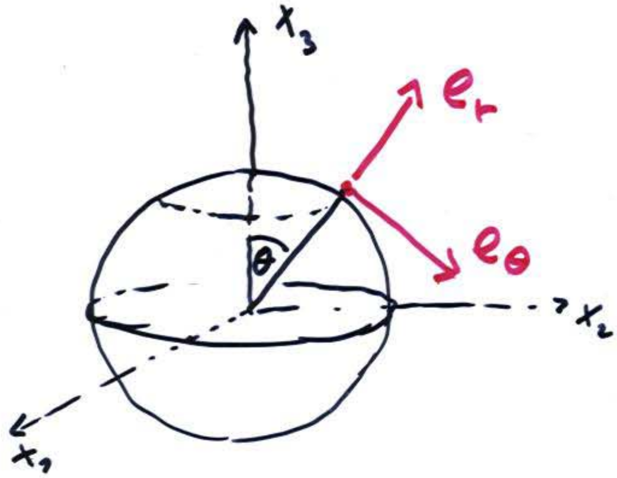
Landau's solutions

(Landau-Lifschitz, p. 82
Batchelor, p. 206)



$$u = \frac{f(\theta)}{r} e_r + \frac{g(\theta)}{r} e_\theta$$

$$p = \frac{\pi(\theta)}{r^2}$$

$$r = |x|$$

$$\left.\begin{aligned} -\Delta u + u\nabla u + \nabla p &= 0 \\ \operatorname{div} u &= 0 \end{aligned}\right\} \Rightarrow \text{ ODE for } f, g, \pi$$

Explicit solutions

(L.D. Landau 1944, H.B. Squire 1951, further works: G. Tian - Z. Xin 1997, M. Cannone - G. Karch 2002)

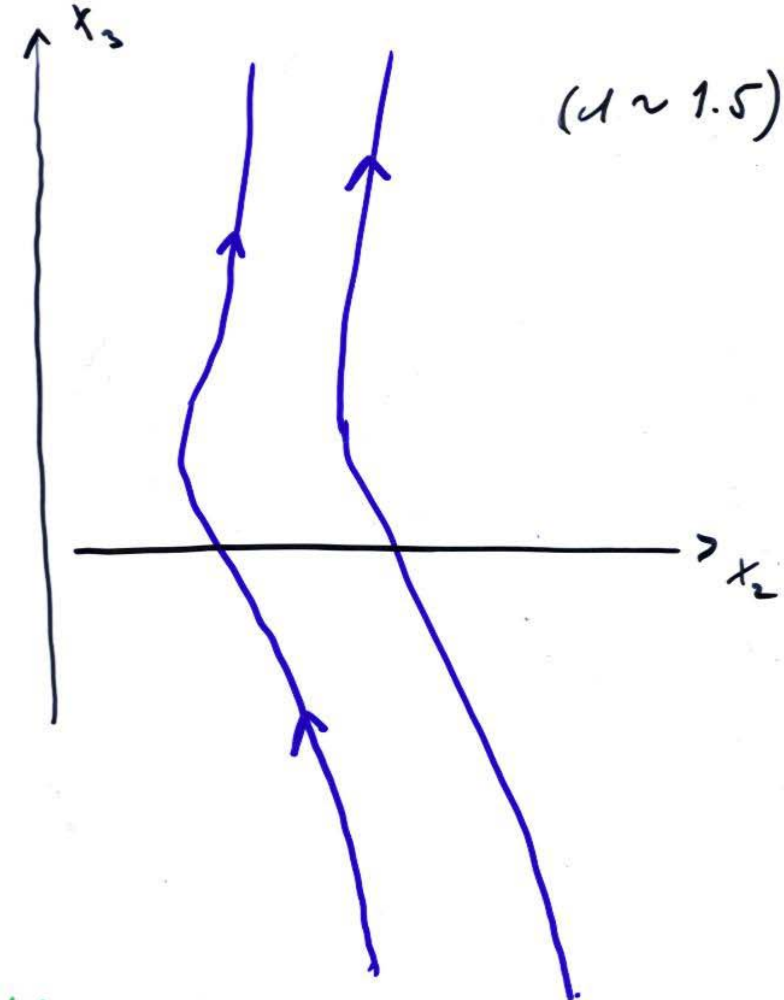
$$f(\theta) = 2\left[\frac{\lambda^2 - 1}{(\lambda - \cos\theta)^2} - 1\right]$$

