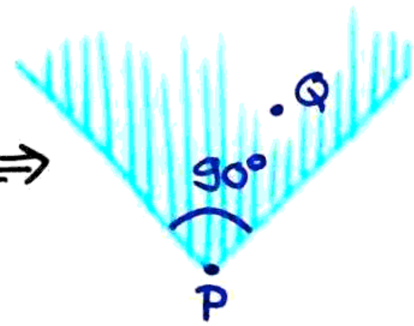
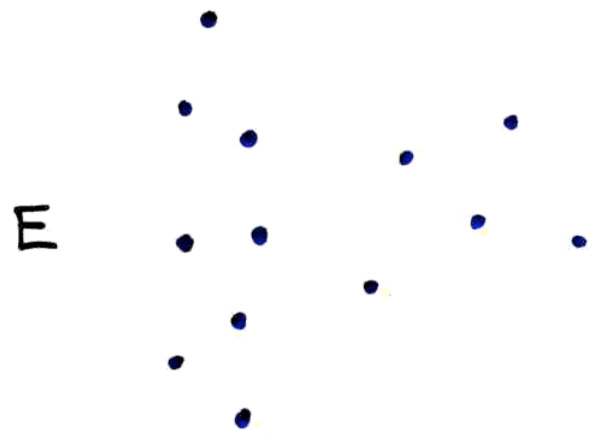
STEP 1. Define in E the partial order $P \lesssim Q \Leftrightarrow$





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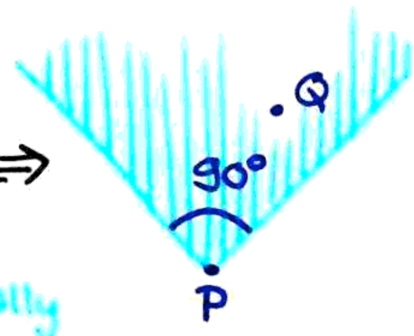

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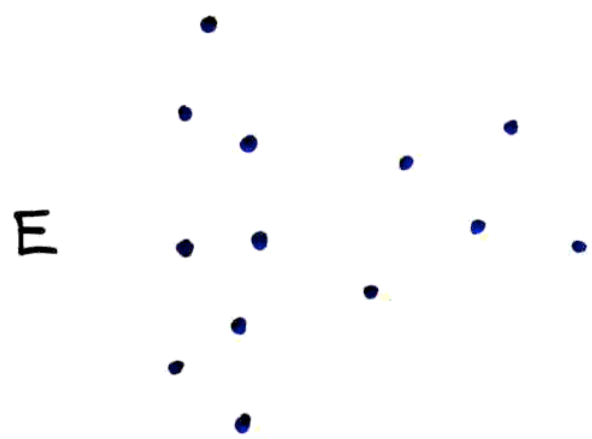
totally ordered subsets



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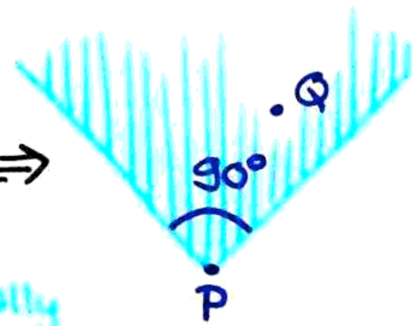

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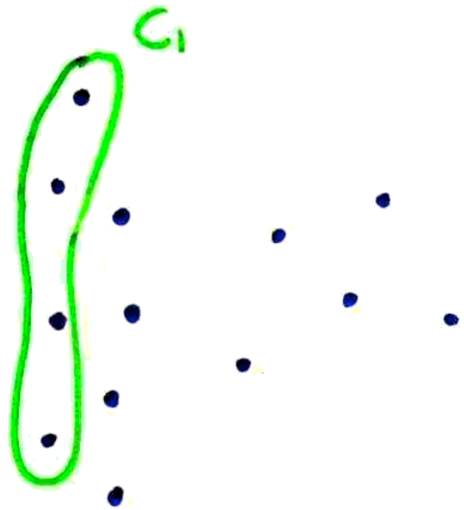

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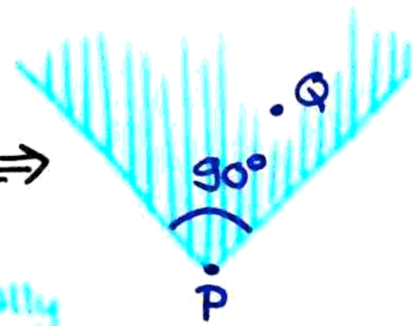

