Dispersive Systems

1) Schrödinger equation
2) Cubic Schrödinger
3) KdV
4) Discreterised hyperbolic equation
5) Discrete systems

KdV

$$u_t + uu_x + \varepsilon^2 u_{xxx} = 0$$

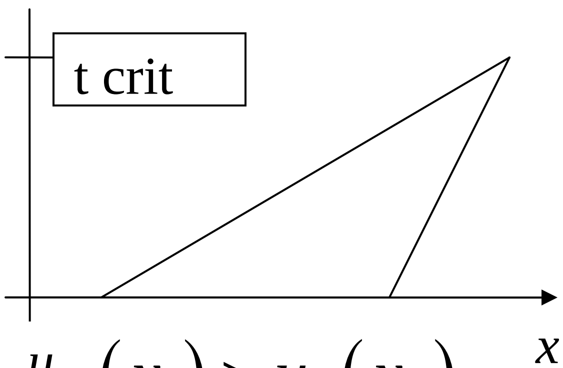
$$u(x,0) = u_0(x)$$

DISCONTINUITY

$$u_t + uu_x = 0$$

$$\frac{d}{dt}u = 0, \frac{dx}{dt} = u.$$

$$u_0(x) = u(x,0) \text{prescribed}$$



$$u_0\left(x_1\right) > u_0\left(x_x\right)$$

Collision of characteristic lines

No continuous solution for $t > t$ crit

Solutions of equ.-s of compressible flow develop discontinuities (shocks).
Stokes, Airy, Riemann:

Conservation laws

$$u_t + f_x + g_y + h_z = 0$$

$$f = f(u),\ g = g(u), etc.$$

Integrate over G:

$$\frac{d}{dt}\int_{\mathbf{G}} u\,dV + \int_{\partial\mathbf{G}} F\cdot n\,dS = 0,$$

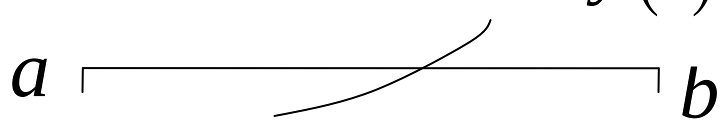
$F = \left(f, g, h\right)$ flux

Weak solutions, distributions

$$\mathrm{u}_{\mathrm{t}} + f_x = 0$$

$$\frac{d}{dt}\int_a^b u\,dx + f_b - f_a = 0.$$

$$y(t), \frac{dy}{dt} = s$$



$$\frac{d}{dt}\left[\int_a^y u\,dx + \int_y^b u\,dx\right] = \int_a^y u_t\,dx + \int_y^b u_t\,dx + s[u_l - u_r] =$$

$$-\int_a^y f_x\,dx - \int_y^b f_x\,dx + s[u_l - u_r] =$$

$$-f_l + f_a - f_b + f_r + s[u_l - u_r],$$

So by the conservation law,

$$\boxed{s = [f]/[u]}$$

Rankine-Hugoniot condition

Example:

$$u_t + uu_x = 0$$

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0$$

$$f(u) = \tfrac{1}{2}u^2$$

$$s = [f]/[u] = \frac{1}{2}\frac{u_e^2 - u_r^2}{u_e - u_r} = \frac{u_e + u_r}{2}$$

Entropy condition: $u_e > u_r$

$$u_t + uu_x = 0$$

$$\tfrac{1}{u} u_t + u_x = 0$$

$$(\log a)_t + u_x = 0$$

RH condition

$$s = \frac{u_e - u_r}{\log u_e - \log u_r}$$

$$\neq \frac{u_e + u_r}{2}$$

Multiply equation

$$2u \,\big|\, u_t + uu_x = 0$$

$$u_t^2 + \left(\tfrac{2}{3}u^3\right)_x = 0$$

R-H condition

$$s = \frac{2}{3}\frac{u_e^3 - u_r^3}{u_e^2 - u_r^2} = \frac{2}{3}\frac{u_e^3 + u_e u_r + u_r^2}{u_e + u_r}$$

Which conservation law??