$$g(\theta) = -2\,\frac{\sin\theta}{\lambda - \cos\theta} \qquad |\lambda| > 1$$

$$\pi(\theta) = 4\,\frac{(1 - \lambda\cos\theta)}{(\lambda - \cos\theta)^2}$$

Streamlines (x_2, x_3 - plane)

symmetric with
respect to rotations
about x_3



Behavior for large λ:

$$u = \frac{1}{\lambda} \left[\begin{array}{l} \text{"fund. solution} \\ \text{of Stokes"} \end{array} \right] + O\left(\frac{1}{\lambda^2} \right)$$

general λ:

$u \sim$ deformation of the fund. solution
of the (linear) Stokes system
by the N-S non-linearity

At the singularity:

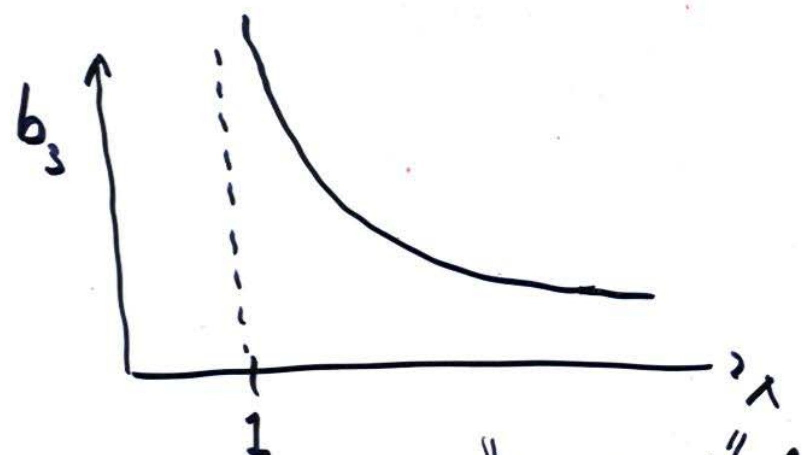
$$-\Delta u + u \cdot \nabla u + \nabla p =$$

$$\operatorname{div}\big(\underbrace{-\nabla u + u \otimes u + p \cdot I}_{\in L^1_{loc}}\big) = \text{well-defined element of } \mathcal{D}'(\mathbb{R}^3)$$

$$= \underset{\in R^3}{b} \cdot \underset{\text{Dirac function}}{\delta(x)}$$

$$b = \begin{pmatrix} 0 \\ 0 \\ b_3(\lambda) \end{pmatrix}$$

$$b_3(\lambda) = \frac{32}{3}\,\frac{\lambda}{\lambda^2 - 1} + 4\lambda^2 \log\left(\frac{\lambda - 1}{\lambda + 1}\right) + 8\lambda \qquad \begin{pmatrix}\text{Batchelor,} \\ \text{p. 209}\end{pmatrix}$$



Possible interpretation: "fund. sol." of steady state NS.

Motivation for further study
of similar classes of solutions:

1. Regularity theory

2. Asymptotic behavior
of solutions of steady-state
NSE as $|x| \longrightarrow \infty$

Regularity theory – a model problem

(ME) $\quad -\Delta u = |u|^{2\delta} u \quad$ in $\quad \Omega = B_1 \subset \mathbb{R}^n$

- well-defined in $\mathcal{D}'$ when $u \in L^{2\delta+1}_{loc}$
- scaling symmetry $\quad u(x) \longrightarrow \lambda^{1/\delta} u(\lambda x)$
- Invariant solution $\quad \bar{u} = \dfrac{A(\delta,n)}{|x|^{1/\delta}}$

  works for $\quad 2 + \frac{1}{\delta} < n$
- $\bar{u} \in L^{\delta n}_{weak}$

Theorem: (standard reg. theory)

$u \in L^{\delta n}_{loc}$ solves (ME) in $\mathcal{D}'$, $\quad 2 + \frac{1}{\delta} < n$

$\Rightarrow$ u is regular

Interpretation: Regularity is governed by the invariant solution.

Steady-state NS, (very) weak formulation:

$$\int_\Omega \left(u_i \Delta \varphi_i + u_j u_i \frac{\partial \varphi_i}{\partial x_j} \right) dx = 0$$

for each $\varphi = (\varphi_1, \dots, \varphi_n) \in \mathcal{D}(\Omega)$, $\operatorname{div} \varphi = 0$.

