Practical Approaches to Calculating CVA for Portfolios of CDSs

Theory and Practice

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Plan

A brief survey of credit literature

Credit Default Swaps (CDSs)

Other Credit Products

Reduced, structural, and hybrid single- and multi- name models

Credit Value Adjustment (CVA)

Examples

Acknowledgements

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Selected References 1

Reduced form model: Chen & Filipovich (2007), Hughston & Turnbull (2001), Jarrow & Turnbull (2000), Lando (1998).

Structural model: Black & Cox (1976), Hilberink & Rogers (2002), Hyer *et al* (1998), Leland (1994), Longstaff & Schwartz (1995), Lipton (2002), Merton (1974), Sepp (2004, 2006), Zhou (2001a), Zhou (2001b).

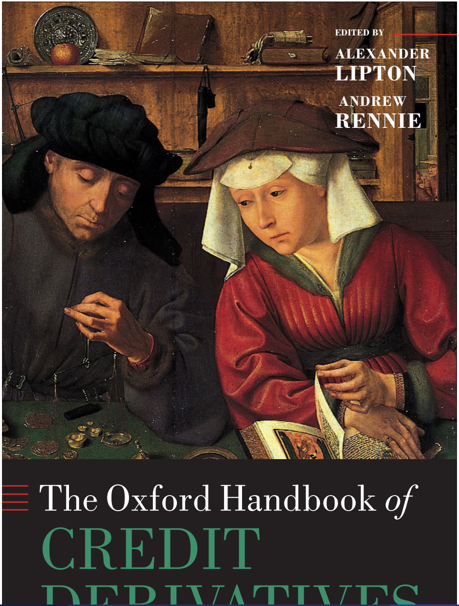
Counterparty risk: Blanchet & Patras (2008), Brigo & Chourdakis (2008), Crepey, Jeanblanc & Zargari (2009), Davis & Lo (2001), Leung & Kwok (2005), Haworth, Reisisnger & Shaw (2006), Hull & White (2000), (2001), Misisrpashayev (2008), Turnbull (2005), Valuzis (2008).

Selected References 2

Numerical methods: Andersen & Andreasen (2000), Boyarchenko & Levendorsky (2000), Cont & Tankov (2004), Cont & Voltchkova (2005), Clift & Forsyth (2008), d'Halluin, Forsyth & Vetzal (2005), Ekstrom, Lotstedt, Tysl (2009),Feng & Linetsky (2008), Lipton (2003), Lipton & Sepp (2009).

Detailed Surveys: Bielecki, Crepey & Herbertsson (2010), Davis (2010), Lipton & Sepp (2010), Lipton & Shelton (2010), all in Lipton & Rennie, The Oxford Handbook of Credit Derivatives (2010).

OUP Handbook Cover



CDS figure

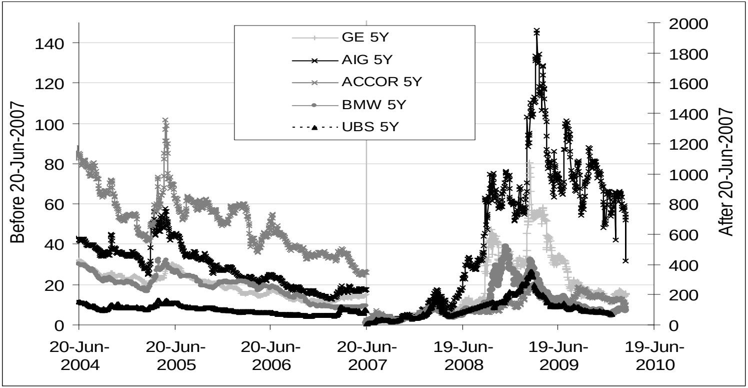


Figure: 5Y CDS Spread in basis points of 5 well known issuers. The spreads are plotted on a different scale before and after 20-Jun-2007. Source: Bank of America - Merrill Lynch Securities.

