Wave equation, Strichartz inequality and Lorentz transformations

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The cubic wave equation

$$\Box := \frac{\partial^2}{\partial t^2} - \Delta_x \ , \ t \in \mathbb{R}, x \in \mathbb{R}^3$$

Study the (real-valued) solutions u of

$$\Box u = \gamma u^3 \ , \ u(0,x) = u_0(x), \ \frac{\partial u}{\partial t}(0,x) = u_1(x)$$

where $\gamma \in \{-1, 1\}$.

Recall :

If $\gamma = -1$: global existence of solutions with sufficiently smooth data u_0, u_1 (say $u_0 \in H^1(\mathbb{R}^3), u_1 \in L^2(\mathbb{R}^3)$) .

If $\gamma = 1$: existence of solutions which blow up in finite time, except if the data are sufficiently small.

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If $\gamma = 1$: existence of solutions which blow up in finite time, except if the data are sufficiently small.

- Problem. Description of nonlinear effects in the high frequency limit ?

The critical regularity

Scale invariance $u^\lambda(t,x) = \lambda u(\lambda t, \lambda x)$,

$$u_0^\lambda(x) = \lambda u_0(\lambda x)\ ,\ u_1^\lambda(x) = \lambda^2 u_1(\lambda x)\ .$$

$\longrightarrow$ Critical regularity of the data :

$$u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\ ,\ u_1 \in \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)\ ,$$

$$\dot{H}^s(\mathbb{R}^d) := \{u \in \mathcal{S}'(\mathbb{R}^d)\ :\ \hat{u}(\xi) \in |\xi|^{-s} L^2(\mathbb{R}^d)\}\ ,\ |s| < \frac{d}{2}\ .$$

If $s < \frac{1}{2}$: even for $\gamma = -1$ (defocusing case),
$\exists$ arbitrary small data in $\dot{H}^s \times \dot{H}^{s-1}$ such that $(u(t), \partial_t u(t))$ is large in $\dot{H}^s \times \dot{H}^{s-1}$ for arbitrarily small time t
(Lebeau 2001 , Christ–Colliander–Tao 2004)

The Strichartz inequality (Strichartz, 1977)

If $\Box v = 0$, $v(0,.) = v_0$, $\partial_t v(0,.) = v_1$, then

$$\|v\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|(v_0, v_1)\|_{\dot{H}^{1/2}(\mathbb{R}^3)\times\dot{H}^{-1/2}(\mathbb{R}^3)} .$$

Inhomogeneous version : if $\Box w = f$, $w(0,.) = \partial_t w(0,.) = 0$,

$$\|w\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} + \sup_{t\in\mathbb{R}} \|(w(t,.), \partial_t w(t,.))\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} \lesssim \|f\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)}$$

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- Consequence $\exists \alpha > 0$ such that, if $\|(u_0, u_1)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} < \alpha$, then the Cauchy problem

$$\Box u = \gamma u^3 \ , \ u(0,x) = u_0(x), \ \frac{\partial u}{\partial t}(0,x) = u_1(x)$$

admits a unique solution

$$u \in L^4(\mathbb{R}\times\mathbb{R}^3) \cap C(\mathbb{R}, \dot{H}^{1/2}(\mathbb{R}^3)), \partial_t u \in C(\mathbb{R}, \dot{H}^{-1/2}(\mathbb{R}^3)) \ .$$

Linearizable data

We fix $\gamma \in \{-1, 1\}$.

Definition

Let (u_0^n, u_1^n) be a sequence in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$, weakly convergent to 0, with $\|(u_0^n, u_1^n)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}} < \alpha$.
We shall say that (u_0^n, u_1^n) is linearizable if the solution u^n to the cubic wave equation $\Box u^n = \gamma (u^n)^3$ with Cauchy data (u_0^n, u_1^n) at $t = 0$ satisfies

$$\|(u^n - v^n, \partial_t(u^n - v^n))\|_{L^\infty(\mathbb{R}, \dot{H}^{1/2} \times \dot{H}^{-1/2})} + \|u^n - v^n\|_{L^4(\mathbb{R} \times \mathbb{R}^3)} \longrightarrow 0 ,$$

where v^n is the solution to the linear wave equation

$$\Box v^n = 0$$

with the same Cauchy data (u_0^n, u_1^n) at $t = 0$.

Problem. Describe the non linearizable data (u_0^n, u_1^n), and the corresponding solutions u^n.

Remark. Sufficient condition to linearizability :

$$\|v^n\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \longrightarrow 0 \ .$$

Is it necessary ?
How to guess it on Cauchy data (u_0^n, u_1^n) ?