- well-defined for $u \in L^2_{loc}$

<u>Question</u>: Regularity of such solutions ?

<u>Motivation:</u>

(i) The real problem : 3+1 -dim NS, <u>with</u> energy estimates

<u>However</u>, the eq. is super-critical and the energy estimates have not found much use for <u>full</u> regularity <u>so far</u>

(ii) 3d steady-state NS with energy est. $\longrightarrow$ no problem (subcritical)

(iii) 3d steady-state NS <u>without</u> energy est.
$\rightarrow$ very weak solutions $u \in L^2_{loc}$

? Is this a goood model problem ?

Theorem: (standard regularity theory)

$u \in L^n_{loc}$ is a very weak solution of steady-state n-dimensional NS, $n \geq 3$,

$\Rightarrow$ u is regular

Scaling symmetry of NS: $u(x) \to \lambda u(\lambda x)$

Question: Is there a scale-invariant (very weak) steady-state solution, smooth in $\mathbb{R}^n \setminus \{0\}$?

If so, the above theorem is more or less optimal.

Theorem: (Frehse-Růžička, Struwe, Tsai-VS)
$n \geq 4$, $u : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n$ is
(-1)-homogeneous, smooth away from 0,
solves steady-state NSE
$\Rightarrow \quad u \equiv 0$

This cannot be true for $n=3$,
due to Landau's solutions

However, Landau's solutions
are not (very weak) solutions of NS
across the origin.

Theorem: (VS)

If $u : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3$ is (-1)-homogeneous, smooth away from 0, and solves the steady-state N-S in $\mathbb{R}^3 \setminus \{0\}$, then u is a Landau solution

(An alternative formulation:
(-1)-homogeneous solution must be axi-symmetric with respect to a suitable axis, and hence a Landau sol.)

Corollary: There are no obstructions to regularity of very weak solutions coming from (-1)-homog. (= scale invariant) solutions (smooth away from 0).

Possible implications for long-range behavior:

$$\left.\begin{aligned} -\Delta u + u\cdot\nabla u + \nabla p &= f \quad \text{comp. supported} \\ \operatorname{div} u &= 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3$$

Problem: behavior of u at ∞

Plausible scenario:

$$u_\lambda(x) = \lambda u(\lambda x)\,, \quad f_\lambda(x) = \lambda^3 f(\lambda x)$$

Note: $f_\lambda \xrightarrow{\lambda\to\infty} \bar{f} = b\cdot\delta\,, \quad b = \int f$

Dirac function

$\underline{\underline{\text{If}}}$ $\quad u_\lambda \xrightarrow{\lambda\to\infty} \bar{u}\,, \quad \bar{u} \in L^3_{loc}(\mathbb{R}^3\setminus\{0\})$

then $\bar{u}$ is (-1) - homogeneous, smooth away from 0 $\Rightarrow$ $\bar{u}$ is a Landau sol.

<u>Conclusion</u>: If a simple asymptotics for u at ∞ exists, it must be given by a Landau solution

Remark: the same situation

$$-\Delta u + u\cdot\nabla u + \nabla p = f$$
$$\operatorname{div} u = 0$$

If $u_\lambda(x) = \lambda u(\lambda x)$ have a limit $\bar{u}$ $(\lambda \to \infty)$ and $\int f = 0$, then $\bar{u} \equiv 0$

$\Rightarrow$ u decays faster than $\frac{1}{|x|}$

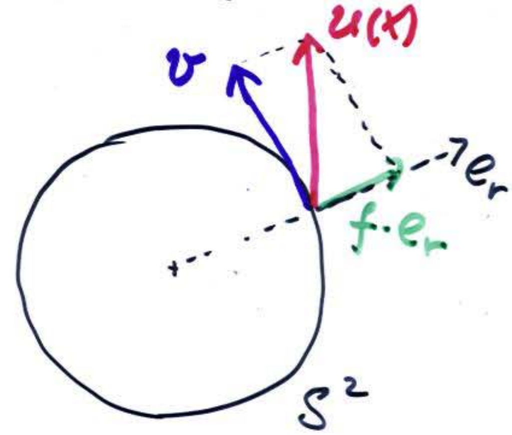
This is standard and expected behavior for the linear Stokes, but it seems it might carry over to the N-S! This seems to be surprising: (In fact, it would be natural to suspect that the non-linear effects can produce a non-symm. invariant solution!)

