Wave equation, Strichartz inequality and Lorentz transformations

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Des équations aux dérivées partielles au calcul scientifique
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The cubic wave equation

\[ \square := \frac{\partial^2}{\partial t^2} - \Delta_x , \ t \in \mathbb{R}, x \in \mathbb{R}^3 \]

Study the (real-valued) solutions \( u \) of

\[ \square u = \gamma u^3 , \ u(0, x) = u_0(x), \ \frac{\partial u}{\partial t}(0, x) = u_1(x) \]

where \( \gamma \in \{-1, 1\} \).

Recall:

If \( \gamma = -1 \) : global existence of solutions with sufficiently smooth data \( u_0, u_1 \) (say \( u_0 \in H^1(\mathbb{R}^3), u_1 \in L^2(\mathbb{R}^3) \)).

If \( \gamma = 1 \) : existence of solutions which blow up in finite time, except if the data are sufficiently small.
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If \( \gamma = 1 \): existence of solutions which blow up in finite time, except if the data are sufficiently small.

- Problem. Description of nonlinear effects in the high frequency limit?
The critical regularity

Scale invariance $u^\lambda(t, x) = \lambda u(\lambda t, \lambda x)$,

$u_0^\lambda(x) = \lambda u_0(\lambda x), \ u_1^\lambda(x) = \lambda^2 u_1(\lambda x)$.

$\rightarrow$ Critical regularity of the data:

$u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3), \ u_1 \in \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3),$

$\dot{H}^s(\mathbb{R}^d) := \{ u \in \mathcal{S}'(\mathbb{R}^d) : \hat{u}(\xi) \in |\xi|^{-s} L^2(\mathbb{R}^d) \}, \ |s| < \frac{d}{2}$.

If $s < \frac{1}{2}$: even for $\gamma = -1$ (defocusing case),

$\exists$ arbitrary small data in $\dot{H}^s \times \dot{H}^{s-1}$ such that $(u(t), \partial_t u(t))$ is large in $\dot{H}^s \times \dot{H}^{s-1}$ for arbitrarily small time $t$

The Strichartz inequality (Strichartz, 1977)

If \( \square v = 0 \), \( v(0, .) = v_0 \), \( \partial_t v(0, .) = v_1 \), then

\[
\|v\|_{L^4(\mathbb{R} \times \mathbb{R}^3)} \lesssim \|(v_0, v_1)\|_{\dot{H}^{1/2}(\mathbb{R}^3) \times \dot{H}^{-1/2}(\mathbb{R}^3)}.
\]

Inhomogeneous version: if \( \square w = f \), \( w(0, .) = \partial_t w(0, .) = 0 \),

\[
\|w\|_{L^4(\mathbb{R} \times \mathbb{R}^3)} + \sup_{t \in \mathbb{R}} \|(w(t, .), \partial_t w(t, .))\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}} \lesssim \|f\|_{L^{4/3}(\mathbb{R} \times \mathbb{R}^3)}.
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Inhomogeneous version: if $\Box w = f$, $w(0, \cdot) = \partial_t w(0, \cdot) = 0$,

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Consequence $\exists \alpha > 0$ such that, if

$$\|(u_0, u_1)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}} < \alpha,$$

then the Cauchy problem

$$\Box u = \gamma u^3, \ u(0, x) = u_0(x), \ \frac{\partial u}{\partial t}(0, x) = u_1(x)$$

admits a unique solution

$$u \in L^4(\mathbb{R} \times \mathbb{R}^3) \cap C(\mathbb{R}, \dot{H}^{1/2}(\mathbb{R}^3)), \ \partial_t u \in C(\mathbb{R}, \dot{H}^{-1/2}(\mathbb{R}^3)).$$
Linearizable data

We fix $\gamma \in \{-1, 1\}$.

**Definition**

Let $(u^0_n, u^1_n)$ be a sequence in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$, weakly convergent to 0, with $\|(u^0_n, u^1_n)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}} < \alpha$.

