Modifications to Newton’s second law and continuum elastodynamics

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Happy Birthday Luc,

and thanks for your great contributions to mathematics and in particular homogenization and for your inspiration
Newton’s second law

\[ F = ma \]

has withstood the centuries but it is a microscopic law (holding at the atomic level) which is frequently applied as a macroscopic law without modification.

Is this correct?

We are not saying Newton’s law is wrong but presenting a different perspective where it seems advantageous to generalize it
Of course (following Einstein) we can rewrite Newton’s second law as

\[ F = \frac{dp}{dt}, \quad p = mv \]

At the macroscopic scale \( p \) can be defined as the sum of momenta of the microscopic masses, and Newton’s law can be trivially made to hold at a macroscopic level if we define the macroscopic velocity to be

\[ v = \frac{\text{total momentum}}{\text{total mass}} \]
But is it always sensible to define the macroscopic velocity in this way?

The “macroscopic velocity” depends on the mass of the cores which are unconnected with the object: more sensible to take the velocity of the matrix as the macroscopic velocity
Maxwell’s Equations:

\[ \frac{\partial}{\partial x_i} \left( C_{ijkl} \frac{\partial E_\ell}{\partial x_k} \right) = \{ \omega^2 \varepsilon \mathbf{E} \}_j \]

\[ C_{ijkl} = \varepsilon_{ijm} \varepsilon_{kln} \{ \mu^{-1} \}_{mn} \]

Continuum Elastodynamics:

\[ \frac{\partial}{\partial x_i} \left( C_{ijkl} \frac{\partial u_\ell}{\partial x_k} \right) = -\{ \omega^2 \rho u \}_j \]

Suggests that \( \varepsilon(\omega) \) and \( \rho(\omega) \) might have similar properties

Specifically a similar dependence on frequency
Sheng, Zhang, Liu, and Chan (2003) found that materials could exhibit a negative effective density over a range of frequencies.

Red=Rubber, Black=Lead, Blue=Stiff Matrix
A simplified one-dimensional model:

\[ \hat{P} = M \hat{V}, \text{ with } M = M_0 + \frac{2Knm}{2K - m\omega^2}, \]

Avila, Griso and Miara (2005) extending work of Bouchitte and Felbacq (2004) essentially proved that materials could have an anisotropic density.
Simplified Model:
General properties of $M(\omega)$

If the velocity was not harmonically varying

$$P(t) = \int dt' H(t-t') V(t'), \quad M(\omega) = \int_{-\infty}^{\infty} ds e^{i\omega s} H(s),$$

Causality implies $M(\omega)$ is analytic for $\text{Im}(\omega) > 0$.

Also since $H$ is real

$$M(-\bar{\omega}) = \overline{M(\omega)},$$

We expect

$$\lim_{\omega \to \infty} M(\omega) = M_0 I.$$
At fixed frequency, with
\[ F(t) = \text{Re}(\hat{F}e^{-i\omega t}), \quad \mathbf{v}(t) = \text{Re}(\hat{\mathbf{v}}e^{-i\omega t}), \quad \text{where} \quad \hat{F} = -i\omega \mathbf{M}(\omega) \hat{\mathbf{v}} \]
the average work done in a cycle of oscillation is

\[
W = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \, F(t) \cdot \mathbf{v}(t)
\]

\[
= \left[ \text{Re}(\hat{F}) \cdot \text{Re}(\hat{\mathbf{v}}) + \text{Im}(\hat{F}) \cdot \text{Im}(\hat{\mathbf{v}}) \right] / 2
\]

\[
= \omega \left[ \text{Re}(\hat{\mathbf{v}}) \cdot \text{Im}(\mathbf{M}) \text{Re}(\hat{\mathbf{v}}) + \text{Im}(\hat{\mathbf{v}}) \cdot \text{Im}(\mathbf{M}) \text{Im}(\hat{\mathbf{v}}) \right].
\]