Proof of the main theorem

$$u(x) = \frac{1}{|x|}\, u\left(\frac{x}{|x|}\right), \qquad p(x) = \frac{1}{|x|^2}\, p\left(\frac{x}{|x|}\right)$$

$\Rightarrow$ can only consider $u\big|_{S^2}$, $p\big|_{S^2}$

On S^2:



$$u(x) = \underbrace{v(x)}_{\text{vector field on } S^2} + \underbrace{f(x)}_{\text{function on } S^2} \cdot e_r(x)$$

Equations on S^2 for v, f, p:

(1) $\quad \operatorname{div} v + f = 0 \qquad$ (continuity)

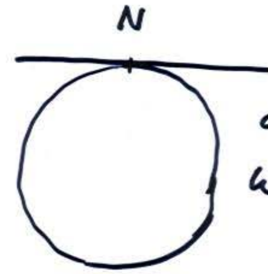
(2) $\quad \Delta f + v \nabla f - f^2 - |v|^2 - 2p = 0 \qquad$ (e_r-comp. of N-S)

(3) $\quad -\Delta_H v + v \cdot \nabla v + \nabla(p - 2f) = 0 \qquad$ (tangential comp. of N-S)

Hodge Laplacian $\quad -\Delta_H = dd^* + d^*d$

Define $\omega = \operatorname{curl} v$

(as a 2-form: $\omega = dv$)



at N:
$\omega = v_{2,1} - v_{1,2}$

Calculation: (take curl of (3))

(VE) $\quad -\Delta\omega + \operatorname{div}(v\omega) = 0$

Lemma 1: $\quad \omega \equiv 0$

Proof:

(i) (VE) $\Rightarrow$ ω does not change sign

(ii) $\displaystyle\int_{S^2} \omega = 0$

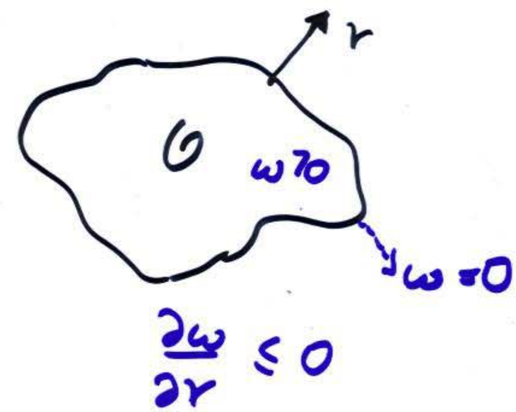
(ii) is easy: $\displaystyle\int_{S^2} \omega = \int_{S^2} dv = 0$

Heuristics for (i):

$\mathcal{O} = \{\omega > 0\}$, assume $\partial\mathcal{O} \neq 0$

$$(VE) \Rightarrow \quad 0 = \int_{\mathcal{O}} \operatorname{div}(-\nabla\omega + v\omega) =$$

$$= -\int_{\partial\mathcal{O}} \underbrace{\frac{\partial\omega}{\partial\nu}}_{\leq 0}$$



$$\Rightarrow \left.\frac{\partial\omega}{\partial\nu}\right|_{\partial\mathcal{O}} = 0$$ $\Rightarrow$ conceptually $\omega \equiv 0$ in $\mathcal{O}$ by strong max. princ. (modulo technicalities)

∎

With $\omega \equiv 0$, (3) simplifies:

$$-\Delta_H v = dd^* v + d^* dv = -\nabla \operatorname{div} v = \nabla f$$

$$v \cdot \nabla v = \frac{1}{2} \nabla |v|^2$$

(3) $\Rightarrow$ $\frac{1}{2}|v|^2 + p - f = c$

(2) now gives

(RE) $-\Delta f - 2f + \operatorname{div}(f \cdot v) = 2c$

Integrate over S^2, use $f = -\operatorname{div} v \Rightarrow c = 0$

Set $v = \nabla\varphi$; $f = -\operatorname{div} v = -\Delta\varphi$

(φE) $\Delta^2\varphi + 2\Delta\varphi - \operatorname{div}(\nabla\varphi\,\Delta\varphi) = 0$

Set $w = 2 - \Delta\varphi$ and re-write (φE) as

(wE) $-\Delta w + \operatorname{div}(\nabla\varphi\, w) = 0$

Lemma 2: The solutions of (wE),

$-\Delta w + \mathrm{div}(\nabla\varphi\, w) = 0$, are exactly the functions $w = f \cdot e^{\varphi}$ (f = const.)