ITX and CDX figure

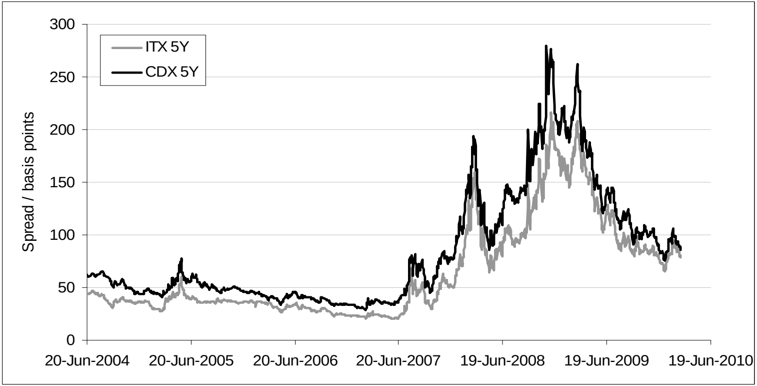


Figure: ITX and CDX historical On-the-Run Index Spreads. Source: Bank of America - Merrill Lynch Securities.

ITX tranches figure

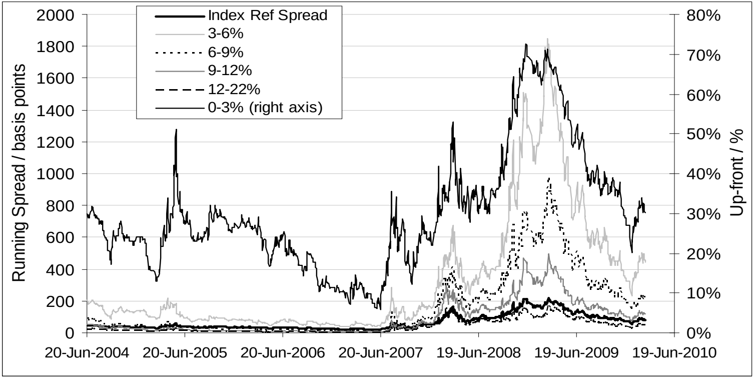


Figure: On-the-Run ITX Tranche Quote History (0-3% maps to right axis and is % up-front. All other tranche prices are quoted as running spreads and map to the left axis. Since Q1 2009 the 3-6% and 6-9% have also been quoted as % up-front with 500bp running but for the purposes of this chart they have been converted to running spreads for the entire time-series). Prior to Series 9 (20-Mar-2008) the on the run index changes every 6 months, subsequently, index

CDS modeling

First, we need to explain how to price CDSs.

Then we need to extend our theory to cover indices, tranches, baskets, etc.

Three complementary approaches to pricing CDSs:

reduced form

structural

hybrid

All of them have pros and cons. Reduced form and structural approaches dominate.

Reduced form model formulation

We represent intensity of the Cox default process in the form

$$\tilde{X}(t) = \alpha(t) + \beta X(t)$$

For analytical convenience (rather than for deeper reasons) it is customary to assume that X is governed by the square-root stochastic differential equation (SDE):

$$dX(t) = \kappa(\theta - X(t))\,dt + \sigma\sqrt{X(t)}dW(t) + JdN(t), \;\; X(0) = X_0$$

with exponential (or hyper-exponential) jump distribution.

Survival probability

For practical purposes it is often more convenient to consider discrete jump distributions with jump values $J_m > 0$, $1 \leq m \leq M$, occurring with probabilities $\pi_m > 0$; such distributions are more flexible than parametric ones because they allow one to place jumps where they are needed.

In this framework, the survival probability $q(t, T)$ and the discounted survival probability $Q(t, T)$ from time t to time T has the form

$$q(t, T) = \mathbb{E}_t \left\{ e^{-\int_t^T \tilde{X}(t')dt'} \right\} = e^{-\int_t^T \alpha(t')dt'} \mathbb{E}_t \left\{ e^{-\int_t^T (\beta X(t'))dt'} \right\}$$

$$Q(t, T) = \mathbb{E}_t \left\{ e^{-\int_t^T \left(r + \tilde{X}(t')\right)dt'} \right\} = e^{-\int_t^T (r+\alpha(t'))dt'} \mathbb{E}_t \left\{ e^{-\int_t^T (\beta X(t'))dt'} \right\}$$