We shall say that $(u^0_n, u^1_n)$ is linearizable if the solution $u^n$ to the cubic wave equation $\square u^n = \gamma (u^n)^3$ with Cauchy data $(u^n_0, u^n_1)$ at $t = 0$ satisfies

$$
\|(u^n - v^n, \partial_t (u^n - v^n))\|_{L^\infty(\mathbb{R}, \dot{H}^{1/2} \times \dot{H}^{-1/2})} + \|u^n - v^n\|_{L^4(\mathbb{R} \times \mathbb{R}^3)} \to 0,
$$

where $v^n$ is the solution to the linear wave equation

$$
\square v^n = 0
$$

with the same Cauchy data $(u^n_0, u^n_1)$ at $t = 0$. 
Problem. Describe the non linearizable data \((u_0^n, u_1^n)\), and the corresponding solutions \(u^n\).

Remark. Sufficient condition to linearizability:

\[
\|v^n\|_{L^4(\mathbb{R} \times \mathbb{R}^3)} \longrightarrow 0.
\]

Is it necessary?
How to guess it on Cauchy data \((u_0^n, u_1^n)\)?
Example of non linearizable data: isotropic concentration

Let \((U_0, U_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}, (U_0, U_1) \neq (0, 0)\). Set

\[
 u_0^\varepsilon(x) = \frac{1}{\varepsilon} U_0 \left( \frac{x}{\varepsilon} \right), \quad u_1^\varepsilon(x) = \frac{1}{\varepsilon^2} U_1 \left( \frac{x}{\varepsilon} \right) \varepsilon \to 0
\]

is not linearizable.
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  u_1^\varepsilon(x) &= \frac{1}{\varepsilon^2} U_1 \left( \frac{x}{\varepsilon} \right) \varepsilon \to 0
\end{align*}
\]

is not linearizable.

- Indeed, by scale invariance,

\[
\begin{align*}
  u^\varepsilon(t, x) &= \frac{1}{\varepsilon} U \left( \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right), \\
  v^\varepsilon(t, x) &= \frac{1}{\varepsilon} V \left( \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right)
\end{align*}
\]

where \(U, V\) are respectively the solutions to the cubic wave equation and to the linear wave equation with Cauchy data \((U_0, U_1)\) at \(t = 0\).
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Linearizability would imply \(U = V\) hence \(U = 0\)!
Other examples of non linearizable data?
Notice that, for the quintic wave equation

\[ \Box u = \gamma u^5, \]

all non linearizable data are obtained from scaling, combined with space-time translations. More precisely, if \( \Box v^n = 0 \) with \((v^n(0), \partial_t v^n(0))\) bounded in \( \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \), then

\[ \| v_n \|_{L^8(\mathbb{R} \times \mathbb{R}^3)} \to 0 \]

if and only if

\[ \forall (g^n) \text{ sequence of translations} - \text{dilations}, \ g^n \cdot v^n \to 0. \]

(Bahouri–PG, 1999) (related to concentration compactness)
Concentration on a plane

Decompose \( x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^2 \).

Consider the solution of \( \Box v^\delta = 0 \) with

\[
v^\delta(0, x) = W_0 \left( \frac{x_1}{\delta}, x' \right), \quad \partial_t v^\delta(0, x) = \frac{1}{\delta} W_1 \left( \frac{x_1}{\delta}, x' \right), \quad \delta \to 0,
\]

with \((W_0, W_1) \in \dot{H}^{1/2}(\mathbb{R}_z \times \mathbb{R}_{x'}) \times \dot{H}^{-1/2}(\mathbb{R}_z \times \mathbb{R}_{x'})\) and

\[
\int_{\mathbb{R} \times \mathbb{R}^3} |\zeta|^{-1} |\hat{W}_1(\zeta, \xi')|^2 \, d\zeta \, d\xi' < +\infty.
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\int_{\mathbb{R} \times \mathbb{R}^3} |\zeta|^{-1} |\hat{W}_1(\zeta, \xi')|^2 \, d\zeta \, d\xi' < +\infty.
\]

First ansatz

\[
v^\delta(x) \simeq \sum_{\pm} W_\pm \left( \frac{x_1 \mp t}{\delta}, x' \right),
\]

\[
\mp 2 \partial_s \partial_z W_\pm - \Delta_{x'} W_\pm = 0,
\]

\[
W_+(0, .) + W_-(0, .) = W_0, \quad -\partial_z W_+(0, .) + \partial_z W_-(0, .) = W_1.
\]
Another ansatz, using Lorentz transformations