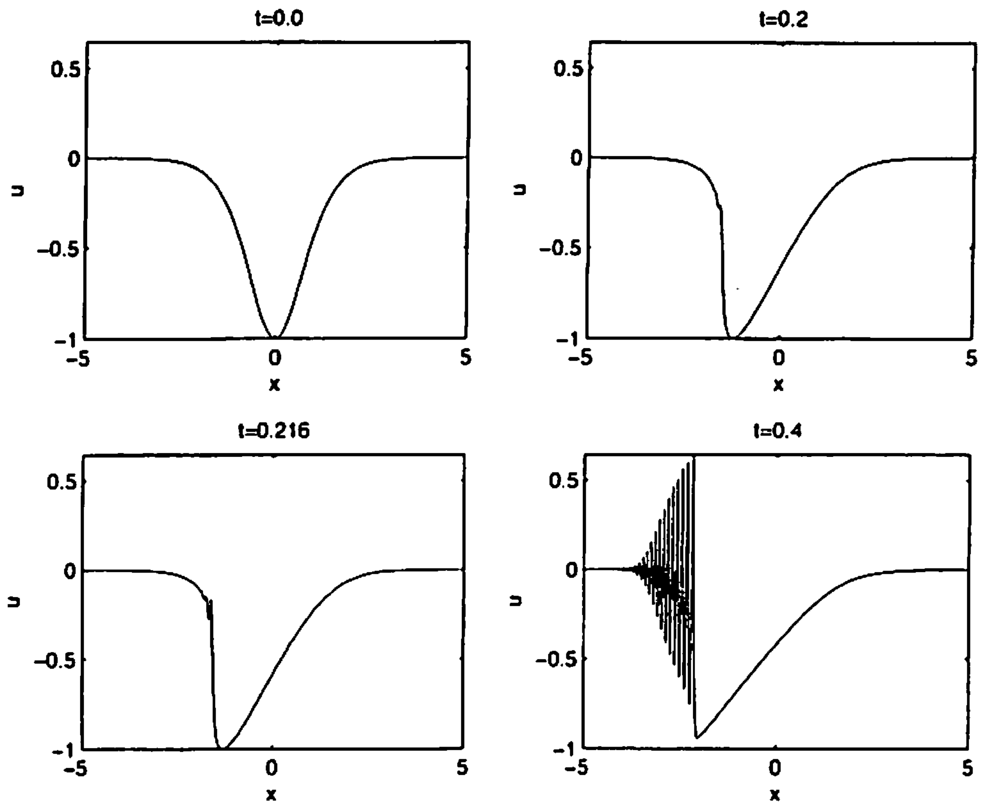


FIGURE 1. The numerical solution of the KdV equation at different times for the initial data $u_0(x) = -1/\cosh^2 x$ and $\epsilon = 10^{-1.5}$.