Calculation of expectations

Probability $Q(t, T)$ can be computed by solving the following partial differential equation (PDE)

$$(\partial_t + \mathcal{L}) V(t, X, T) - (r + \alpha + \beta X) V(t, X, T) = 0$$

$$V(T, X, T) = 1$$

where

$$\mathcal{L}V \equiv \kappa(\theta - X) V_X + \tfrac{1}{2}\sigma^2 X V_{XX} + \lambda \left[\phi \int_0^\infty V(X + J) e^{-\phi J} dJ - V(X)\right]$$

or

$$\mathcal{L}V \equiv \kappa(\theta - X) V_X + \tfrac{1}{2}\sigma^2 X V_{XX} + \lambda \sum_m \pi_m \left[V(X + J_m) - V(X)\right]$$

Affine ansatz 1

The corresponding solution can be written in the so-called affine form:

$$V(t, X, T) = e^{-\int_t^T (r+\alpha(t'))dt' + a(t,T) + b(t,T)X}$$

When $\beta = 1$, the corresponding a, b have the form ($\gamma = \sqrt{\kappa^2 + 2\sigma^2}$)

$$\begin{aligned} a &= -\frac{2\kappa\theta}{\sigma^2} \ln\left(\frac{\left((\kappa+\gamma)\,e^{(-\kappa+\gamma)\tau/2} + (-\kappa+\gamma)\,e^{(-\kappa-\gamma)\tau/2}\right)}{2\gamma}\right) \\ &\quad + \frac{2\lambda\tau}{(\phi(-\kappa+\gamma)-2)} - \frac{2\lambda\phi}{(\phi^2\sigma^2 - 2\phi\kappa - 2)} \\ &\quad \times \ln\left(\frac{(\phi(\kappa+\gamma)+2)\,e^{\gamma\tau} + (\phi(-\kappa+\gamma)-2)}{2\gamma\phi}\right) \end{aligned}$$

$$b = -\frac{2\left(e^{\gamma\tau}-1\right)}{(\kappa+\gamma)\,e^{\gamma\tau} + (-\kappa+\gamma)}$$

Affine ansatz 2

When $\beta \neq 1$, we have to apply scaling

$$\theta \rightarrow \theta\beta, \quad \sigma \rightarrow \sigma\sqrt{\beta}, \quad \phi \rightarrow \phi/\beta, \quad b \rightarrow b\beta$$

The survival probability $q(0, T)$, discounted survival probability $Q(0, T)$ and default probability $p(0, T)$ have the form

$$\begin{aligned} Q(0, T) &= e^{-\int_t^T (r+\alpha(t'))dt' + a(T) + b(T)X_0} \\ q(0, T) &= e^{-\int_t^T \alpha(t')dt' + a(T) + b(T)X_0} \\ p(0, T) &= 1 - q(0, T) \end{aligned}$$

CDS valuation 1

The value U of a credit default swap (CDS) paying an up-front amount υ and a coupon s in exchange for receiving $L = 1 - R$ (where R is the default recovery) on default as follows:

$$U = -\upsilon + V\left(0, X_0\right)$$

Here $V\left(t, X\right)$ solves the following pricing problem

$$\left(\partial_t + \mathcal{L}\right) V\left(t, X\right) - \left(r + \alpha + \beta X\right) V\left(t, X\right) = s - L\left(\alpha + \beta X\right)$$

$$V\left(T, X\right) = 0$$

Using Duhamel's principle and integration by parts, we obtain the following expression for V:

$$V\left(t, X\right) = -\left(s + rL\right) A\left(t, T\right) + L\left[1 - Q\left(t, T\right)\right]$$

where $A\left(t, T\right)$ is the annuity factor between two times t, T

$$A\left(t, T\right) = \int_t^T e^{-\int_t^{t'}\left(r+\alpha\left(t''\right)\right)dt'' + a(t,t') + b(t,t')X} dt' = \int_t^T Q\left(t, t'\right) dt$$

CDS valuation 2

For a given up-front payment υ, we can represent the corresponding par spread $\hat{s}$ (i.e. the spread which makes the value of the corresponding CDS zero) as follows:

$$\hat{s}(T) = \frac{-\upsilon + L\left[1 - Q(0,T) - rA(0,T)\right]}{A(0,T)}$$

Conversely, for a given spread we can represent the par up-front payment in the form

$$\hat{\upsilon} = -(s + rL)\,A(0,T) + L\left[1 - Q(0,T)\right]$$

In these formulas we implicitly assumed that the corresponding CDS is fully collateralized, so that in the event of default L is readily available.