So \( \text{Im} \mathbf{M}(\omega) \) must be positive semidefinite for all real \( \omega > 0 \).
Generalization of the model which can mimic any response function
If each of the springs behaves according to a Maxwell model of a dashpot in series with a purely elastic spring then:

$$\frac{1}{K_j} = \frac{1}{k_j} + i/(\omega \eta_j), \quad \frac{1}{L_j} = \frac{1}{\ell_j} + i/(\omega \nu_j),$$

and the material will have a mass:

$$M(\omega) = M_0 I + \sum_{j=1}^{n} R_j^T \begin{pmatrix} 2k_j/(2k_j/m_j - \omega^2 - i\omega/\eta_j) & 0 \\ 0 & 2\ell_j/(2\ell_j/m_j - \omega^2 - i\omega/\nu_j) \end{pmatrix} R_j.$$
One could argue that the failure of Newton’s law in these models is due to the failure to take into account the internal “hidden mass” which is not moving in phase with the motion of the rest of the body. But there is always some internal mass vibrating at microscopic levels.

**Key point:** Rather than look to deeper and deeper levels of the microstructure (perhaps far beyond the reach of experiments) to explain everything using Newton’s law, it seems much more sensible to allow modifications of Newton’s second law to explain the response of bodies.
Why would one want anisotropic density? One reason: Transformation based cloaking
This was first discovered by Greenleaf, Lassas and Uhlmann (2003) for conductivity.

It was later extended to 2-dimensional geometric optics by Leonhardt (2006) and to the time harmonic Maxwell equations by Pendry, Schurig and Smith (2006).

The conductivity or Maxwell equations retain their form under coordinate transformations and the key idea of Greenleaf, Lassas and Uhlmann is to use a transformation which maps a point to a sphere (the cloaking region) and which is the identity mapping outside a larger sphere.
From Pendry, Schurig and Smith (2006)

Putting an object in the orange cloaking region is equivalent to disturbing the moduli inside a point in the equivalent problem, which will not disturb the surrounding fields, in particular outside the blue shell where the mapping is the identity.
Both Greenleaf, Lassas and Uhlmann (2003) and Pendry, Schurig and Smith (2006) use the same cloaking transformation:

\[ r' = a + r(b-a)/b \]
\[ \theta' = \theta \]
\[ \phi' = \phi \]

for \( r < b \), with \( r' = r, \theta' = \theta, \phi' = \phi \) for \( r > b \).

The moduli of the cloak are given by:

\[ \sigma'_r = \varepsilon'_r = \mu'_r = \frac{b}{b-a} \left( \frac{(r'-a)}{r'} \right)^2 \]
\[ \sigma'_\theta = \varepsilon'_\theta = \mu'_\theta = \frac{b}{b-a} \]
\[ \sigma'_\phi = \varepsilon'_\phi = \mu'_\phi = \frac{b}{b-a} \]

with the results for \( \sigma' \) being due to Greenleaf, Lassas and Uhlmann and the results for \( \varepsilon' \) and \( \mu' \) being due to Pendry, Schurig and Smith.
Cloaking for elasticity (joint work with Marc Briane and John Willis)

Idea: Apply the Greenleaf, Lassas and Uhlmann/Pendry, Schurig, Smith method of cloaking to elastodynamics

Requires new materials with new behavior, in particular, materials with anisotropic density

Cummer and Schurig found that materials with anisotropic density are also needed for two-dimensional acoustic cloaking
Transformation of the elastodynamic equations

\[-\nabla \cdot \sigma = \omega^2 \varrho \mathbf{u}, \quad \sigma = C \nabla \mathbf{u}\]

under the coordinate transformation
\[x \to x'(x), \quad u \to u'(x') \quad \text{with}\]
\[u'(x') = (A^T)^{-1} u(x) \quad A_{ij} = \frac{\partial x'_i}{\partial x_j}\]

transform to
\[\nabla' \cdot \sigma' = D' \nabla' u' - \omega^2 \varrho' \mathbf{u}', \quad \sigma' = C' \nabla' u' + S' \mathbf{u}'\]