For $\beta \in ]-1, 1[$, define

$$\mathcal{L}_\beta F(t, x) := F\left( \frac{t - \beta x_1}{\sqrt{1 - \beta^2}}, \frac{x_1 - \beta t}{\sqrt{1 - \beta^2}}, x' \right)$$
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$$\| \mathcal{L}_\beta F \|_{L^4(\mathbb{R} \times \mathbb{R}^3)} = \| F \|_{L^4(\mathbb{R} \times \mathbb{R}^3)}$$
Another ansatz, using Lorentz transformations

For $\beta \in ]-1, 1[$, define

$$L_\beta F(t, x) := F\left(\frac{t - \beta x_1}{\sqrt{1 - \beta^2}}, \frac{x_1 - \beta t}{\sqrt{1 - \beta^2}}, x'\right)$$

- $\|L_\beta F\|_{L^4(\mathbb{R} \times \mathbb{R}^3)} = \|F\|_{L^4(\mathbb{R} \times \mathbb{R}^3)}$
- $\Box v = 0 \Rightarrow \Box L_\beta v = 0$ and $E_{1/2}(L_\beta v) = E_{1/2}(v)$ with

$$E_{1/2}(v) := \|v(t, .)\|_{H^{1/2}}^2 + \|\partial_t v(t, .)\|_{H^{-1/2}}^2.$$
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$$E_{1/2}(v) := \|v(t, .)\|_{H^{1/2}}^2 + \|\partial_t v(t, .)\|_{H^{-1/2}}^2.$$

- Leads to a new ansatz with “Lorentz” profiles,

$$v^\delta = \sum_{\pm} L_{\pm \sqrt{1 - \delta^2}} V_\pm + o(1),$$

$$V_\pm(t, x) = W_\pm \left(\pm(x_1 + t), \frac{x_1 - t}{2}, x'\right), \quad \square V_\pm = 0.$$
Proposition (nonlinear Lorentz profile)

Let \( V : \Box V = 0, (V(0, .), \partial_t V(0, .)) \in \dot{H}^{1/2} \times \dot{H}^{-1/2} \) small enough.

There exists a unique \( U : \Box U = \gamma U^3, U \in L^4(\mathbb{R} \times \mathbb{R}^3), (U, \partial_t U) \in C(\mathbb{R}, \dot{H}^{1/2} \times \dot{H}^{-1/2}) \), such that the solution of

\[
\Box u_\beta = \gamma u_\beta^3, \quad (u_\beta(0, .), \partial_t u_\beta(0, .)) = (\mathcal{L}_\beta V(0, .), \partial_t \mathcal{L}_\beta V(0, .))
\]

satisfies, as \( |\beta| \to 1 \),

\[
u_\beta = \mathcal{L}_\beta U + o(1)
\]

for the \( L^4 \) and \( L^\infty(\mathbb{R}, \dot{H}^{1/2} \times \dot{H}^{-1/2}) \) norms.

Basic ingredient: If \( f \in L^{4/3}(\mathbb{R} \times \mathbb{R}^3) \) and \( \Box w_\beta = f \in L^{4/3}, (\mathcal{L}_\beta w_\beta(0, .), \partial_t \mathcal{L}_\beta w_\beta(0, .)) = (0, 0) \), show that, as \( |\beta| \to 1 \),

\[
(w_\beta(0, .), \partial_t w_\beta(0, .)) \to (\Gamma_0(f), \Gamma_1(f)) \quad \text{in} \quad \dot{H}^{1/2} \times \dot{H}^{-1/2}.
\]
The nonlinear ansatz for concentration on a plane

Given $W_0, W_1$ with small norms, the solution of $\Box u^\delta = \gamma (u^\delta)^3$ with

$$ (u^\delta(0, x), \partial_t u^\delta(0, x)) = \left( W_0 \left( \frac{x_1}{\delta}, x' \right), \frac{1}{\delta} W_1 \left( \frac{x_1}{\delta}, x' \right) \right) $$

is given, as $\delta \to 0$, by

$$ u^\delta = \sum_{\pm} \mathcal{L}_{\pm\sqrt{1-\delta^2}} U_\pm + o(1), $$

where $U_\pm$ is the nonlinear Lorentz profile associated to $V_\pm$. Hence these data are not linearizable if $(W_0, W_1) \neq (0, 0)$.