We can use these formulas to calibrate $\alpha(t)$ to a given spread curve $s(t)$.

Two name correlation 1

It is very tempting to extend the above framework to cover several correlated names. For example, consider two credits, $1, 2$ and assume for simplicity that their default intensities have the form,

$$X_i(t) = \alpha_i(t) + \beta_i X(t)$$

and their losses given default are L_i.

For a given maturity T, the default event correlation ρ is defined as follows

$$\rho(0,T) = \frac{p_{12}(0,T) - p_1(0,T)\,p_2(0,T)}{\sqrt{p_1(0,T)\,(1-p_1(0,T))\,p_2(0,T)\,(1-p_2(0,T))}}$$

where τ_1, τ_2 are the default times, and

$$\begin{aligned} p_1(0,T) &= P(\tau_1 \le T), \qquad p_1(0,T) = P(\tau_2 \le T) \\ p_{12}(0,T) &= P(\tau_1 \le T, \tau_2 \le T) \end{aligned}$$

Two name correlation 2

It is clear that

$$p_i(0,T) = 1 - e^{-\int_0^T \alpha_i(t')dt' + a(0,T;\beta_i) + b(0,T;\beta_i)X_0}$$

Simple calculation yields

$$p_{12}(0,T) = 1 - q_1(0,T) - q_2(0,T) + q_{12}(0,T)$$

where

$$q_{12}(0,T) = e^{-\int_0^T (\alpha_1(t') + \alpha_2(t'))dt' + a(0,T;\beta_1+\beta_2) + b(0,T;\beta_1+\beta_2)X_0}$$

Thus

$$\rho(0,T) = \frac{q_{12}(0,T) - q_1(0,T)\,q_2(0,T)}{\sqrt{p_1(0,T)\,q_1(0,T)\,p_2(0,T)\,q_2(0,T)}}$$

In the absence of jumps, the corresponding event correlation is very low. If large positive jumps are added (while overall survival probability is preserved), then correlation can increase all the way to one. We illustrate this observation below.

Event correlation Figure

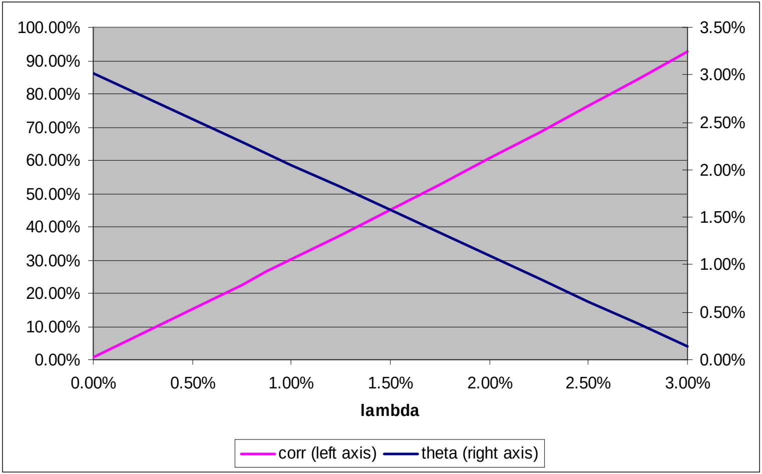


Figure: Correlation ρ and mean-reversion level $\theta = X_0$ as functions of jump intensity λ. Other parameters are as follows: $T = 5y$, $\kappa = 0.5$, $\sigma = 7\%$, and $J = 5.0$.

FTD-STD

In the two-name portfolio, we can define two types of CDSs which depend on the correlation: (A) the first-to-default swap (FTD); (B) the second-to-default swap (STD). The corresponding par spreads $\hat{s}_1, \hat{s}_2$ can be found by using discounted survival probabilities of the form (we are assuming that $L_1 = L_2 = L$)

$$Q_{12}(0, T) = e^{-\int_0^T (r+\alpha_1(t')+\alpha_2(t'))dt' + a(0,T;\beta_1+\beta_2)+b(0,T;\beta_1+\beta_2)X_0}$$

$$\hat{Q}_{12}(0, T) = Q_1(0, T) + Q_2(0, T) - Q_{12}(0, T)$$

It is clear that the relative values of $\hat{s}_1, \hat{s}_2$ very strongly depend on whether or not jumps are present in the model.