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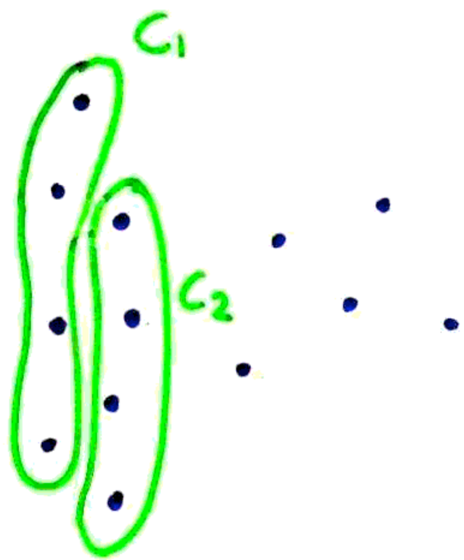

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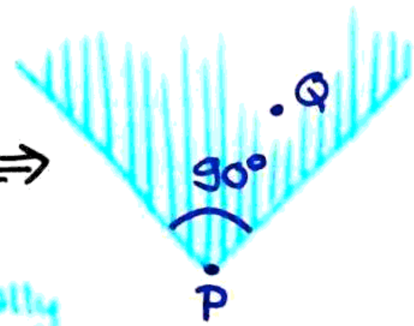

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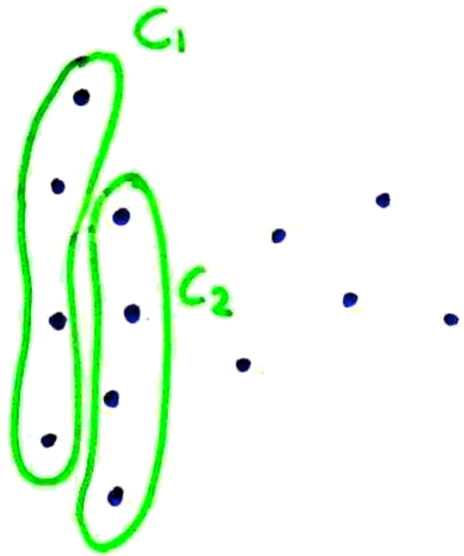

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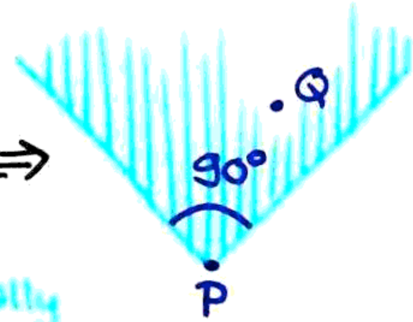

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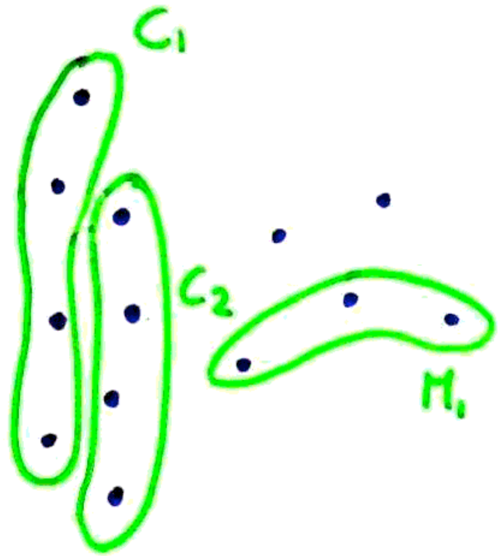