Example of non linearizable data : isotropic concentration

Let $(U_0, U_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$, $(U_0, U_1) \neq (0,0)$. Set

$$u_0^\varepsilon(x) = \frac{1}{\varepsilon} U_0\left(\frac{x}{\varepsilon}\right) \ , \ u_1^\varepsilon(x) = \frac{1}{\varepsilon^2} U_1\left(\frac{x}{\varepsilon}\right) \ \varepsilon \to 0$$

is not linearizable.

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is not linearizable.

- Indeed, by scale invariance,

$$u^\varepsilon(t,x) = \frac{1}{\varepsilon} U\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \ , \ v^\varepsilon(t,x) = \frac{1}{\varepsilon} V\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$

  where U, V are respectively the solutions to the cubic wave equation and to the linear wave equation with Cauchy data (U_0, U_1) at $t = 0$.

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  where U, V are respectively the solutions to the cubic wave equation and to the linear wave equation with Cauchy data (U_0, U_1) at $t = 0$.

- Linearizability would imply $U = V$ hence $U = 0$!

Other examples of non linearizable data ?

Notice that, for the quintic wave equation

$$\Box u = \gamma u^5 \ ,$$

all non linearizable data are obtained from scaling, combined with space-time translations. More precisely, if $\Box v^n = 0$ with $(v^n(0), \partial_t v^n(0))$ bounded in $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, then

$$\|v_n\|_{L^8(\mathbb{R}\times\mathbb{R}^3)} \longrightarrow 0$$

if and only if

$$\forall (g^n) \text{ sequence of translations -- dilations, } \ g^n.v^n \rightharpoonup 0 \ .$$

(Bahouri–PG, 1999) (related to concentration compactness)

Concentration on a plane

Decompose $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^2$.
Consider the solution of $\Box v^\delta = 0$ with

$$v^\delta(0,x) = W_0\left(\frac{x_1}{\delta}, x'\right)\ ,\ \partial_t v^\delta(0,x) = \frac{1}{\delta} W_1\left(\frac{x_1}{\delta}, x'\right)\ ,\ \delta \to 0\ ,$$

with $(W_0, W_1) \in \dot{H}^{1/2}(\mathbb{R}_z \times \mathbb{R}^2_{x'}) \times \dot{H}^{-1/2}(\mathbb{R}_z \times \mathbb{R}^2_{x'})$ and

$$\int_{\mathbb{R}\times\mathbb{R}^3} |\zeta|^{-1} |\hat{W}_1(\zeta, \xi')|^2 \, d\zeta \, d\xi' < +\infty\ .$$

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- First ansatz

$$v^\delta(x) \simeq \sum_{\pm} W_\pm\left(\delta t, \frac{x_1 \mp t}{\delta}, x'\right) \ ,$$

$$\mp 2\partial_s \partial_z W_\pm - \Delta_{x'} W_\pm = 0 \ ,$$

$$W_+(0,.) + W_-(0,.) = W_0 \ , \ -\partial_z W_+(0,.) + \partial_z W_-(0,.) = W_1 \ .$$

Another ansatz, using Lorentz transformations

For $\beta \in]-1,1[$, define

$$\mathcal{L}_\beta F(t,x) := F\left(\frac{t-\beta x_1}{\sqrt{1-\beta^2}}, \frac{x_1-\beta t}{\sqrt{1-\beta^2}}, x'\right)$$

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- $\|\mathcal{L}_\beta F\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} = \|F\|_{L^4(\mathbb{R}\times\mathbb{R}^3)}$

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- $\|\mathcal{L}_\beta F\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} = \|F\|_{L^4(\mathbb{R}\times\mathbb{R}^3)}$
- $\Box v = 0 \Rightarrow \Box \mathcal{L}_\beta v = 0$ and $E_{1/2}(\mathcal{L}_\beta v) = E_{1/2}(v)$ with

$$E_{1/2}(v) := \|v(t,.)\|^2_{\dot{H}^{1/2}} + \|\partial_t v(t,.)\|^2_{\dot{H}^{-1/2}} \ .$$

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- Leads to a new ansatz with "Lorentz" profiles,

$$v^\delta = \sum_{\pm} \mathcal{L}_{\pm\sqrt{1-\delta^2}} V_\pm + o(1) \ ,$$

$$V_\pm(t,x) = W_\pm\left(\pm(x_1+t), \frac{x_1-t}{2}, x'\right) \ , \ \Box V_\pm = 0.$$

Proposition (nonlinear Lorentz profile)