Main observation: $\mathcal{L}_{\sqrt{1-\delta^2}} U_+$ and $\mathcal{L}_{-\sqrt{1-\delta^2}} U_-$ do not interact (concentration on two different hyperplane tangent to the wave cone).
Concentration on null rays

By combining the previous concentration on a plane with an isotropic concentration,

\[(u^{\delta,\varepsilon}(0, x), \partial_t u^{\delta,\varepsilon}(0, x)) = \left( \frac{1}{\varepsilon} W_0 \left( \frac{x_1}{\varepsilon \delta}, \frac{x'}{\varepsilon} \right), \frac{1}{\varepsilon^2 \delta} W_1 \left( \frac{x_1}{\varepsilon \delta}, \frac{x'}{\varepsilon} \right) \right) \]

one describes linear and nonlinear solutions which concentrate on two generating lines \( \{x_1 = \pm t, x' = 0\} \) of the wave cone as \( \delta \to 0, \varepsilon \to 0 \).

Once again such data are not linearizable if \((W_0, W_1) \neq (0, 0)\).
The main result

Denote by $\mathcal{H}$ the Hilbert space of solutions $v$ to $\Box v = 0$, with $(v(0, .), \partial_t v(0, .)) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$, equipped with the norm $\sqrt{E_{1/2}(v)}$. Let $G$ be the group of isometries of $\mathcal{H}$ generated by space-time translations, dilations, Lorentz transformations $L_\beta$, rotations and symmetries in the $x$ variable.

Theorem

Given a bounded sequence $(v^n)$ of $\mathcal{H}$, the following are equivalent:

1. $\|v^n\|_{L^4(\mathbb{R} \times \mathbb{R}^3)} \to 0$.
2. For every sequence $(g_n)$ of $G$, $g_n \cdot v^n \rightharpoonup 0$ weakly.
The main result

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Theorem

*Given a bounded sequence $(v^n)$ of $\mathcal{H}$, the following are equivalent:

- $\|v^n\|_{L^4(\mathbb{R} \times \mathbb{R}^3)} \to 0$.
- For every sequence $(g^n)$ of $G$, $g^n \cdot v^n \rightharpoonup 0$ weakly.*
Profile decompositions : abstract setting

Let $\mathcal{H}$ be a Hilbert space.
Let $G$ be a group of isometries of $\mathcal{H}$ such that every sequence of $G$ which is not weakly convergent to 0 has a subsequence which converges strongly in $G$.
Let us say that two sequences $(g^n),(\tilde{g}^n)$ are orthogonal if

$$(g^n)^* \tilde{g}^n \rightharpoonup 0 \text{ weakly}.$$ 

For every bounded sequence $v = (v^n)$ of $\mathcal{H}$, we set

$$\eta(v) := \sup\{\| V \| : \exists (n_k, g^k), g^k . v^{n_k} \rightharpoonup V \}.$$ 

Claim, roughly speaking : Every bounded sequence of $\mathcal{H}$ can be written, up to a subsequence, as an almost orthogonal superposition of sequences $g^n . V$, where $V \in \mathcal{H}$ and $(g^n)$ is a sequence of $G$, with a remainder term $r^n$ such that $\eta(r)$ is small.
Theorem (Schindler–Tintarev, 2002)

For every bounded sequence \((v^n)\) of \(\mathcal{H}\), there exist elements \((V_j)_{j \geq 1}\) of \(\mathcal{H}\) and pairwise orthogonal sequences \(((g_j^n))_{j \geq 1}\) of \(G\), such that, up to a subsequence, for every \(\ell \geq 1\),

\[
v^n = \sum_{j=1}^{\ell} g_j^n \cdot V_j + r^n_\ell,
\]

\[
\|v^n\|^2 = \sum_{j=1}^{\ell} \|V_j\|^2 + \|r^n_\ell\|^2 + o(1), \quad n \to \infty.
\]

and \(\eta(r^n_\ell) \to 0\) as \(\ell \to \infty\).