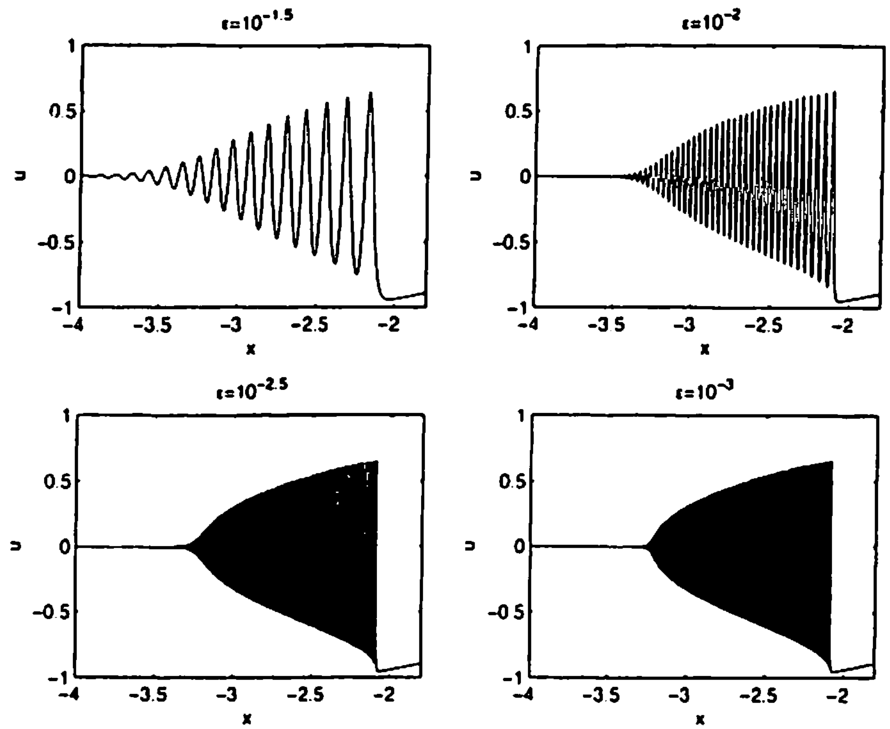


FIGURE 4. The numerical solution of the KdV equation for the initial data $u_0(x) = -1/\cosh^2 x$, for different values of ϵ and for fixed time $t = 0.4$.

Goodman-Lax; in

$u_t + uu_x = 0$ discretise space:

$u(k\Delta, t) \simeq u_k(t)$

Replace u_x by symmetric difference quotient:

$$\dot{u}_k + u_k \frac{u_{k+1} - u_{k-1}}{2\Delta} = 0$$

$$\bullet = \frac{d}{dt}.$$

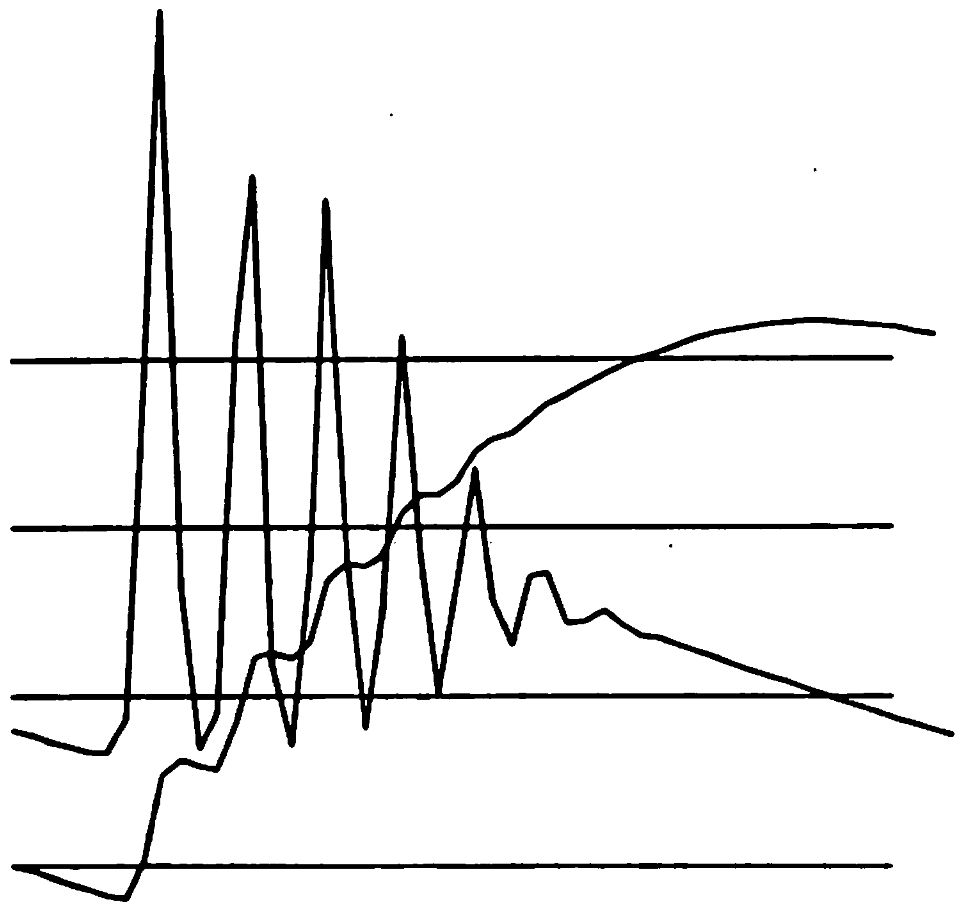
Dispersive Conservation form

$$\dot{u}_k + \frac{f_{k+1/2} - f_{k+1/2}}{\Delta} = 0,$$