Let V : $\Box V = 0$, $(V(0,.), \partial_t V(0,.)) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$ small enough.
There exists a unique U : $\Box U = \gamma U^3$, $U \in L^4(\mathbb{R} \times \mathbb{R}^3)$, $(U, \partial_t U) \in C(\mathbb{R}, \dot{H}^{1/2} \times \dot{H}^{-1/2})$, such that the solution of

$$\Box u_\beta = \gamma u_\beta^3 \ , \ (u_\beta(0,.), \partial_t u_\beta(0,.)) = (\mathcal{L}_\beta V(0,.), \partial_t \mathcal{L}_\beta V(0,.))$$

satisfies, as $|\beta| \to 1$,

$$u_\beta = \mathcal{L}_\beta U + o(1)$$

for the L^4 and $L^\infty(\mathbb{R}, \dot{H}^{1/2} \times \dot{H}^{-1/2})$ norms.

Basic ingredient: If $f \in L^{4/3}(\mathbb{R} \times \mathbb{R}^3)$ and $\Box w_\beta = f \in L^{4/3}$, $(\mathcal{L}_\beta w_\beta(0,.), \partial_t \mathcal{L}_\beta w_\beta(0,.)) = (0,0)$, show that , as $|\beta| \to 1$,

$$(w_\beta(0,.), \partial_t w_\beta(0,.)) \to (\Gamma_0(f), \Gamma_1(f)) \text{ in } \dot{H}^{1/2} \times \dot{H}^{-1/2} \ .$$

The nonlinear ansatz for concentration on a plane

Given W_0, W_1 with small norms, the solution of $\Box u^\delta = \gamma(u^\delta)^3$ with

$$(u^\delta(0,x), \partial_t u^\delta(0,x)) = \left(W_0\left(\frac{x_1}{\delta}, x'\right) \ , \ \frac{1}{\delta} W_1\left(\frac{x_1}{\delta}, x'\right) \right)$$

is given, as $\delta \to 0$, by

$$u^\delta = \sum_{\pm} \mathcal{L}_{\pm\sqrt{1-\delta^2}} U_\pm + o(1),$$

where $U_\pm$ is the nonlinear Lorentz profile associated to $V_\pm$. Hence these data are not linearizable if $(W_0, W_1) \neq (0,0)$.
Main observation: $\mathcal{L}_{\sqrt{1-\delta^2}} U_+$ and $\mathcal{L}_{-\sqrt{1-\delta^2}} U_-$ do not interact (concentration on two different hyperplane tangent to the wave cone).

Concentration on null rays

By combining the previous concentration on a plane with an isotropic concentration,

$$(u^{\delta,\varepsilon}(0,x), \partial_t u^{\delta,\varepsilon}(0,x)) = \left(\frac{1}{\varepsilon} W_0\left(\frac{x_1}{\varepsilon\delta}, \frac{x'}{\varepsilon}\right)\ ,\ \frac{1}{\varepsilon^2\delta}\, W_1\left(\frac{x_1}{\varepsilon\delta}, \frac{x'}{\varepsilon}\right)\right)$$

one describes linear and nonlinear solutions which concentrate on two generating lines $\{x_1 = \pm t, x' = 0\}$ of the wave cone as $\delta \to 0, \varepsilon \to 0$.

Once again such data are not linearizable if $(W_0, W_1) \neq (0,0)$.

The main result

Denote by $\mathcal{H}$ the Hilbert space of solutions v to $\Box v = 0$, with $(v(0,.), \partial_t v(0,.)) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$, equipped with the norm $\sqrt{E_{1/2}(v)}$. Let G be the group of isometries of $\mathcal{H}$ generated by space-time translations, dilations, Lorentz transformations $\mathcal{L}_\beta$, rotations and symmetries in the x variable.

Theorem

Given a bounded sequence (v^n) of $\mathcal{H}$, the following are equivalent :

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Given a bounded sequence (v^n) of $\mathcal{H}$, the following are equivalent :

- $\|v^n\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \to 0$.

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Theorem

Given a bounded sequence (v^n) of $\mathcal{H}$, the following are equivalent :

- $\|v^n\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \to 0$.
- *For every sequence (g^n) of G, $g^n.v^n \rightharpoonup 0$ weakly.*

Profile decompositions : abstract setting

Let $\mathcal{H}$ be a Hilbert space.

Let G be a group of isometries of $\mathcal{H}$ such that every sequence of G which is not weakly convergent to 0 has a subsequence which converges strongly in G.