Proof: (i) e^{φ} solves (wE)

(ii) write $w = f(x)\, e^{\varphi}$ to get

$$-\Delta f + b_j f_{,j} = 0 \implies f \equiv \text{const}$$

(max. princ.)

▨

Also, $\int_{S^2} f e^{\varphi} = \int_{S^2} 2 - \Delta\varphi = 8\pi > 0 \implies f > 0$

Therefore

$$-\Delta\varphi + 2 = f e^{\varphi} \qquad (f = \text{const.} > 0)$$

Change $\varphi \longrightarrow \varphi + \text{const.}$ to get

(GE) $$\boxed{-\Delta\varphi + 2 = 2e^{\varphi}}$$

Meaning of (GE):

$S^2, \bar{g}$ (canonical metric)

New metric: $g = \bar{g} e^{\varphi}$

(GE): Gauss curvature of g is $\equiv 1$,

or: g is isometric to $\bar{g}$

$\Rightarrow \exists h : S^2 \to S^2$, a diffeomorphism with

$$e^{\varphi} \bar{g} = h^* g$$

$\Rightarrow$ h is conformal or anti-conformal, and

$$\varphi(x) = \log |h'(x)|^2$$

Conformal diffeom. of S^2 are known explicitly:

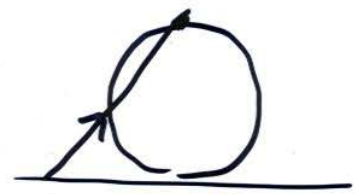
the nontrivial ones can be obtained by composing

stereographic projection

$\mathbb{C} \to \mathbb{C}$

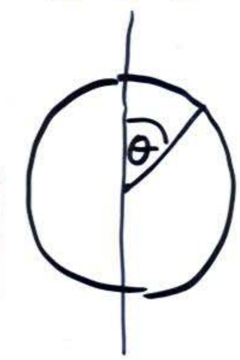
$z \to \lambda z$

(dilation)



inverse stereogr. projection

The corresponding φ (in suitable co-ord.)



$$\boxed{\varphi(x) = -2\log(\cosh\varkappa - \sinh\varkappa\cos\theta)}$$

$\varkappa \in R$ is a parameter

Gives exactly the Landau solutions

Remark:

Let $h: S^2 \to S^2$ be a conformal map (perhaps with branch points where $h'(x) = 0$)

Then $\varphi = \log |h'(x)|^2$ is well-defined in $S^2 \setminus \{a_1, \dots, a_m\}$, and gives a (-1)-homogeneous solution of N.S. in $\mathbb{R}^3 \setminus \{$finitely many rays from $0\}$

Beyond (-1)-homogeneous solutions

(-1)-homog. functions $\longrightarrow$ continuous (scaling) symmetry group

$$\mu f(\mu t) = f(t)$$

Consider $f(t) = \dfrac{1}{t^{1+i\alpha}}$ $\longrightarrow \left\{ \begin{array}{l} t f(t) \\ \text{is periodic} \\ \text{in } \log t \end{array}\right.$

discrete (scaling) symmetry group

$$\mu_0^k f(\mu_0^k t) = f(t) \quad , \quad \mu_0 = e^{\frac{2\pi}{\alpha}}$$

<u>Example:</u> $-\Delta u = u^3$ in $\mathbb{R}^3$

The invariant solution $\bar{u} = \dfrac{A}{|x|}$ can bifurcate into non-trivial, discretely invariant solutions (periodic in $\log r$)

<u>Question:</u> Is such behavior possible for Landau solutions?

Numerical evidence: <u>No</u>

Conclusions :

The study of Landau solution suggests:

(i) non-trivial regularity results for very weak solutions of steady state NS

(ii) conjectures for long-range behavior of st. state sol. in $\mathbb{R}^3$

(iii) Landau sol. are rigid, perhaps even in the class of discretely invariant solutions.
(numerically this seems to be true for infinitesimal perturbations)