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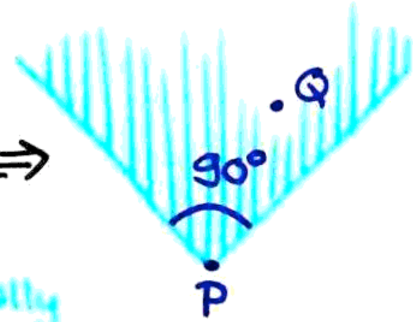
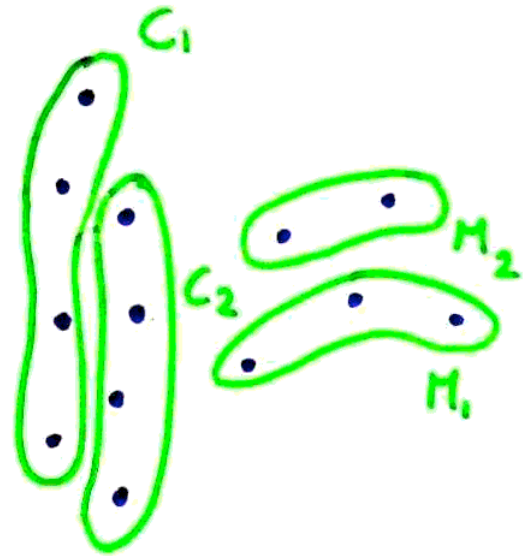
STEP 1. Define in E the partial order $P \lesssim Q \Leftrightarrow$



STEP 2. Take away from E chains C_i with at least $\sqrt{n}$ elements.... ← totally ordered subsets

STEP 3. From what is left take away the first stratum (set of minima), then the second, ...



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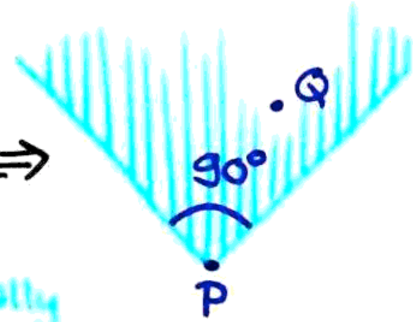

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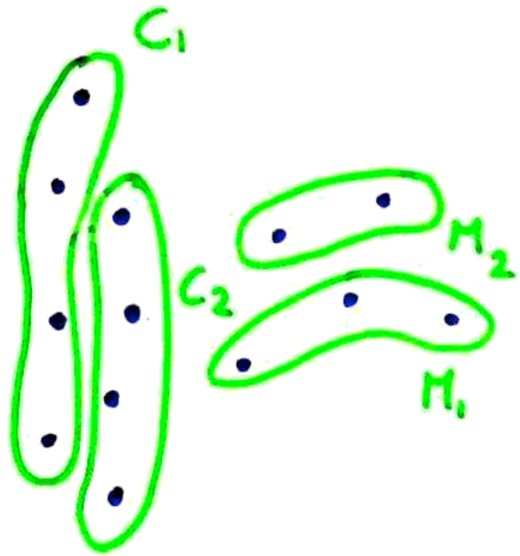

STEP 4. Each chain C_i is contained in a vertical graph. There are at most $\sqrt{n}$ chains

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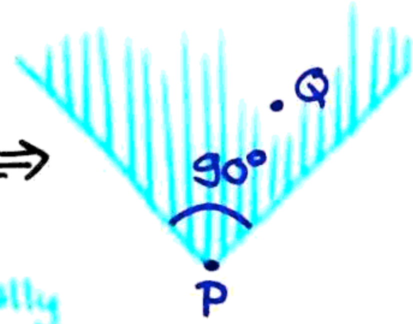

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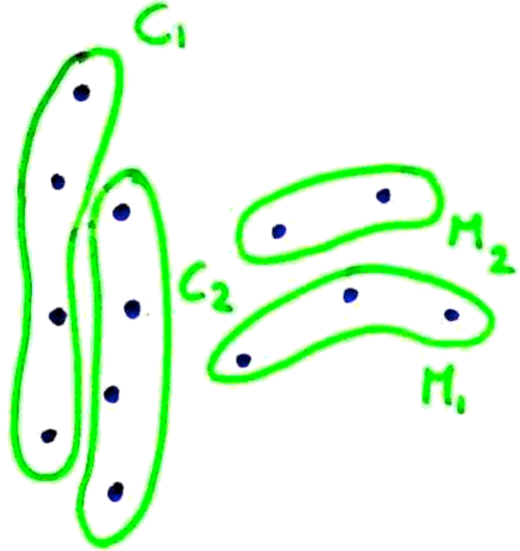
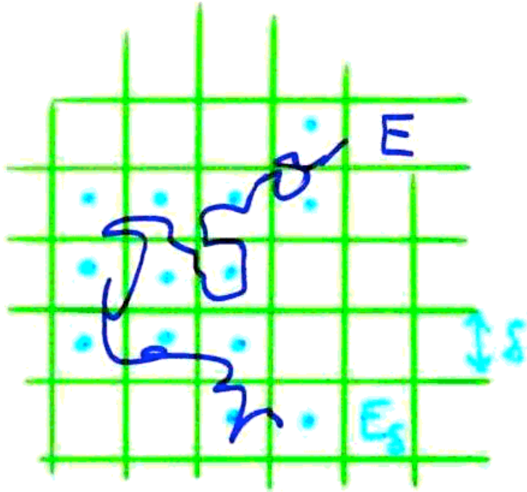