Let us say that two sequences (g^n) ,$(\tilde{g}^n)$ are orthogonal if

$$(g^n)^* \tilde{g}^n \rightharpoonup 0 \text{ weakly } .$$

For every bounded sequence $\mathbf{v} = (v^n)$ of $\mathcal{H}$, we set

$$\eta(\mathbf{v}) := \sup\{\|V\| : \exists (n_k, g^k), g^k.v^{n_k} \rightharpoonup V\} .$$

Claim, roughly speaking : Every bounded sequence of $\mathcal{H}$ can be written, up to a subsequence, as an almost orthogonal superposition of sequences $g^n.V$, where $V \in \mathcal{H}$ and (g^n) is a sequence of G, with a remainder term r^n such that $\eta(\mathbf{r})$ is small.

Theorem (Schindler–Tintarev, 2002)

For every bounded sequence (v^n) *of* $\mathcal{H}$*, there exist elements* $(V_j)_{j\geq 1}$ *of* $\mathcal{H}$ *and pairwise orthogonal sequences* $((g_j^n))_{j\geq 1}$ *of* G*, such that, up to a subsequence, for every* $\ell \geq 1$*,*

$$v^n = \sum_{j=1}^{\ell} g_j^n . V_j + r_\ell^n \ ,$$

$$\|v^n\|^2 = \sum_{j=1}^{\ell} \|V_j\|^2 + \|r_\ell^n\|^2 + o(1) \ , \ n \to \infty \ .$$

and $\eta(\mathbf{r}_\ell) \to 0$ *as* $\ell \to \infty$*.*

Problem. Smallness of r_ℓ^n for some continuous norm $v \mapsto N(v)$ on $\mathcal{H}$?

Previous results in the literature.

- $\mathcal{H} = \dot{H}^s(\mathbb{R}^d)$, $0 < s < d/2$,
  G = translations–dilations :
  $N(v) = \|u\|_{L^p(\mathbb{R}^d)}$, $p = 2d/(d-2s)$.
  (PG, 1998, $\simeq$ concentration-compactness).
- $\mathcal{H} = \{v : \Box v = 0, (v(0), \partial_t v(0)) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\}$,
  G = space-time translations–dilations :
  $N(v) = \|v\|_{L^8(\mathbb{R}\times\mathbb{R}^3)}$. (Bahouri-PG, 1999) and similar results by Keraani (2000) for the Schrödinger equation.
- $\mathcal{H} = \{v : i\partial_t v + \Delta v = 0, v(0) \in L^2(\mathbb{R}^2)\}$,
  G = space-time translations-dilations-Galilean transformations :
  $N(v) = \|v\|_{L^4(\mathbb{R}\times\mathbb{R}^2)}$. (Bourgain, 1998, Merle-Vega, 1998, Carles-Keraani and Begout-Vargas (other dimensions)).

Corollary (sequences of $\dot H^{1/2}$ solutions to $\Box u = \gamma u^3$)

Every sequence (v^n) : $\Box v^n = 0$, $\sup_n E_{1/2}(v^n) < \infty$, *can be written, , up to a subsequence, with* $\Box V_j = 0$,

$$\forall \ell \geq 1,\ \ v^n = \sum_{j=1}^{\ell} g_j^n . V_j + r_\ell^n \ ,\ \limsup_{n\to\infty} \|r_\ell^n\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \underset{\ell\to\infty}{\longrightarrow} 0\ .$$

If $\Box u^n = \gamma (u^n)^3,\ (u^n(0), \partial_t u^n(0)) = (v^n(0), \partial_t v^n(0))$, *then*

$$\forall \ell \geq 1,\ \ u^n = \sum_{j=1}^{\ell} g_j^n . U_j + r_\ell^n + \rho_\ell^n \ ,$$

where $\Box U_j = \gamma U_j^3$ *(nonlinear profile associated to* V_j*), and*

$$\limsup_{n\to\infty} \|(\rho_\ell^n, \partial_t \rho_\ell^n)\|_{L^\infty(\mathbb{R}, \dot H^{1/2}\times \dot H^{-1/2})} + \|\rho_\ell^n\|_{L^4} \underset{\ell\to\infty}{\longrightarrow} 0\ .$$

Consequence :

$$(v^n(0), \partial_t v^n(0)) \textit{linearizable} \Longleftrightarrow \forall j, V_j = 0 \Longleftrightarrow \|v^n\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \underset{n\to\infty}{\longrightarrow} 0$$

Consequence :

$$(v^n(0), \partial_t v^n(0)) \mathit{linearizable} \iff \forall j, V_j = 0 \iff \|v^n\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \underset{n\to\infty}{\longrightarrow} 0$$

- Corollary (a test using H–measures.)