with
\[g'_{pq} = \frac{\partial x'_p}{\partial x_i} \frac{\partial x'_q}{\partial x_i} \quad + \frac{1}{a} \frac{\partial^2 x'_p}{\partial x_i \partial x_j} C_{ijkl} \frac{\partial^2 x'_q}{\partial x_k \partial x_l}\]
\[C'_{pqrst} = \frac{1}{a} \frac{\partial x'_p}{\partial x_i} \frac{\partial x'_q}{\partial x_j} C_{ijkl} \frac{\partial x'_r}{\partial x_k} \frac{\partial x'_t}{\partial x_l}\]
\[S'_{pqr} = \frac{1}{a} \frac{\partial x'_p}{\partial x_i} \frac{\partial x'_q}{\partial x_j} C_{ijkl} \frac{\partial^2 x'_r}{\partial x_k \partial x_l} = S'_{qrp}\]
\[D'_{pqr} = S'_{qrp}\]
\[a = \det(A)\]
These new equations don't look like elastodynamic equations at all!
But in fact they are a special case of the Willis equations:

\[
\text{div}\langle \sigma \rangle = \langle \dot{p} \rangle, \quad \langle e \rangle = [\nabla \langle u \rangle + (\nabla \langle u \rangle)^T] \\
\langle \sigma \rangle = C_{\text{eff}} \ast \langle e \rangle + S_{\text{eff}} \ast \langle \dot{u} \rangle \\
\langle p \rangle = (S_{\text{eff}})^{\dagger} \ast \langle e \rangle + g_{\text{eff}} \ast \langle \dot{u} \rangle
\]

Brackets \( \langle \rangle \) denote an ensemble average
* denotes a convolution with respect to time
\( C_{\text{eff}}, S_{\text{eff}} \) and \( g_{\text{eff}} \) are operators which are generally non-local in space.

So the (ensemble averaged) stress depends not only on the strain but also on the velocity
and the momentum depends not only on the velocity but also on the strain.
The Black circles have positive mass
The White circles have negative mass
Origin at center

\[ x_A = (-h, 0), \quad x_B = (0, -h), \quad x_C = (h, 0), \quad x_D = (0, h) \]
\[ x_E = (-ch, 0), \quad x_F = (ch, 0) \]
Displacements at vertices of unit cell are assumed to derive from some smooth complex displacement field in the limit \( h \) to 0:

\[
\begin{align*}
\mathbf{u}_A &= \Re\{\mathbf{u}(\mathbf{x}_A)e^{-i\omega t}\} \approx \Re\{(\mathbf{u}_0 - h\mathbf{q})e^{-i\omega t}\} \\
\mathbf{u}_B &= \Re\{\mathbf{u}(\mathbf{x}_B)e^{-i\omega t}\} \approx \Re\{(\mathbf{u}_0 - h\mathbf{w})e^{-i\omega t}\} \\
\mathbf{u}_C &= \Re\{\mathbf{u}(\mathbf{x}_A)e^{-i\omega t}\} \approx \Re\{(\mathbf{u}_0 + h\mathbf{q})e^{-i\omega t}\} \\
\mathbf{u}_D &= \Re\{\mathbf{u}(\mathbf{x}_D)e^{-i\omega t}\} \approx \Re\{(\mathbf{u}_0 + h\mathbf{w})e^{-i\omega t}\}
\end{align*}
\]

where

\[
\mathbf{u}_0 = \mathbf{u}(\mathbf{x}_0), \quad q = \frac{\partial \mathbf{u}}{\partial x_1} \bigg|_{\mathbf{x}=\mathbf{x}_0}, \quad w = \frac{\partial \mathbf{u}}{\partial x_2} \bigg|_{\mathbf{x}=\mathbf{x}_0}
\]
Displacements at interior points are internal hidden variables

\[ u_E \approx \Re \left[ \left( u_0 + h s \right) e^{-i\omega t} \right] \]
\[ u_F \approx \Re \left[ \left( u_0 - h s \right) e^{-i\omega t} \right] \]

From the geometry:

\[ s_1 = \frac{w_2}{c} = \frac{1}{c} \frac{\partial u_2}{\partial x_2}, \quad s_2 = c w_1 = c \frac{\partial u_1}{\partial x_2}. \]
Let the masses at the interior points be:

\[ m_E = hm, \quad m_F = -hm + \delta h^2 \]

Then the physical momentum density is:

\[ \Re \{ -i \omega [m_E (u_0 + hs) + m_F (u_0 - hs)] e^{-i \omega t} \} \]
\[ \approx h^2 \Re \{ -i \omega [2ms + \delta u_0] e^{-i \omega t} \} \]

So the associated complex momentum density is

\[ p = -i \omega ms + (\delta/2)(-i \omega u_0) \]

Depends on the displacement gradient through \( s \)
Acceleration of the material will generate stress.

