Existence results for some micro-macro models of polymeric flows

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Outline

Introduction

Systems coupling fluids and polymers are of great interest in many branches of applied physics, chemistry and biology.
There are many models to describe them :

- The FENE (Finite Extensible Nonlinear Elastic) dumbbell model. In this model, a polymer is idealized as an “elastic dumbbell” consisting of two “beads” joined by a spring. The microscopic variable is $R \in B(0, R_0)$.
- The Hooke model is the case when $R_0 = \infty$.
- The Doi model (or Rigid model): The polymers have a fixed length and $R \in \mathbb{S}^{N-1}$

At the level of the polymeric liquid, we get a system coupling the Navier-Stokes equation for the fluid velocity with a Fokker-Planck equation describing the evolution of the polymer density.

- Bird, Curtis, Amstrong and Hassager
- Doi and Edwards,
- Ottinger

The FENE model

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = \mathrm{div}\tau, \quad \mathrm{div} u = 0, \\ \partial_t \psi + u.\nabla \psi = \mathrm{div}_R \Big[-\nabla u \cdot R\psi + \beta \nabla_R \psi + \nabla_R \mathcal{U} \psi \Big]. \\ \tau_{ij} = \int_B (R_i \otimes \partial_{R_j} \mathcal{U}) \psi(t, x, R) \; dR \\ (\nabla_R \mathcal{U} \psi + \beta \nabla_R \psi).n = 0 \text{ on } \partial B(0, R_0). \end{cases}$$

We will take $\beta = 1$.

Here, $\psi(t, x, R)$ is the distribution function for the internal configuration and $F(R) = \nabla\mathcal{U}$ is the spring force which derives from a potential $\mathcal{U}$:

$$\mathcal{U}(R) = -k|R_0|^2 \log(1 - |R|^2/|R_0|^2)$$

for some constant $k > 0$. We take $R_0 = 1$ and we denote

$$\psi_\infty = \frac{e^{-\mathcal{U}}}{\int_B e^{-\mathcal{U}}} = \frac{(1 - |R|^2)^k}{Z}.$$

The Fokker Planck equation can also be written

$$\partial_t \psi + u.\nabla\psi = \mathrm{div}_R \Big[-\nabla u \cdot R\psi + \psi_\infty \nabla \frac{\psi}{\psi_\infty} \Big].$$

- If $R_0 = \infty$, we take $\mathcal{U}(R) = kR^2$ and we get the Hooke model which yields the Oldroyd B model.
- If we replace ∇u by $W(u) = \frac{\nabla u - {}^t\nabla u}{2}$ in the second equation, we get the co-rotational model.
- We have to add a boundary condition for u. We take Dirichlet boundary condition, namely $u = 0$ on $\partial\Omega$ where Ω is a bounded of $\mathbb{R}^N$

We can think of the distribution function ψ as the density of a random variable R which solves

$$dR + u.\nabla R dt = (\nabla u R - \nabla_R \mathcal{U}(R))dt + \sqrt{2} dW_t$$

where the stochastic process W_t is the standard Brownian motion in $\mathbb{R}^N$ and the additional stress tensor is given by the following expectation $\tau = \mathbb{E}(R_i \otimes \partial_{R_j} \mathcal{U})$. Of course, we may need to add a boundary condition when R reaches the boundary of B.

The Doi model

$$
\begin{cases}
\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = \text{div}\tau, \quad \text{div} u = 0, \\
\partial_t \psi + u.\nabla \psi = \mathrm{div}_R \Big[- P_{R^\perp} (\nabla u \cdot R) \psi \Big] - \Delta_R \psi \\
\tau_{ij} = \int_{\mathbb{S}^{N-1}} N(R_i \otimes R_j) \psi(t, x, R) \; dR + \\
\qquad\qquad b \nabla_k u_l : \int_{\mathbb{S}^{N-1}} R_k R_l R_i R_j \psi \; dR,
\end{cases}
$$

$P_{R^\perp}$ is the orthogonal projection on the tangent space to the sphere at R, namely $P_{R^\perp}(\nabla u R) = \nabla u R - (R.\nabla u.R) R$ and b is a parameter.