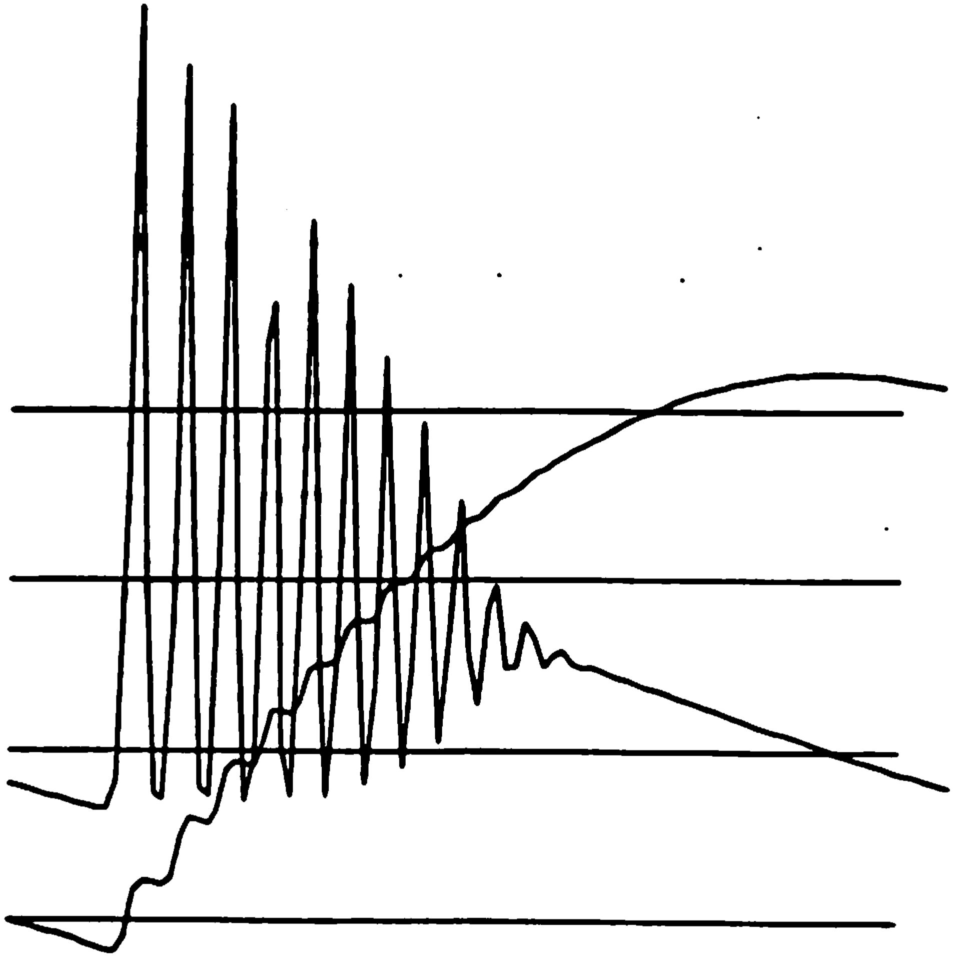
$$f_{k+1/2} = u_k u_{k+1/2}$$

$$u(x,0) = \frac{3}{2} + \cos x$$

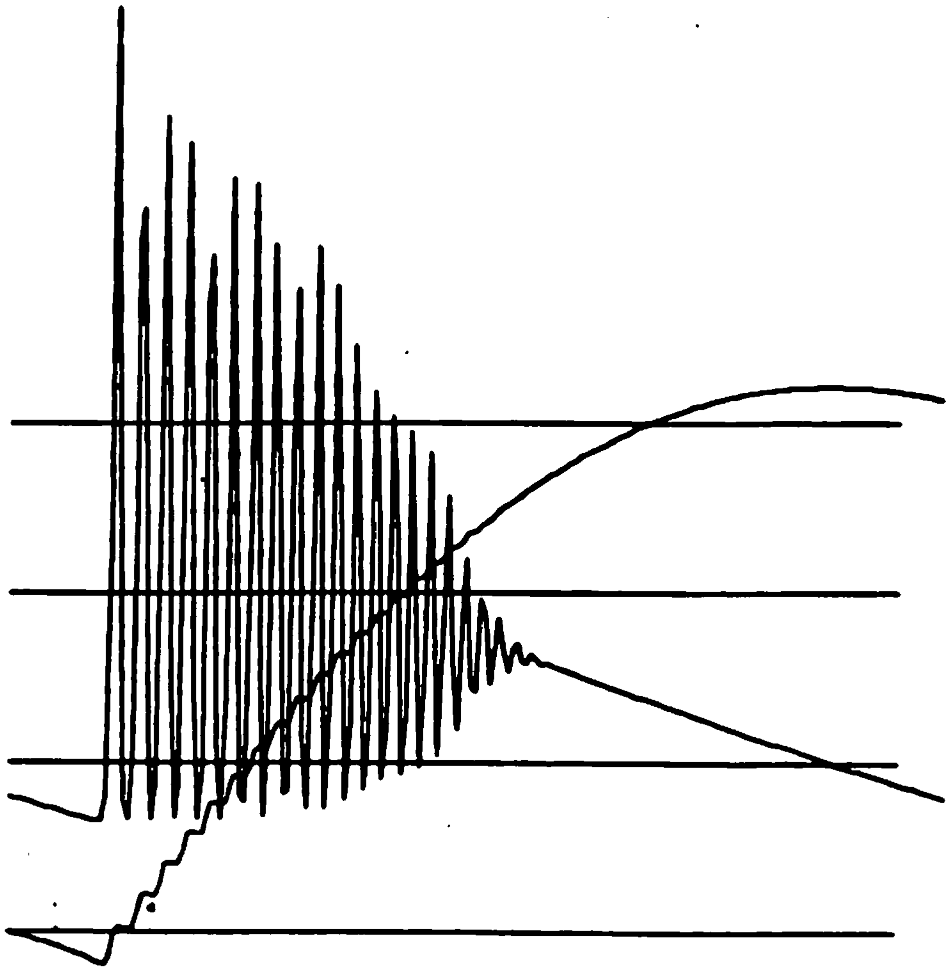
Numerical experiments reveal oscillations and weak convergence for $t > t$ crit