To leading order in \( h \) the masses at E and F must be accelerated by complex forces:

\[
F = -\omega^2 mh u \quad \text{and} \quad -F
\]

This must be provided by tension in the rods.

Forgetting for the moment the springs and surrounding material a force

\[
h t = (cF_2, F_1/c) = -\omega^2 mh(cu_2, u_1/c)
\]

needs to act on the vertex D to maintain the motions. Similarly a force \(-ht\) needs to act on the vertex B.
To compensate for the inertial stress $\sigma_I$ caused by the acceleration a traction of $t$ per unit length needs to be applied to the top boundary of a small sample, and a traction of $-t$ to the bottom boundary, and no tractions to the side boundaries $\sigma_{Ins}$.

Equating these tractions with $\sigma$ gives:

$$\sigma_I = \begin{pmatrix} 0 & t_1 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega^2 mc u_2 \\ 0 & -\omega^2 mu_1/c \end{pmatrix}$$

Total stress: $\sigma = \sigma_E + \sigma_I$

with $\sigma_E = C[\nabla u + (\nabla u)^T]$
Constitutive law:

\[
\begin{pmatrix}
\sigma \\
p
\end{pmatrix} =
\begin{pmatrix}
\mathbf{C} & \mathbf{S} \\
\mathbf{D} & \rho
\end{pmatrix}
\begin{pmatrix}
\nabla \mathbf{u} \\
\mathbf{v}
\end{pmatrix}
\]

in a basis with

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{21} \\
\sigma_{12} \\
\sigma_{22}
\end{pmatrix}, \quad
\begin{pmatrix}
p_1 \\
p_2
\end{pmatrix}, \quad
\begin{pmatrix}
\partial u_1 / \partial x_1 \\
\partial u_2 / \partial x_1 \\
\partial u_1 / \partial x_2 \\
\partial u_2 / \partial x_2
\end{pmatrix}, \quad
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
\]

the tensors \( \mathbf{S} \) and \( \mathbf{D} \) are

\[
\mathbf{S} =
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -i\omega mc & 0 \\
0 & 0 & 0 & -i\omega mc
\end{pmatrix}, \quad
\mathbf{D} =
\begin{pmatrix}
0 & 0 & 0 & -i\omega m/c \\
0 & 0 & -i\omega mc & 0 \\
0 & -i\omega mc & 0 & 0
\end{pmatrix}
\]
Generalization:

\[ x_G = (-c'h, 0), \quad x_H = (c'h, 0), \quad m_G = -hm' + \delta h^2, \quad m_H = hm' \]
One gets:

\[
S = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
i\omega(m'/c' - m/c) & i\omega(m'/c' - m/c)
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & i\omega(m'/c' - m/c) \\
0 & i\omega(m'/c' - m/c) & 0
\end{pmatrix}.
\]

When \( m'/c' = mc \) the stress is symmetric and the momentum only depends on the strain, not on local rotations. The constitutive law then has the general form required for elasticity cloaking.
Besides transformation based cloaking there is also cloaking based on anomalous localized resonance (joint work with Nicorovici, McPhedran and Botton)

Clusters of polarizable dipoles near superlenses get cloaked. In contrast to transformation based cloaking, the cloaking region is outside the cloaking device.
A "polarizable molecule" outside the cloaking region, in the presence of a dipolar source.