Existence results

For Oldroyd B model :

- Renardy
- Guillopé and Saut (1990)
- Fernández-Cara, Guillén and Ortega (1997)
- Chemin and Masmoudi 2001
- Lions and Masmoudi 2001
- Lin, Liu and Zhang 2005

For micro-macro models :

- Renardy
- W. E, Li and Zhang
- Jourdain, Lelievre and Le Bris
- Zhang and Zhang
- Barrett, Schwab and Suli
- Lin, Liu and Zhang
- Otto and Tzavaras
- Constantin, Fefferman, Titi and Zarnescu

For numeric results :

- Keunigs
- Ottinger
- Jourdain, Lelievre and Le Bris
- P. Zhang

Main results

Three types of results

- Local well-posedness for the FENE model (and global well-posedness for small data).
- Global existence of weak solutions for the co-rotational FENE model and for the Doi model (with P.-L. Lions).
- Global existence of regular solution for the Doi model in 2D (with P. Constantin) and for the co-rotational FENE model (with P. Zhang and Z. Zhang).

A priori estimates

The free energy for FENE

$$\frac{\partial}{\partial t}\left[\int_\Omega \frac{|u|^2}{2}\right] = -\nu \int_\Omega |\nabla u|^2 - \int_\Omega \tau : \nabla u.$$

$$\frac{\partial}{\partial t}\left[\int_{\Omega\times B} \psi \log \frac{\psi}{\psi_\infty}\right] = -\int_{\Omega\times B} |\nabla_R \sqrt{\frac{\psi}{\psi_\infty}}|^2 \psi_\infty + \int_\Omega \tau : \nabla u.$$

Hence

$$\frac{\partial}{\partial t}\left[\int_\Omega \frac{|u|^2}{2} + \int_{\Omega\times B} \psi \log \frac{\psi}{\psi_\infty}\right] = -\int_{\Omega\times B} |\nabla_R \sqrt{\frac{\psi}{\psi_\infty}}|^2 - \nu \int_\Omega |\nabla u|^2$$

For the co-rotational FENE model, we get

$$\frac{\partial}{\partial t}\left[\int_{\Omega\times B}\psi\log\frac{\psi}{\psi_\infty}\right] = -\int_{\Omega\times B}|\nabla_R\sqrt{\frac{\psi}{\psi_\infty}}|^2\psi_\infty$$

More generally, for $p > 0$, we have

$$\partial_t\int_B\psi_\infty\left(\frac{\psi}{\psi_\infty}\right)^p dR + u.\nabla\int_B\psi_\infty\left(\frac{\psi}{\psi_\infty}\right)^p dR = -\frac{4(p-1)}{p}\int_B\psi_\infty\left|\nabla_R\left(\frac{\psi}{\psi_\infty}\right)^{p/2}\right|^2 dR.$$

The free energy for the Doi model

$$\partial_t \left[\int_\Omega \frac{|u|^2}{2} + \int_{\Omega \times \mathbb{S}^{N-1}} \psi \log \psi - \psi + 1 \right] =$$
$$-\nu \int_\Omega |\nabla u|^2 + 4 \int_{\Omega \times \mathbb{S}^{N-1}} |\nabla_R \sqrt{\psi}|^2$$
$$+ b \int_\Omega \nabla_k u_l : \int_{\mathbb{S}^{N-1}} R_k R_l R_i R_j \psi dR : \nabla_i u_j$$

To make sure that the free energy is dissipated, we have to assume that $b > -\frac{N}{N-1}\nu$.

Higher order derivatives

We use the notations

$$|u|_s^2 = \sum_{|\alpha|\leq s} \int_\Omega |\partial^\alpha u|^2 dx$$

$$|\psi|_s^2 = \sum_{|\alpha|\leq s} \int_\Omega \int_B |\partial^\alpha \psi|^2 \frac{dR}{\psi_\infty} dx$$

$$|\psi|_{s,1}^2 = \sum_{|\alpha|\leq s} \int_\Omega \int_B \psi_\infty |\partial^\alpha \nabla_R \frac{\psi}{\psi_\infty}|^2 dR dx$$

From the first equation of FENE system, we deduce that

$$\partial_t |u|_s^2 + \nu |u|_{s+1}^2 \leq C|u|_s^3 + \frac{C}{\nu}|\tau|_s^2.$$