STEP 4. Each chain C_i is contained in a vertical graph. There are at most $\sqrt{n}$ chains

STEP 5. Each stratum M_j is contained in a horizontal graph. There are at most $\sqrt{n}$ strata

PROOF OF THE COVERING THEOREM

(not complete, and just for E compact)



STEP 1. Discretize E as follows: for every $\delta > 0$, E_δ are the centers of the squares in the mesh that intersect E. Then $\# E_\delta = O(1/\delta^2)$.

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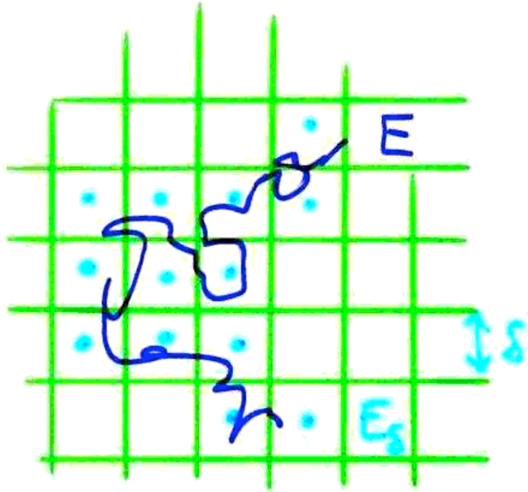

STEP 2. Cover E_δ with $O(1/\delta)$ horiz. graphs f_i and vertical graphs g_j using Dillworth's lemma.

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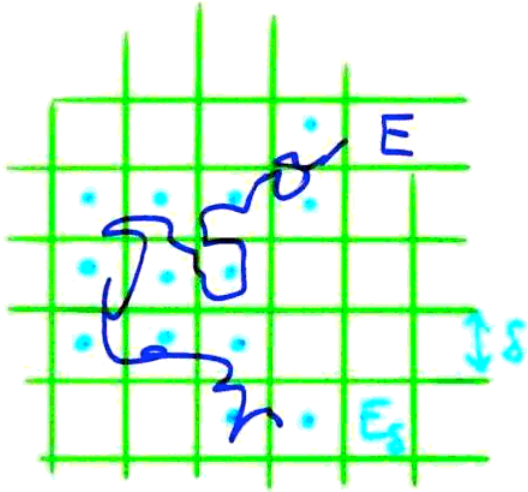

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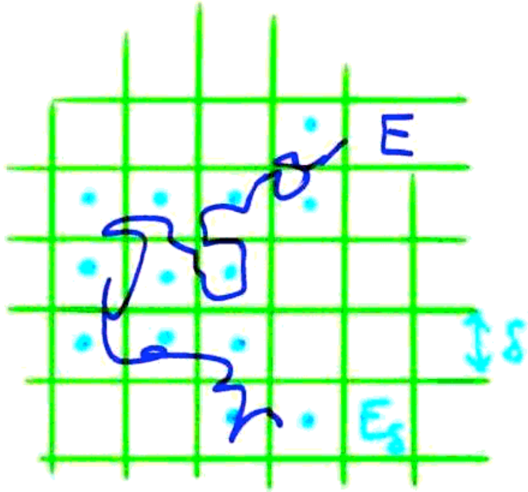

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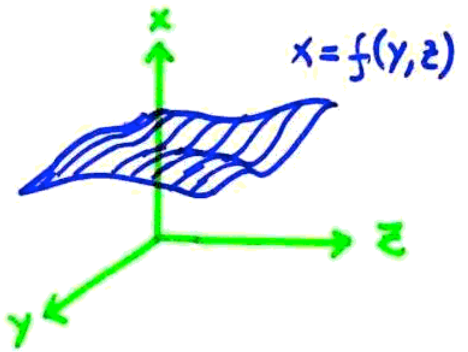
STEP 4. Choose δ s.t. $o(1) \leq \varepsilon$.