FTD-STD Figure

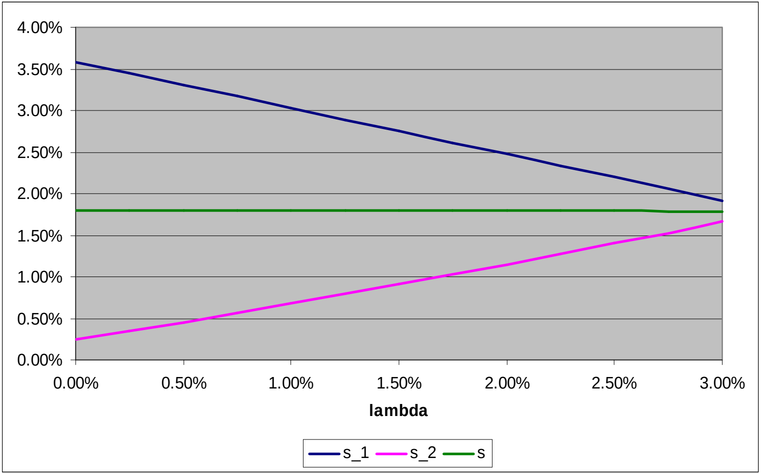


Figure: FTD spread $\hat{s}_1$, STD spread $\hat{s}_2$, and single name CDS spread $\hat{s}$ as functions of jump intensity λ. Other parameters are the same as in Fig.1. It is clear that jumps are necessary to have $\hat{s}_1$ and $\hat{s}_2$ of similar magnitudes.

Counter-party effects 1

An important application of the above model is to the evaluation of counter-party effects on fair CDS spreads. Consider three names, $1, 2, 3$, and assume that name 2 has written a CDS on reference name 3, which is bought by name 1. For simplicity, we assume that their spreads have the form

$$X_i = \alpha_i + \beta_i X$$

It is clear that the pricing problem for the value of the uncollateralized CDS $V^{\{2,3\}}$ can be written as follows

$$\mathcal{L}V^{\{2,3\}}(t,X) - (r + \alpha_2 + \alpha_3 + (\beta_2 + \beta_3) X) V^{\{2,3\}}(t,X)$$
$$= s - L_3 (\alpha_3 + \beta_3 X) - \left(R_2 U_+^{\{3\}}(t,X) + U_-^{\{3\}}(t,X)\right)(\alpha_2 + \beta_2 X)$$

where $U^{\{3\}}$ is the value of a fully collateralized CDS on name 3.
It is clear that the discount rate is increased from $r + \alpha_3 + \beta_3 X$ to $r + \alpha_2 + \alpha_3 + (\beta_2 + \beta_3) X$, since there are two cases when the uncollateralized CDS can be terminated due to default: when the reference name 3 defaults; when the issuer 2 defaults.

Alternatively, the pricing equation can be written in terms of CVA_1 as follows

$$\mathcal{L}W^{\{2,3\}}(t,X) - (r + \alpha_2 + \alpha_3 + (\beta_2 + \beta_3)X)\, W^{\{2,3\}}(t,X)$$

$$= L_2 U_{+}^{\{3\}}(t,X)\,(\alpha_2 + \beta_2 X)$$

where $W^{\{2,3\}} = V^{\{2,3\}} - U^{\{3\}}$. It is clear that in this formulation in addition to coupon payments at rate s; the buyer 1 has to account for additional losses incurred when the issuer 2 defaults.

Although this equation is no longer analytically solvable, it can be solved numerically via, say, an appropriate modification of the classical Crank-Nicholson method. It turns out that in the presence of jumps the value of the fair par spread goes down dramatically.