Assume (v_0^n, v_1^n) *supported in a fixed compact subset of* $\mathbb{R}^3$ *and for every pseudodifferential operator* A *of order* 0,

$$(A|D|^{1/2}v_0^n, |D|^{1/2}v_0^n)_{L^2} + (A|D|^{-1/2}v_1^n, |D|^{-1/2}v_1^n)_{L^2} \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^3\times S^2} \sigma_0(A)(x,\xi)\, d\mu(x,\xi) \ ,$$

then (v_0^n, v_1^n) *is linearizable if the measure* μ *is singular to the following measures :*

$$\delta(x-x_0)\, d\sigma(\xi) \ , \ \delta(|x-x_0|-R)\,\delta\left(\xi \mp \frac{x-x_0}{R}\right) \ ,$$

$$\delta((x-x_0).\xi_0)\,\delta(\xi\mp\xi_0) \ , \ \delta(x-x_0)\,\delta(\xi-\xi_0) \ .$$

Sketch of proof (adaptation of arguments by Bourgain, 1998 for Schrödinger).

Let $v : i\partial_t v + \sqrt{-\Delta} v = 0,\ v(0) = v_0$.

We shall use three refinements of the Strichartz inequality on v which correspond to

- Dilations
- Lorentz transformations
- Space-time translations

Step 1. Extracting dilations

$$\|v\|^4_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \sum_{p\in\mathbb{Z}} \left(\int_{2^p\leq|\xi|\leq 2^{p+1}} |\xi| |\hat{v}_0(\xi)|^2 \, d\xi \right)^2$$

$$\lesssim \|v_0\|^2_{\dot{H}^{1/2}} \left(\sup_{p\in\mathbb{Z}} \int_{2^p\leq|\xi|\leq 2^{p+1}} |\xi| |\hat{v}_0(\xi)|^2 \, d\xi \right) .$$

(see Klainerman–Machedon, 1993-. . .)

Step 2. Extracting Lorentz parameters

Assume $\hat{v}_0$ supported in $\{1 \leq |\xi| \leq 2\}$.
Let $\delta \in \{2^{-j}, j \in \mathbb{Z}_+\}$ and $\mathcal{C}_\delta$ be a collection of tubes

$$T(\omega, \delta) = \left\{\xi \in \mathbb{R}^3 : 1 \leq |\xi| \leq 2, \left|\frac{\xi}{|\xi|} - \omega\right| \leq \delta\right\}, \ \omega \in \mathbb{S}^2,$$

which covers $\{1 \leq |\xi| \leq 2\}$ with bounded overlapping. Then, with $p = 8/5$, $\theta = 1/20$,

$$\|v\|^4_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \sum_\delta \sum_{T_\delta \in \mathcal{C}_\delta} \left(\frac{1}{|T_\delta|^{\frac{1}{p}-\frac{1}{2}}} \|\hat{v}_0\|_{L^p(T_\delta)}\right)^4$$

$$\lesssim \|\hat{v}_0\|^{4(1-\theta)}_{L^2} \left(\sup_\delta \sup_{T_\delta} \frac{1}{|T_\delta|^{1/2}} \int_{T_\delta} |\hat{v}_0| \, d\xi\right)^{4\theta}$$

see Moyua-Vargas-Vega, 1999, for the case of $2D$ Schrödinger ($\mathcal{C}_\delta$ is a collection of squares of size δ).

Step 3. Extracting space-time translations

Assume $\hat{v}_0$ supported in $\{1 \leq |\xi| \leq 2\}$, and that

$$\|\hat{v}_0\|_{L^\infty} \simeq \|\hat{v}_0\|_{L^2}$$

Apply Wolff's restriction theorem (2001)

$$\forall p > 3, \; \|v\|_{L^p(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\hat{v}_0\|_{L^p(\mathbb{R}^3)} \; .$$

therefore, for some $\theta \in]0,1[$,

$$\|v\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\hat{v}_0\|^{\theta}_{L^p(\mathbb{R}^3)} \|v\|^{1-\theta}_{L^\infty(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\hat{v}_0\|^{\theta}_{L^2(\mathbb{R}^3)} \|v\|^{1-\theta}_{L^\infty(\mathbb{R}\times\mathbb{R}^3)} \; .$$

Then observe that, for a bounded sequence (v^n) of such v,

$$\limsup_{n\to\infty} \|v^n\|_{L^\infty(\mathbb{R}\times\mathbb{R}^3)} \lesssim$$
$$\sup\{\|V(0,.)\|_{L^2(\mathbb{R}^3)} : \exists (n_k, t_k, x_k), v^{n_k}(t+t_k, x+x_k) \rightharpoonup V\} \; .$$