7. OPEN PROBLEMS (EXTENSION TO HIGHER DIM. / $n=3$)

NOTATION for sets in the space



surface of type x
and lipschitz const. L

i.e. graph of a
function $x=f(y,z)$
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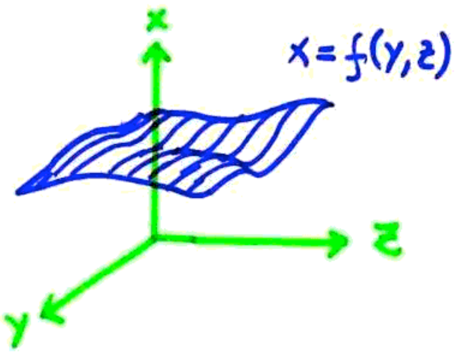

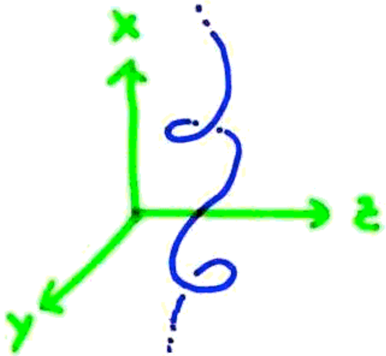

surface of type x and lipschitz const. L

curve of type x and Lip. const. L

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i.e. graph of a map $(y,z) = \phi(x)$ with $\mathrm{Lip}(\phi) \le L$

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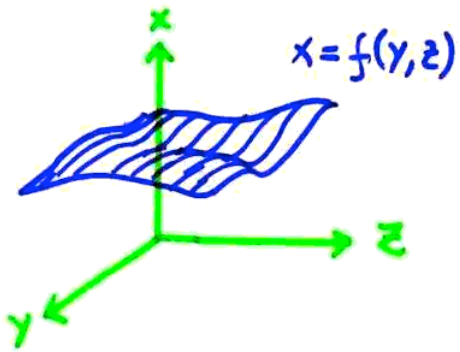

surface of type x and lipschitz const. L

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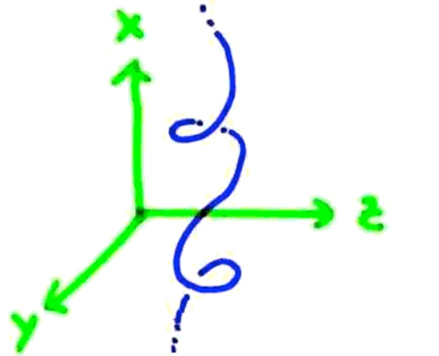

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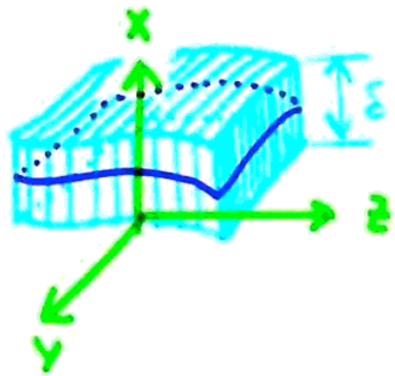

slab of type x and...

i.e. $V^x(f,\delta) := \left\{ |x - f(y,z)| \le \frac{\delta}{2} \right\}$

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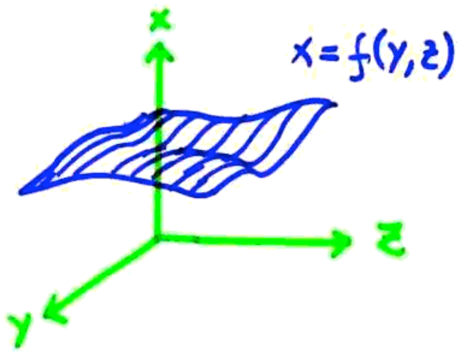

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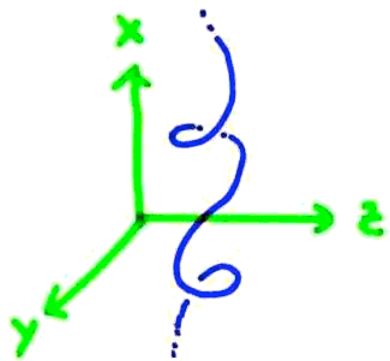

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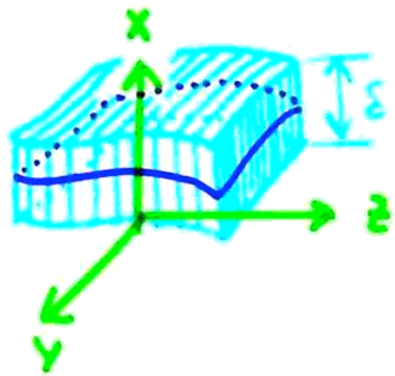

slab of type x and...