Problem. Smallness of \(r^n_\ell\) for some continuous norm \(v \mapsto N(v)\) on \(\mathcal{H}\)?
Previous results in the literature.

- \( \mathcal{H} = \dot{H}^s(\mathbb{R}^d), 0 < s < d/2, \)
  \( G = \text{translations–dilations} : \)
  \( N(v) = \| u \|_{L^p(\mathbb{R}^d)}, \ p = 2d/(d - 2s). \)
  (PG, 1998, \( \simeq \) concentration-compactness).

- \( \mathcal{H} = \{ v : \Box v = 0, (v(0), \partial_t v(0)) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \} , \)
  \( G = \text{space-time translations–dilations} : \)
  \( N(v) = \| v \|_{L^8(\mathbb{R} \times \mathbb{R}^3)}. \) (Bahouri-PG, 1999) and similar results by Keraani (2000) for the Schrödinger equation.

- \( \mathcal{H} = \{ v : i\partial_t v + \Delta v = 0, v(0) \in L^2(\mathbb{R}^2) \} , \)
  \( G = \text{space-time translations-dilations-Galilean transformations} : \)
  \( N(v) = \| v \|_{L^4(\mathbb{R} \times \mathbb{R}^2)}. \) (Bourgain, 1998, Merle-Vega, 1998, Carles-Keraani and Begout-Vargas (other dimensions)).
Corollary (sequences of $H^{1/2}$ solutions to $\Box u = \gamma u^3$)

Every sequence $(v^n) : \Box v^n = 0$, $\sup_n E_{1/2}(v^n) < \infty$, can be written, up to a subsequence, with $\Box V_j = 0$,

$$\forall \ell \geq 1, \ v^n = \sum_{j=1}^\ell g^n_j \cdot V_j + r^n_\ell, \ \limsup_{n \to \infty} \| r^n_\ell \|_{L^4(\mathbb{R} \times \mathbb{R}^3)} \rightarrow 0.$$ 

If $\Box u^n = \gamma (u^n)^3$, $(u^n(0), \partial_t u^n(0)) = (v^n(0), \partial_t v^n(0))$, then

$$\forall \ell \geq 1, \ u^n = \sum_{j=1}^\ell g^n_j \cdot U_j + r^n_\ell + \rho^n_\ell,$$

where $\Box U_j = \gamma U_j^3$ (nonlinear profile associated to $V_j$), and

$$\limsup_{n \to \infty} \|(\rho^n_\ell, \partial_t \rho^n_\ell)\|_{L^\infty(\mathbb{R}, \dot{H}^{1/2} \times \dot{H}^{-1/2})} + \| \rho^n_\ell \|_{L^4} \rightarrow 0.$$
Consequence:

\((v^n(0), \partial_t v^n(0))\) linearizable \iff \forall j, V_j = 0 \iff \|v^n\|_{L^4(\mathbb{R} \times \mathbb{R}^3)} \xrightarrow{n \to \infty} 0\)
Consequence:

\[(v^n(0), \partial_t v^n(0))\text{linearizable } \iff \forall j, V_j = 0 \iff \|v^n\|_{L^4(\mathbb{R} \times \mathbb{R}^3)} \to 0\]

**Corollary (a test using \(H\)-measures.)**

Assume \((v^n_0, v^n_1)\) supported in a fixed compact subset of \(\mathbb{R}^3\) and for every pseudodifferential operator \(A\) of order 0,

\[
(A|D|^{1/2}v^n_0, |D|^{1/2}v^n_0)_{L^2} + (A|D|^{-1/2}v^n_1, |D|^{-1/2}v^n_1)_{L^2} \to_{n \to \infty} \int_{\mathbb{R}^3 \times S^2} \sigma_0(A)(x, \xi) \, d\mu(x, \xi) ,
\]

then \((v^n_0, v^n_1)\) is linearizable if the measure \(\mu\) is singular to the following measures:

\[
\delta(x - x_0) \, d\sigma(\xi) , \delta(|x - x_0| - R) \delta \left( \xi \mp \frac{x - x_0}{R} \right) ,
\]

\[
\delta((x - x_0) \cdot \xi_0) \delta(\xi \mp \xi_0) , \delta(x - x_0) \delta(\xi - \xi_0) .
\]
Sketch of proof (adaptation of arguments by Bourgain, 1998 for Schrödinger).