From the second equation, we get

$$\begin{aligned}\partial_t \int_B \frac{\psi^2}{\psi_\infty} dR \quad + \quad & u.\nabla \int_B \frac{\psi^2}{\psi_\infty} dR + \int_B \psi_\infty \left| \nabla_R \frac{\psi}{\psi_\infty} \right|^2 \\ & \leq |Du| \left(\int_B \frac{\psi^2}{\psi_\infty} \right)^{1/2} \left(\int_B \psi_\infty \left| \nabla_R \frac{\psi}{\psi_\infty} \right|^2 \right)^{1/2} \\ & \leq C|Du|^2 \left(\int_B \frac{\psi^2}{\psi_\infty} \right) + \frac{1}{2} \left(\int_B \psi_\infty \left| \nabla_R \frac{\psi}{\psi_\infty} \right|^2 \right)\end{aligned}$$

We define the flow Φ by

$$\begin{cases} \partial_t \Phi(t,x) = u(t,\Phi(t,x)) \\ \Phi(0,x) = x \end{cases}$$

Integrating along the flow, we get

$$\sup_x \int_B \frac{\psi^2(t)}{\psi_\infty} dR + \sup_x \int_0^t \int_B \psi_\infty \left| \nabla_R \frac{\psi}{\psi_\infty} \right|^2 (s,\Phi(s,x)) ds$$
$$\leq \sup_x \int_B \frac{\psi_0^2}{\psi_\infty} \, e^{C \int_0^t |Du|^2_{L^\infty}}$$

$$\partial_t \int_B \frac{(\partial^s \psi)^2}{\psi_\infty} + u.\nabla \int_B \frac{(\partial^s \psi)^2}{\psi_\infty} + \int_B \psi_\infty \left| \nabla_R \frac{\partial^s \psi}{\psi_\infty} \right|^2 =$$

$$= - \sum_{|\alpha|+|\beta| \le s} \int_B div_R(\partial^\alpha D u R \partial^\beta \psi) \frac{\partial^{\alpha+\beta} \psi}{\psi_\infty}$$

Integrating in the x variable, we get

$$\partial_t |\psi|_s^2 + \frac{1}{2} |\psi|_{s,1}^2 \le C \left(|Du|_{L^\infty}^2 |\psi|_s^2 + |u|_{s+1}^2 \sup_x \int \frac{\psi^2}{\psi_\infty} dR \right)$$

Global existence of weak solutions for co-FENE

Theorem

(with P.-L. Lions) Take $u_0 \in L^2(\Omega)$ *and* ψ_0 *such that* $\int \psi_0 dR = 1$ *a.e in* x *and* $\int_B \frac{\psi_0^2}{\psi_\infty} dR \in L_x^\infty$. *Then, there exists a global weak solution* (u, ψ) *of co-FENE with*

$$u \in L^\infty(0, T; L^2) \cap L^2_{loc}(0, T; H^1) \quad \textit{and}$$

$$\psi \in L^\infty(0, T; L^\infty(L^2(\frac{dR}{\psi_\infty}))).$$

Proof: Stability of weak solutions:

Take (u^n, ψ^n) a sequence of weak solutions with initial data (u_0^n, ψ_0^n) and such that (u_0^n, ψ_0^n) converges strongly to (u_0, ψ_0) in $L^2(dx) \times L^2(\frac{dR}{\psi_\infty} dx)$.

We extract a subsequence such that u^n converges weakly to u in $L^p((0,T); L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$ and ψ^n converges weakly to ψ in $L^p((0,T) \times \Omega; L^2(\frac{dR}{\psi_\infty}))$ for each $p < \infty$.

We would like to prove that (u, ψ) is still a solution of co-FENE.