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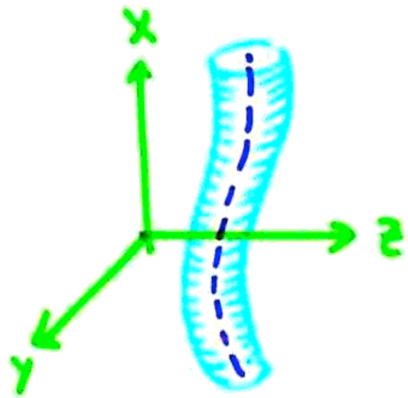

tube of type x and...

i.e. $U^x(\phi,\delta) := \left\{ |(y,z) - \phi(x)| \le \frac{\delta}{2} \right\}$

STATEMENT

Let E be a null set in $\mathbb{R}^3$. Then $E = E^x \cup F^x$ where

i) for every $\varepsilon > 0$, E_x is covered by slabs V_i^x with Lip. const. 1, thickness δ_i s.t. $\sum_i \delta_i \le \varepsilon$.

ii) for every $\varepsilon > 0$, F^x is covered by tubes U_j^x with Lip. const. 1, thickness η_j s.t. $\sum_j \eta_j^2 \le \varepsilon$.

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but the real question is:

Given a null set E in $\mathbb{R}^3$ can we cover it with slabs V_i of type x, y, z and Lip. const. L, thickness δ_i s.t. $\sum \delta_i \le \varepsilon$?

independent of E!

In $\mathbb{R}^3$ the required version of Dillworth's lemma fails (ACP)

STATEMENT A. $\forall\, L, M < +\infty$, $\exists\, E \subset \mathbb{R}^3$ ($n := \# E$) that cannot be covered by $M \cdot n^{1/3}$ surfaces of type x, y, z and Lipschitz constant L.

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STATEMENT B. $\forall\, L < +\infty$, $\delta > 0$, $\exists\, E \subset \mathbb{R}^3$ $(n := \# E)$ such that $\sigma_{x,L}(E)\,;\ \sigma_{y,L}(E)\,;\ \sigma_{z,L}(E) \leq \delta\, n^{2/3}$

where $\sigma_{x,L}(E) := \max\{\#(E \cap S) : S \text{ surf. of type } x, \text{ Lip. const. } L\}$

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STATEMENT C. $\forall\, L < +\infty$, $\exists\, E \subset \mathbb{R}^3$ $(n := \#E)$ such that $\sigma_{x,L}(E)$; $\sigma_{y,L}(E)$; $\sigma_{z,L}(E) < n^{2/3}$

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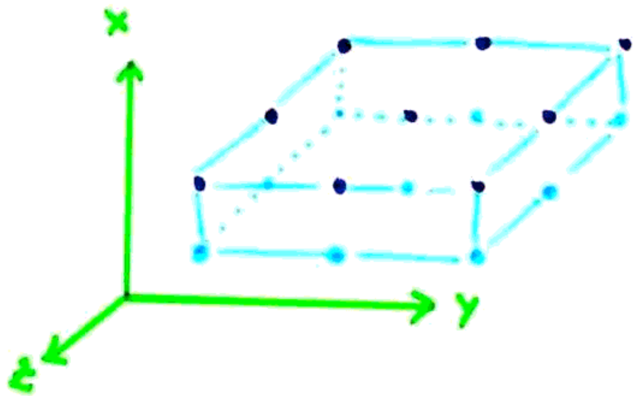
STATEMENT D. $\forall\, L < +\infty$, $\exists\, E \subset \mathbb{R}^3$ $(n := \# E)$ such that $\sigma_{x,L}(E) \cdot \sigma_{y,L}(E) \cdot \sigma_{z,L}(E) < n^2$

There is an easy example that almost proves STATEMENT D.