Real Part

Polarizable Molecule Clearly Visible

\[ \varepsilon_m = \varepsilon_c = 1, \quad \varepsilon_s = -1 + 10^{-12} i \]

"molecule" has polarizability \( \alpha = 2 \) and induced dipole moment

\[ 0.5 + 8.39 \times 10^{-8} i \]
A "polarizable molecule" inside the cloaking region in the presence of a dipolar source

\[ \varepsilon_m = \varepsilon_c = 1, \quad \varepsilon_s = -1 + 10^{-12} \imath \]

"molecule" has polarizability \( \alpha = 2 \) and induced dipole moment

\[-6.31 \times 10^{-7} + 2.30 \times 10^{-6} \imath\]
The effect of cloaking is greatest when the "polarizable molecule" is near the surface of the coated cylinder.

\[ E_m = E_c = 1, \quad E_S = -1 + 0.01i \]

"molecule" has polarizability \( \alpha = 2 \)
and induced dipole moment
\[ 0.000012 - 0.00066i \]
"molecule" outside the cloaking region

\[ \varepsilon_m = \varepsilon_c = 1, \quad \varepsilon_s = -1 + 0.01 i \]

"molecule" has polarizability \( \alpha = 2 \) and induced dipole moment \(-1.68 - 0.74 i\)
Effective equations for a random media: the ensemble averaging approach of Willis.

Summary of the approach initiated by Willis (1981)

For each realization:

\[ \text{div } \sigma + f = \dot{\rho}, \quad \sigma = C \ast e, \quad \rho = \rho \dot{u}. \]

Ensemble averaging:

\[ \text{div} \langle \sigma \rangle + f = \langle \dot{\rho} \rangle. \]

The equations of motion could be solved if there were constitutive laws:

\[ \langle \sigma \rangle = C^{\text{eff}} \ast \langle e \rangle + S^{\text{eff}} \ast \langle \dot{u} \rangle, \quad \langle \rho \rangle = S^{\text{eff}} \ast \langle e \rangle + \rho^{\text{eff}} \ast \langle \dot{u} \rangle. \]
Introduce a comparison medium and define “polarization fields”

\[ \tau = (C - C_0) \ast e, \quad \pi = (\rho - \rho_0) \dot{u}. \]

The equation of motion then implies:

\[ \text{div}(C_0 \ast e) + \mathbf{f} + \text{div} \tau - \dot{\pi} = \rho_0 \ddot{u}. \]

which has the solution

\[ u = u_0 + G \ast (\text{div} \tau - \dot{\pi}), \]

where \( G \) is the relevant Green’s function. Then we have:

\[ e = e_0 - S_x \ast \tau - M_x \ast \pi, \]
\[ \dot{u} = \dot{u}_0 - S_t \ast \tau - M_t \ast \pi, \quad \text{and} \quad S_t = M_x^\dagger \]

in the sense

\[ \int \pi \ast (S_t \ast \tau) \, dx = \int \tau \ast (M_x \ast \pi) \, dx. \]
Expressed in terms of the polarization fields:

\[
(C - C_0)^{-1} \ast \tau + S_x \ast \tau + M_x \ast \pi = e_0,
\]

\[
(\rho - \rho_0)^{-1} \pi + S_t \ast \tau + M_t \ast \pi = \dot{u}_0.
\]

and by ensemble averaging:

\[
(C - C_0)^{-1} \ast \tau + S_x \ast (\tau - \langle \tau \rangle) + M_x \ast (\pi - \langle \pi \rangle) = \langle e \rangle,
\]

\[
(\rho - \rho_0)^{-1} \pi + S_t \ast (\tau - \langle \tau \rangle) + M_t \ast (\pi - \langle \pi \rangle) = \langle \dot{u} \rangle.
\]

which has the formal solution

\[
\begin{pmatrix}
\tau \\
\pi
\end{pmatrix}
= T \ast 
\begin{pmatrix}
\langle e \rangle \\
\langle \dot{u} \rangle
\end{pmatrix}.
\]

from which it follows that

\[
\begin{pmatrix}
C^{\text{eff}} & S^{\text{eff}} \\
S^{\text{eff}\dagger} & \rho^{\text{eff}}
\end{pmatrix}
= \begin{pmatrix}
C_0 & 0 \\
0 & \rho_0 I
\end{pmatrix} + \langle T \rangle.
\]
Extension allowing a weighted average.