Take $N = 2$:

$$(\psi^n - \psi)^2 \to \eta, \qquad |\nabla(u^n - u)|^2 \to \mu, \qquad \psi^n \nabla u^n \to \psi \nabla u + \beta$$

$$|\nabla_R(\psi^n - \psi)|^2 \to \kappa, \qquad |\tau^n - \tau|^2 \to \alpha$$

We can prove that

$$\begin{aligned} \nu\mu &= \int \beta_{ij} R_i \nabla_j \phi dR \leq C\sqrt{\mu}\sqrt{\alpha}, \qquad |\beta_{ij}| \leq \sqrt{\mu}\sqrt{\eta} \\ \mu &\leq C\alpha \leq C \int \left(\psi_\infty \kappa + \frac{\eta}{\psi_\infty} \right) dR. \end{aligned}$$

And

$$
\begin{aligned}
\partial_t \int_B \frac{\eta}{\psi_\infty} + u.\nabla \int_B \frac{\eta}{\psi_\infty} \\
&\leq C\sqrt{\mu} \int_B \sqrt{\eta} \left| \nabla \frac{\psi}{\psi_\infty} \right| - \int_B \psi_\infty \kappa \\
&\leq C\sqrt{\mu} \left(\int_B \frac{\eta}{\psi_\infty} \int_B \psi_\infty \left| \nabla \frac{\psi}{\psi_\infty} \right|^2 \right)^{1/2} - \int_B \psi_\infty \kappa \\
&\leq C \left(1 + \int_B \psi_\infty \left| \nabla \frac{\psi}{\psi_\infty} \right|^2 \right) \int_B \frac{\eta}{\psi_\infty}
\end{aligned}
$$

Local existence for FENE

We take, $s > \frac{N}{2} + 1$.

Theorem

Take $u_0 \in H^s(\mathbb{R}^N)$ *and* $\psi_0 \in H^s(\mathbb{R}^N; L^2(\frac{dR}{\psi_\infty}))$ *with* $\int \psi_0 dR = 1$ *a.e in* x. *Then, there exists a time* T^* *and a unique solution* (u, ψ) *of FENE system in* $C([0, T^*); H^s) \times C([0, T^*); H^s(\mathbb{R}^N; L^2(\frac{dR}{\psi_\infty})))$. *Moreover,* $u \in L^2_{loc}([0, T^*); H^{s+1})$ *and* $\psi \in L^2_{loc}([0, T^*); H^s(\mathbb{R}^N; \mathcal{H}^1))$

where we denote $\mathcal{H} = L^2(\frac{dR}{\psi_\infty})$ and

$$\mathcal{H}^1 = \left\{ \psi \mid \quad \int \psi_\infty \left| \nabla_R \frac{\psi}{\psi_\infty} \right|^2 + \frac{\psi^2}{\psi_\infty} \, dR < \infty \right\}.$$

Proof

We have

$$\partial_t |u|_s^2 + \nu |u|_{s+1}^2 \le C|u|_s^3 + \frac{C}{\nu}|\tau|_s^2.$$

$$\partial_t |\psi|_s^2 + \frac{1}{2}|\psi|_{s,1}^2 \le C\left(|Du|_{L^\infty}^2 |\psi|_s^2 + |u|_{s+1}^2 \sup_x \int \frac{\psi^2}{\psi_\infty} dR \right)$$

We have

$$|\tau|_s^2 \leq \epsilon |\psi|_{s,1}^2 + C_\epsilon |\psi|_s^2$$

for each $\epsilon > 0$, since

$$\left(\int_B \frac{|\psi|}{1-|R|} dR \right)^2 \leq \epsilon \int_B \psi_\infty \left| \nabla_R \frac{\psi}{\psi_\infty} \right|^2 dR + C_\epsilon \int_B \frac{|\psi|^2}{\psi_\infty} dR$$

We choose T such that

$$\int_0^T |u|_s^2 + |Du|_{L^\infty}^2 + |u|_s \leq A$$

for some fixed constant A. Hence,

$$|\psi(t)|_s^2 + \frac{1}{2}\int_0^t |\psi|_{s,1}^2 \leq |\psi_0|_s^2 e^{CA} + Ce^{2CA}\int_0^t |u|_{s+1}^2.$$

Moreover,

$$|u(t)|_s^2 + \nu \int_0^t |u|_{s+1}^2 \leq (|u_0|_s^2 + \int_0^t |\tau|_s^2) e^{C\int_0^t |u|_s}$$