Counter-party effects 3

If the buyer is risky, the pricing problem for $V^{\{1,3\}}$ can be written as follows

$$\mathcal{L}W^{\{1,3\}}(t,X) - (r + \alpha_1 + \alpha_3 + (\beta_1 + \beta_3)X)\, W^{\{1,3\}}(t,X)$$

$$= L_1 U_-^{\{3\}}(t,X)(\alpha_1 + \beta_1 X)$$

If all three names are risky, the pricing problem is symmetric and quite intuitive

$$\mathcal{L}W^{\{1,2,3\}}(t,X) - (r + \alpha_1 + \alpha_2 + \alpha_3 + (\beta_1 + \beta_2 + \beta_3)X)\, W^{\{1,2,3\}}(t,X)$$

$$= L_1 U_-^{\{3\}}(t,X)(\alpha_1 + \beta_1 X) + L_2 U_+^{\{3\}}(t,X)(\alpha_2 + \beta_2 X)$$

Counterparty Figure

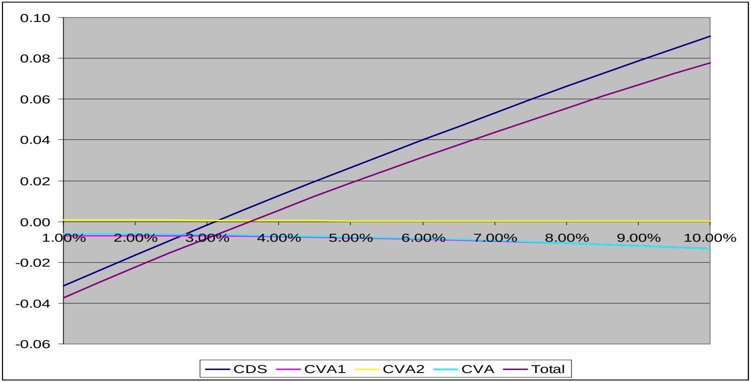


Figure: $r = 3\%, s_1 = 3\%, R_1 = 40\%, \beta_1 = 1, x_1 = 3\%, s_2 = 3\%, R_2 = 40\%, \beta_2 = 1.5, x_2 = 3\%, s_3 = 3\%, R_3 = 40\%, \beta_3 = 1.6, x_3 = 2.5\%, \kappa = 0.5, \theta = 2\%, \sigma = 10\%, \lambda = 2\%, \phi = 1, T = 5Y$

Counter-party effects 4

In case of a *portfolio* of $N-2$ CDSs the pricing problem becomes extraordinary complex since $2^{N-2}-1$ pricing problems need to be solved. In practice, with proper parallelization we can handle $N \sim 20$ via PDEs. For larger N we have to use MC. Let's assume that $N=4$, i.e. we deal with two CDSs. Then

$$\mathcal{L}W^{\{1,2,3,4\}}(t,X) - (r + X_1 + X_2 + X_3 + X_4) W^{\{1,2,3,4\}}(t,X)$$

$$= L_1 X_1 U_-^{\{3,4\}}(t,X) + L_2 X_2 U_+^{\{3,4\}}(t,X) - X_3 W^{\{1,2,4\}} - X_4 W^{\{1,2,3\}}$$

Thus, in order to compute $W^{\{1,2,3,4\}}$ we need to compute $W^{\{1,2,4\}}$, $W^{\{1,2,3\}}$ first.

Counter-party effects 5

We can split the problem as follows. Define the following combinations

$$R_{\pm}^{\{3,4\}} = U_{\pm}^{\{3,4\}} - U_{\pm}^{\{3\}} - U_{\pm}^{\{4\}}$$

$$S^{\{1,2,3,4\}} = W^{\{1,2,3,4\}} - W^{\{1,2,3\}} - W^{\{1,2,4\}}$$

we obtain the following system of decoupled equations which emphasizes portfolio effects

$$\mathcal{L}S^{\{1,2,3,4\}}(t,X) - (r + X_1 + X_2 + X_3 + X_4) S^{\{1,2,3,4\}}(t,X)$$

$$= L_1 X_1 R_{-}^{\{3,4\}}(t,X) + L_2 X_2 R_{+}^{\{3,4\}}(t,X)$$

Finally

$$W^{\{1,2,3,4\}} = S^{\{1,2,3,4\}} + S^{\{1,2,3\}} + S^{\{1,2,4\}}$$

Numerical Methods 1

Traditionally, square-root process causes a lot of confusion when boundary conditions are discussed. However, recently it was clarified (see, e.g., Ekstrom *et al.* 2009)