Let $E = E_x \times E_y \times E_z$ product of three sets in $\mathbb{R}$

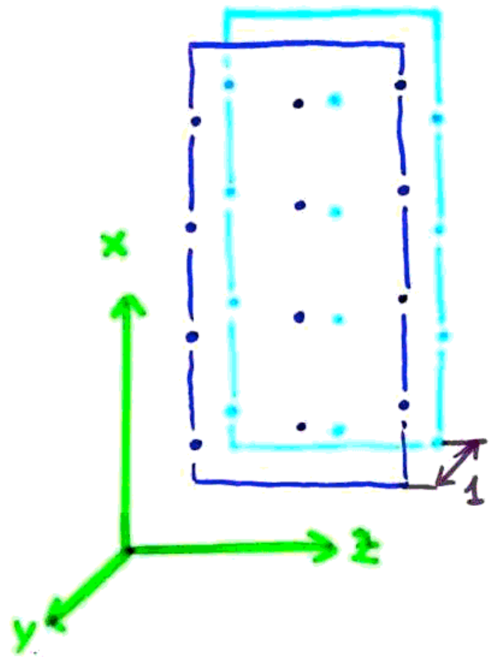


Then, for every L,

$$\sigma_{x,L}(E) = \#(E_y) \cdot \#(E_z)$$
$$\sigma_{y,L}(E) = \#(E_x) \cdot \#(E_z)$$
$$\sigma_{z,L}(E) = \#(E_x) \cdot \#(E_y)$$

and then $\sigma_{x,L}(E) \cdot \sigma_{y,L}(E) \cdot \sigma_{z,L}(E) = (\# E)^2 = n^2$

By perturbation we obtain an example that proves STATEMENT D



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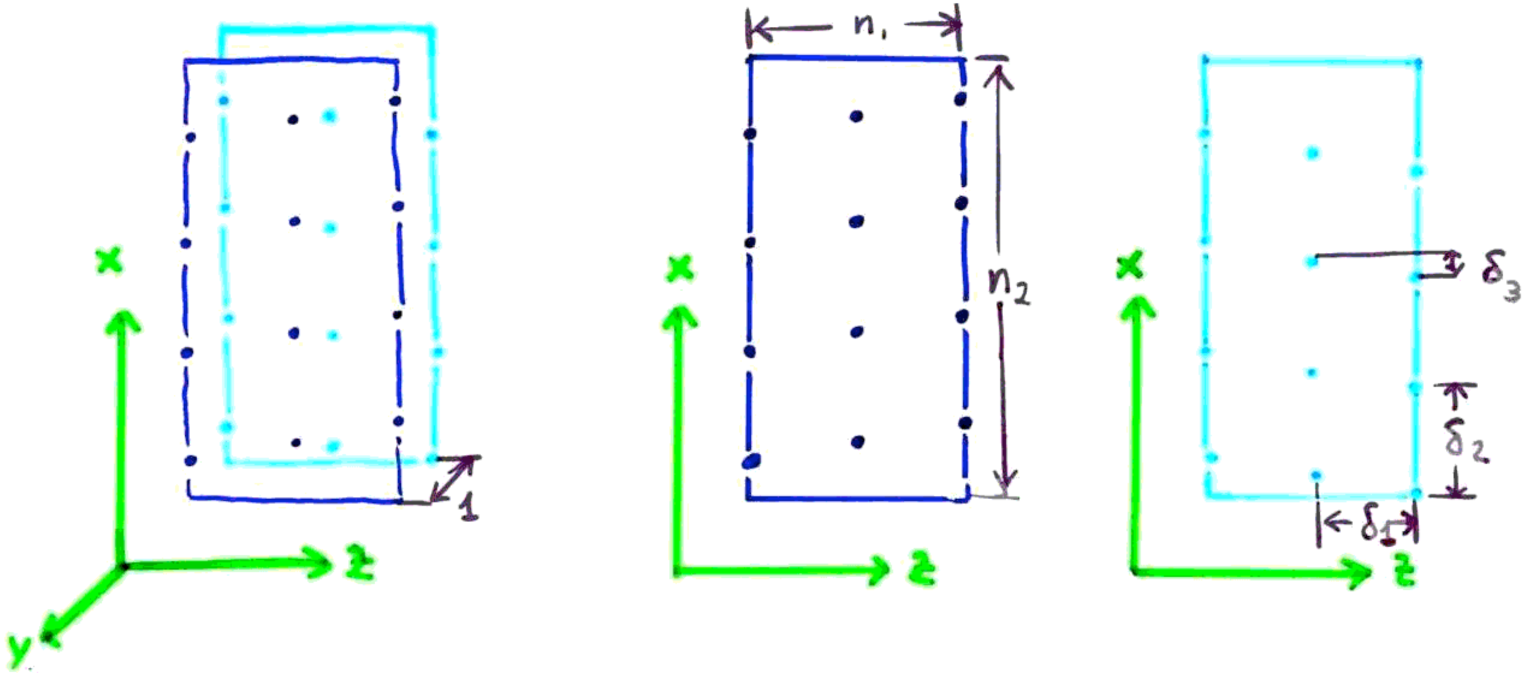