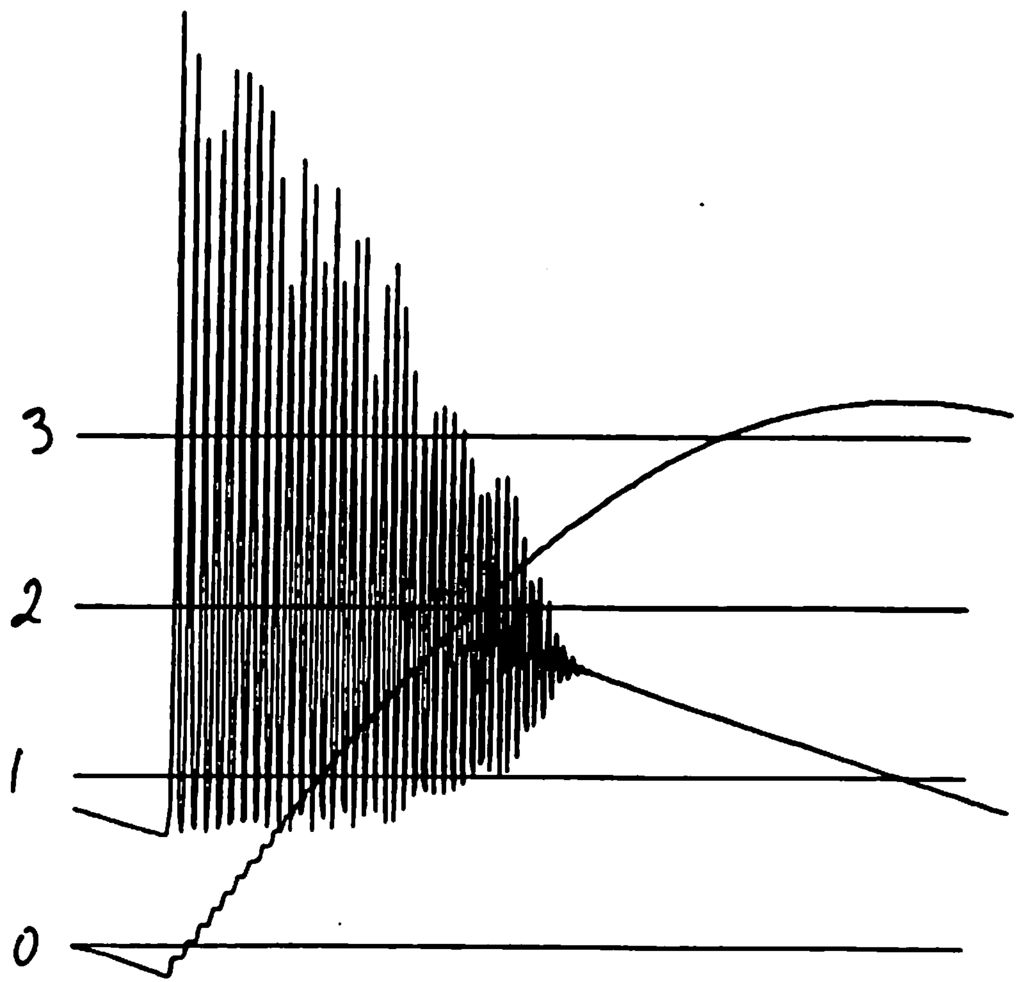
$t = 2$
$n = 50$



$\Delta = 2\pi/100$



$\Delta = 2\pi/200$



$\Delta = 2\pi/400$

1. Explain oscillatory behavior
2. Prove weak convergence
3. Determine weak limit
4. What equation does the weak limit satisfy?

$$\dot{u}_k + u_k \frac{u_{k+1} - u_k}{2\Delta} = 0$$

$$\dot{u}_k + \frac{f_{k+1/2} - f_{k-1/2}}{\Delta} = 0$$

$$f_{k+1/2} = \tfrac{1}{2} u_k u_{k+1}$$

Divide by u_k :

$$\dot{\log} u_k + \frac{u_{k+1} - u_{k-1}}{2\Delta} = 0$$

$$\dot{\log}\, u_k + \frac{g_{k+1/2} - g_{k-1/2}}{\Delta} = 0$$

$$g_{k+1/2} = \frac{u_k + u_{k+1}}{2}$$

If $u(\Delta)$ would tend strongly to u, both

$$u_t + f(u)_x = 0$$

and

$$(\log u)_t + g(u)_x = 0$$

would be satisfied.
But they are incompatible for discontinuous solutions.

The solution of the difference equation desperately tries to satisfy two incompatible conservation laws. Hence the oscillations.

Limits of Semigroups.

Denote by $U\left(t,\varepsilon\right)$ the solution operator for the KdV equation

$$u_t + uu_x + \varepsilon u_{xxx=0,} \qquad (1)$$

that is, the operator

$$U\left(t,\varepsilon\right): u\left(x,0;\varepsilon\right) \to u\left(x,t;\varepsilon\right)$$

that relates initial values of solutions of (1) to their values at t. Clearly, the operators

$U(t,\varepsilon)$form a semigroup (even a group) in the parameter t:

$$U(s,\varepsilon)U(t,\varepsilon)=U(s+t,\varepsilon) \qquad (2)$$

Lax & Levermore have shown that the limit $\varepsilon \to 0$ of $U(t,\varepsilon)$ exits in the weak but for $t > t$ crit not in the strong topology:

$$U(t,\varepsilon)u_0 \to U(t)u_{0.} \qquad (3)$$

Note that since equation (1), written in conservation form

$$u_t + \tfrac{1}{2}\left(u^2\right)_x + \varepsilon u_{xxx} = 0,$$

is nonlinear, a weak limit of its solutions that is not a strong limit is <u>not</u> a solution of the limiting equation

$$u_t + \tfrac{1}{2}\left(u^2{}_x = 0.\right) \qquad (4)$$

So $U(t)u_0$ in (2) is <u>not</u> the solution of (4) with initial value u_0.