The so-called Fichera condition suggest that no boundary condition is needed when

$$\lim_{X \to 0} \left(\text{drift}\,(x) - \frac{1}{2} \frac{d}{dx} \left(\text{variance}\,(x) \right) \right) \geq 0$$

For square-root process this is equivalent to the Feller condition. Still, modern practice suggest that using equation as boundary condition always works:

$$V_t\,(t,0) + \kappa\,(\theta - X)\,V_X\,(t,0) = 0$$

We also use equation as a far-field condition.

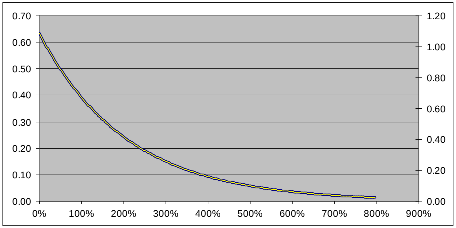
Numerical Methods 2

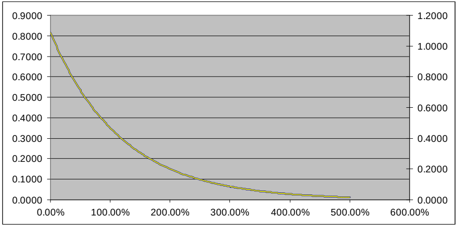
We derive the usual FD discretization which is almost tri-diagonal but the left and right blocks are 3x3 and 4x4 matrices, respectively, which can easily be handled.
We treat the jump term fully explicitly as in Lipton (2003). Specifically

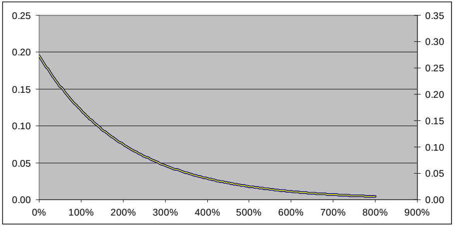
$$I(x) = e^{-\phi dx} I(x + dx) + \frac{e^{-\phi dx} - 1 + \phi dx}{\phi dx} V(x) + \frac{1 - (1 + \phi dx) e^{-\phi dx}}{\phi dx} V($$

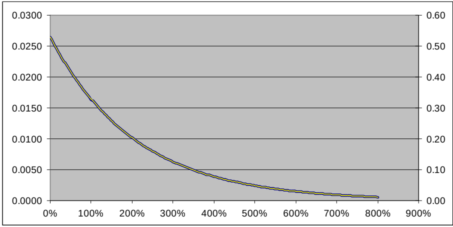
The corresponding scheme is very fast and robust.

Numerical Graphs









Structural model with jumps 1

Consider a company and assume that its asset value is driven by the following SDE

$$\frac{dV(t)}{V(t)} = (r - d - \kappa\lambda(t))\,dt + \sigma(t)\,dW(t) + \left(e^{J} - 1\right) dN(t) \qquad V_i(0) =$$

$$\kappa = \mathbb{E}\left\{e^{J} - 1\right\}$$

Default boundary is driven by the following ODE

$$\frac{dB(t)}{B(t)} = (r - d)\,dt \qquad B(0) = b$$

In log coordinates

$$x(t) = \ln\left(\frac{V(t)}{B(t)}\right)$$

$$dx(t) = \mu(t)\,dt + \sigma(t)\,dW(t) + J dN(t) \qquad x(0) = \ln\left(\frac{v}{b}\right) = \xi$$