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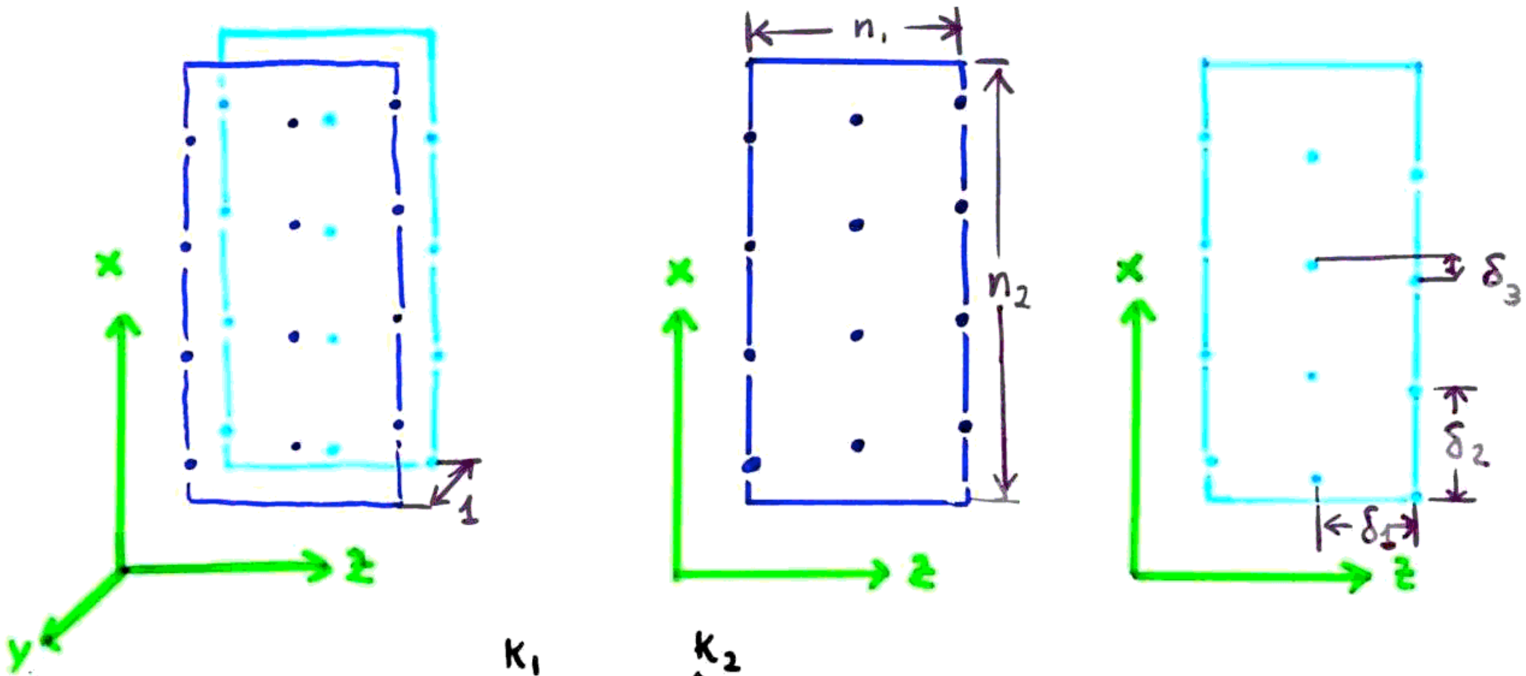

Then $n = \overbrace{\left(\frac{n_1}{\delta_1}+1\right)}^{K_1} \cdot \overbrace{\left(\frac{n_2}{\delta_2}+1\right)}^{K_2} \cdot 2$

Choosing the parameters $n_1, n_2, \delta_1, \delta_2, \delta_3$ carefully one obtains that

$\sigma_{y,L}(E) = K_1 \cdot K_2 \quad \sigma_{z,L}(E) = 2 \cdot K_1 \quad \underline{\sigma_{x,L}(E) < 2 \cdot K_2}$!

Final Remarks

The required covering result for discrete sets does not hold in dimension 3.

But what about the covering result for null sets?

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Other covering result for discrete sets could be helpful (and true...)