<u>Question.</u> Do the limit operators $U(t)$ defined in (3) form a semigroup:

$$U(t)U(s) \doteq U(t+s). \qquad (5)$$

L & L have derived an explicit formula for $U(t)$, which shows that $U(t)$ is not continuous in the weak topology. That makes it very dubious that the limit as ε tends zero of (2) is (5). The question can be easily decided by a single calculation.

There is an analogue of this question in the theory of turbulence. Here operators to be considered are the solutions operators $U(t,R)$ for the Navier-Stokes equation. I surmise that as the Reynolds number $R \to \infty$, the operators $U(t,R)$ tend weakly but not strongly to a limit $U(t)$.

Since the limiting Euler equations are nonlinear conservation laws, $U(t)$ is <u>not</u> the solution operator for the incompressible Euler equation but a description of turbulent flow. In analogy with the zero dispersion limit of the KdV equation I would surmise that the operators $U(t)$, the limits of $U(t,R)$, do <u>not</u> form a semigroup, as conjectured by Heinz Kreiss.

Compactness

$$u_t + uu_x = 0, \quad t \geq 0,$$

can be written as conservation law

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0.$$

Initial value $u(x,0) = u_0(x)$, bounded and zero outside a finite interval.

Viscosity method

$$u_t + uu_x = \varepsilon u_{xx}, \quad \varepsilon > 0, u = u(x,t;\varepsilon),$$

$u(x,0;\varepsilon)=u_0(x).\ \lim u(x,t;\varepsilon)=u(x,t),\varepsilon\to 0$
boundedly, in L^1 norm.
How does u depend on initial u_0?

Define $U_0(x)=\int_{-\infty}^{x}u_0(y)dy,$

and set

$$|u_0|_{-1}=\sup|U_0|.$$

For any to solutions u,υ with initial values $u_0,\upsilon_0,$

$$\left|u(\cdot,t)-\upsilon(\cdot,t)\right|_{-1} \le \left|u_0-\upsilon_0\right|_{-1}$$

Proof. Integrate viscous equation:

$$U_t + \tfrac{1}{2}U_x^2 = \varepsilon U_{xx},$$

and

$$V_t + \tfrac{1}{2}V_x^2 = \varepsilon V_{xx}.$$

Subtract these equations, and denote $U-V$ as D:

$$D_t + \tfrac{1}{2}\left(U_x+V_x\right)D = \varepsilon D$$

By maximum principle,

$$\left|D(x,t)\right| \leq \sup\left|D_0\right|,$$

as claimed.

Let $u_0^{(n)}$ be any sequence of initial date that tend weakly, that is in the sense of the $|\ |_{-1}$ nom, to u_0; we saw that then $u^{(n)}(x,t)$ tends weakly to $u(x,t)$ for every $t > 0$.

$u^{(n)}$ and u satisfy the basic differential equation

$$u_t^{(n)} + \tfrac{1}{2}\left(u^{(n)^2}\right)_x = 0,$$

$$u_t + \tfrac{1}{2}\left(u^2\right)_x = 0.$$

$u_t^{(n)}$ tends to u_t in the sense of distribution. Therefore $\left(u^n\right)_x^2$ tends to u_x^2 in the sense of distribution. So $\left(u^n\right)^2$ tends to u^2 + const in the sense of distribution. Since both u^n and u tend to zero as $x \to \infty$, the constant is zero. But then u^n must tend to u strongly, that is in the L^1 norm.

Summary If the initial values of $u^{(n)}$ tend to the initial values of u weakly, $u^n(x,t)$ tends to $u(x,t)$ strongly, for every $t > 0$.

The mapping from initial values $u(x,0)$ to their values $u(x,t)$, $t > 0$, is a compact mapping.