Hence,

$$\int_0^t |\tau|_s^2 \leq \epsilon \int_0^t |\psi|_{s,1}^2 + C_\epsilon \int_0^t |\psi|_s^2 \leq$$
$$(\epsilon + C_\epsilon T) e^{2CA} \left(C + \int_0^t |u|_{s+1}^2\right)$$

and if ϵ and T are chosen small enough, we get

$$|u(t)|_s^2 + \frac{\nu}{2} \int_0^t |u|_{s+1}^2 \leq (|u_0|_s^2 + C) e^{CA}.$$

Remark : The linearized problem and boundary condition

$$L\psi = -div(\psi_\infty \nabla \frac{\psi}{\psi_\infty})$$

on the space $\mathcal{H} = L^2(\frac{dR}{\psi_\infty})$ with domain

$$D(L) = \left\{ \psi \in \mathcal{H}^2, \quad \psi_\infty \nabla \frac{\psi}{\psi_\infty}|_{\partial B} = 0 \right\}$$

where $\mathcal{H}^1$ and $\mathcal{H}^2$ are given by

$$\mathcal{H}^1 = \left\{ \psi \,| \quad \int \psi_\infty \left| \nabla \frac{\psi}{\psi_\infty} \right|^2 + \frac{\psi^2}{\psi_\infty} \; dR < \infty \right\}$$

$$\mathcal{H}^2 = \left\{ \psi \in \mathcal{H}^1 \,| \quad \int \left(div(\psi_\infty \nabla \frac{\psi}{\psi_\infty}) \right)^2 \frac{dR}{\psi_\infty} < \infty \right\}$$

If $k \geq 1$ then

$$\overline{C_0^\infty}^{\mathcal{H}^1} = \mathcal{H}^1 \tag{1}$$

and $D(L) = \mathcal{H}^2$.

However, (1) does not hold when $k < 1$ since ψ_∞ is not in $\overline{C_0^\infty}^{\mathcal{H}^1}$ and, $D(L) \subset \mathcal{H}^2$ is strict. Indeed, for $k < 1$, $\psi_\infty^{1/k} \in \mathcal{H}^2$ but does not satisfy the boundary condition and hence it is not in $D(L)$.

This is related to Jourdain and Lelievre who proved that when $k \geq 1$, then the stochastic process R_t does not reach the boundary and when $k < 1$, it reaches the boundary a.s.

Global existence in 2D for co-Hooke

Theorem

(with Zhang and Zhang) Let $1 < s < 2$*. Let* $u_0 \in H^1(\mathbb{R}^2) \cap C^s(\mathbb{R}^2), \psi_0 \in H^1(\mathbb{R}^2; L^2(\mathbb{R}^2)) \cap C^{s-1}(\mathbb{R}^2; L^2(\mathbb{R}^2))$*, and* $|R| f_0 \in L^\infty(\mathbb{R}^2; L^2(\mathbb{R}^2))$*. Then co-Hooke has a unique global solution* (u, ψ) *such that for any* $T > 0$*, there holds*

$$u \in C\Big([0, +\infty); H^1(\mathbb{R}^2) \cap C^s(\mathbb{R}^2)\Big) \cap L^2((0, T); H^2(\mathbb{R}^2)),$$

$$\psi \in C\Big([0, +\infty); H^1(\mathbb{R}^2; L^2(\mathbb{R}^2)) \cap C^{s-1}(\mathbb{R}^2; L^2(\mathbb{R}^2))\Big),$$

Furthermore, there holds

$$\|u(t)\|_{C^s} + \|f(t)\|_{s-1} \leq C_0(C + \|u_0\|_{C^s} + \|f_0\|_{s-1})^{\exp(C_0 t)}, \quad \forall t < \infty$$

where C_0 *only depends on*
$\|u_0\|_{L^2}^2 + \|f_0\|_{L^2}^2 + \|(1 + |R|) f_0\|_{L^\infty(\mathbb{R}^2; L^2(\mathbb{R}^2))}^2.$

- We have a similar type of result for the Doi model (with P. Constantin)
- The proof is based on losing regularity type of estimates (Bahouri and Chemin)
- In Chemin and Masmoudi, it was proved that if $\tau \in L^\infty$, then we get global existence in 2D for the Oldroyd B model.