Let $v : i\partial_t v + \sqrt{-\Delta} v = 0, \ v(0) = v_0$.

We shall use three refinements of the Strichartz inequality on $v$ which correspond to

- Dilations
- Lorentz transformations
- Space-time translations
Step 1. Extracting dilations

\[ \| v \|_{L^4(\mathbb{R} \times \mathbb{R}^3)}^4 \lesssim \sum_{p \in \mathbb{Z}} \left( \int_{2^p \leq |\xi| \leq 2^{p+1}} |\xi| |\hat{v}_0(\xi)|^2 \, d\xi \right)^2 \]

\[ \lesssim \| v_0 \|_{\dot{H}^{1/2}}^2 \left( \sup_{p \in \mathbb{Z}} \int_{2^p \leq |\xi| \leq 2^{p+1}} |\xi| |\hat{v}_0(\xi)|^2 \, d\xi \right) . \]

(see Klainerman–Machedon, 1993–...)
Step 2. Extracting Lorentz parameters

Assume $\hat{v}_0$ supported in $\{1 \leq |\xi| \leq 2\}$. Let $\delta \in \{2^{-j}, j \in \mathbb{Z}_+\}$ and $C_\delta$ be a collection of tubes

$$T(\omega, \delta) = \{\xi \in \mathbb{R}^3 : 1 \leq |\xi| \leq 2, \left|\frac{\xi}{|\xi|} - \omega\right| \leq \delta\}, \quad \omega \in S^2,$$

which covers $\{1 \leq |\xi| \leq 2\}$ with bounded overlapping. Then, with $p = 8/5$, $\theta = 1/20$,

$$\|v\|^4_{L^4(\mathbb{R} \times \mathbb{R}^3)} \lesssim \sum_\delta \sum_{T_\delta \in C_\delta} \left(\frac{1}{|T_\delta|^{1/p - 1/2}} \|\hat{v}_0\|_{L^p(T_\delta)}\right)^4 \lesssim \|\hat{v}_0\|^{4(1-\theta)}_{L^2} \left(\sup_\delta \sup_{T_\delta} \frac{1}{|T_\delta|^{1/2}} \int_{T_\delta} |\hat{v}_0| \, d\xi\right)^{4\theta}$$

see Moyua-Vargas-Vega, 1999, for the case of 2D Schrödinger ($C_\delta$ is a collection of squares of size $\delta$).
Step 3. Extracting space-time translations

Assume \( \hat{v}_0 \) supported in \( \{1 \leq |\xi| \leq 2\} \), and that

\[
\| \hat{v}_0 \|_{L^\infty} \simeq \| \hat{v}_0 \|_{L^2}
\]

Apply Wolff’s restriction theorem (2001)

\[
\forall p > 3, \quad \| v \|_{L^p(\mathbb{R} \times \mathbb{R}^3)} \lesssim \| \hat{v}_0 \|_{L^p(\mathbb{R}^3)}.
\]

therefore, for some \( \theta \in ]0, 1[ \),

\[
\| v \|_{L^4(\mathbb{R} \times \mathbb{R}^3)} \lesssim \| \hat{v}_0 \|_{L^p(\mathbb{R}^3)} \| v \|_{L^\infty(\mathbb{R} \times \mathbb{R}^3)}^{1-\theta} \lesssim \| \hat{v}_0 \|_{L^2(\mathbb{R}^3)} \| v \|_{L^\infty(\mathbb{R} \times \mathbb{R}^3)}^{1-\theta}.
\]

Then observe that, for a bounded sequence \((v^n)\) of such \(v\),

\[
\limsup_{n \to \infty} \| v^n \|_{L^\infty(\mathbb{R} \times \mathbb{R}^3)} \lesssim \\
\sup \{ \| V(0, .) \|_{L^2(\mathbb{R}^3)} : \exists (n_k, t_k, x_k), v^{n_k}(t + t_k, x + x_k) \to V \}.
\]