If there are voids then \( u \) is not defined there and \( \langle u \rangle \) has no meaning. To allow for this and the presence of "hidden regions" it makes sense to introduce a weighted displacement field \( u_w(x) = w(x)u(x) \) and find the equation governing its ensemble averaged behavior

\[
\begin{pmatrix}
  e \\
  u
\end{pmatrix} = \begin{pmatrix}
  (C - C_0)^{-1} & 0 \\
  0 & H \ast (\rho - \rho_0)^{-1}
\end{pmatrix} \ast \begin{pmatrix}
  T_{11} & T_{12} \\
  T_{21} & T_{22}
\end{pmatrix} \ast \begin{pmatrix}
  \langle e \rangle \\
  \langle \dot{u} \rangle
\end{pmatrix}
\]

\( H \) is the Heaviside function and we assume

\[ u = 0 \text{ for } t \leq 0. \]
Then
\[ e_w = \left[ \nabla \otimes \mathbf{u}_w + (\nabla \otimes \mathbf{u}_w)^T \right] / 2 = w e + \left[ (\nabla w) \otimes \mathbf{u} + \mathbf{u} \otimes \nabla w \right] / 2 \]
\[ \langle (\nabla w) \otimes \mathbf{u} + \mathbf{u} \otimes \nabla w \rangle = \langle [H \ast (\rho - \rho_0)^{-1} \ast [T_{21} \ast \langle e \rangle + T_{22} \ast \langle \dot{u} \rangle] \otimes (\nabla w) \rangle \]
\[ + \langle [H \ast (\rho - \rho_0)^{-1} \ast [T_{21} \ast \langle e \rangle + T_{22} \ast \langle \dot{u} \rangle] \otimes (\nabla w) \rangle^T \]
\[ \equiv Y_1^w \ast \langle e \rangle + Y_2^w \ast \langle \dot{u} \rangle, \]

so
\[ \begin{pmatrix} \langle e_w \rangle \\ \langle \dot{u}_w \rangle \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \ast \begin{pmatrix} \langle e \rangle \\ \langle \dot{u} \rangle \end{pmatrix} \equiv R_w \ast \begin{pmatrix} \langle e \rangle \\ \langle \dot{u} \rangle \end{pmatrix}, \]

where
\[ R_{11} = \langle w(C - C_0)^{-1} \ast T_{11} \rangle + Y_1^w, \]
\[ R_{12} = \langle w(C - C_0)^{-1} \ast T_{12} \rangle + Y_2^w, \]
\[ R_{21} = \langle w(\rho - \rho_0)^{-1} \ast T_{21} \rangle, \]
\[ R_{22} = \langle w(\rho - \rho_0)^{-1} \ast T_{22} \rangle. \]
Giving

\[
\begin{pmatrix}
  \langle \sigma \rangle \\
  \langle p \rangle \\
\end{pmatrix} = \begin{pmatrix}
  C^\text{eff}_w & S^\text{eff}_w \\
  D^\text{eff}_w & \rho^\text{eff}_w \\
\end{pmatrix} \ast \begin{pmatrix}
  \langle e_w \rangle \\
  \langle \dot{u}_w \rangle \\
\end{pmatrix},
\]

\[
\begin{pmatrix}
  C^\text{eff}_w & S^\text{eff}_w \\
  D^\text{eff}_w & \rho^\text{eff}_w \\
\end{pmatrix} \equiv \begin{pmatrix}
  C^\text{eff} & S^\text{eff} \\
  S^\text{eff} & \rho^\text{eff} \\
\end{pmatrix} \ast (R_w)^{-1}.
\]

which with \( \langle e_w \rangle = [\nabla \otimes \langle u_w \rangle + (\nabla \otimes \langle u_w \rangle)^T]/2 \),
and \( \text{div} \langle \sigma \rangle + f = \langle \dot{p} \rangle \). determine \( \langle u_w \rangle \)

This analysis breaks down if voids are present but we still expect the final equations to hold.
Of course this only gives the expected (ensemble averaged behavior, not the behavior in a particular realization.

For more details see:


Thank-you for your attention!