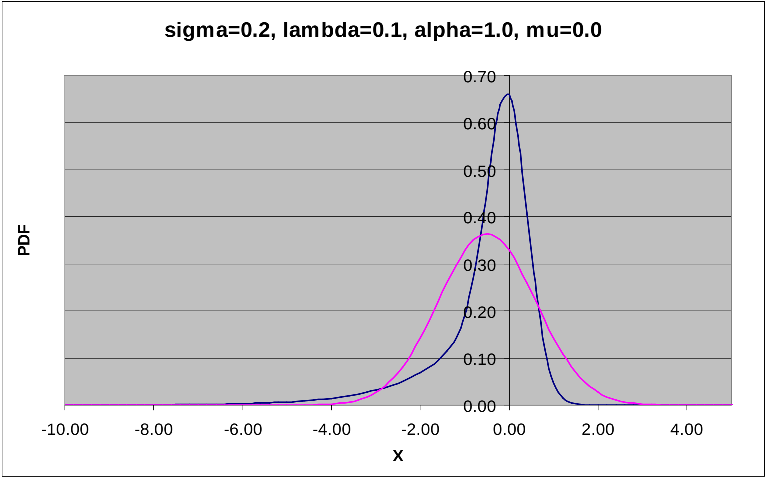
$$\mu = -\tfrac{1}{2}\sigma^2 - \kappa\lambda$$

The corresponding default boundary is flat

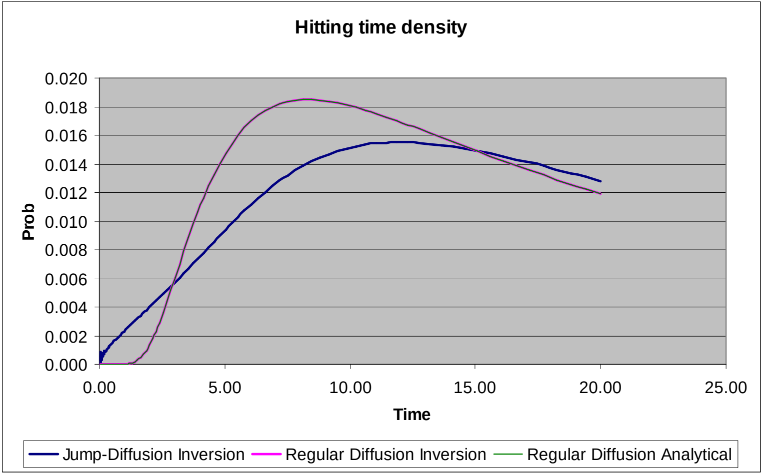
Structural model with jumps 2

Thus, the asset value is governed by a combination of a Wiener process and a Poisson process with exponentially distributed jumps. The firm defaults if the value $x(t)$ crosses zero barrier $b(t) = 0$. It was realized early on that without jumps (or/and curvilinear or uncertain barriers) it is impossible to explain the short end of the CDS curve within the structural framework. When all the relevant parameters are constant, the problem can be solved analytically via the Laplace transform. However, in general this approach does not work.

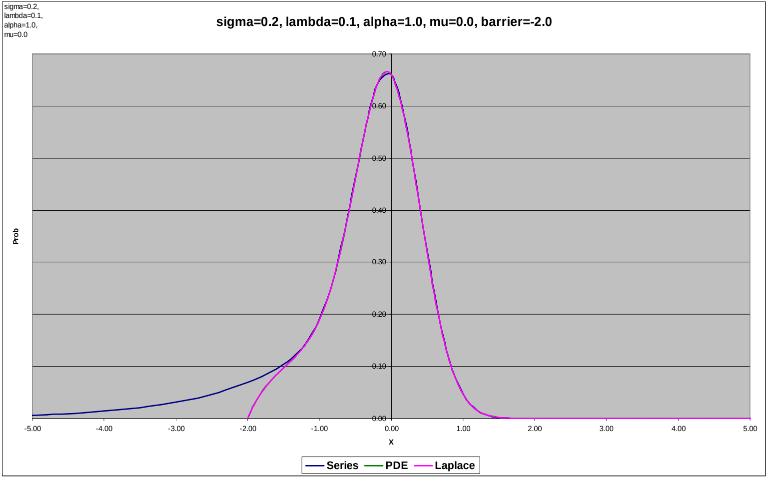
Unbounded PDF figure



Hitting time density figure



Barrier PDF figure



Model Comparison

Comparison of numerical and analytical t.p.d.f. as well as the solution of the barrier problem with non-constant barrier is given below.

By bootstrapping the barrier, we can reproduce term structure of CDS for most names. In addition, we can price equity derivatives and produce a respectable volatility skew.

The barrier can assumed to be stochastic as in Duffie and Lando (2001) or CreditGrades (2000). Reduction of the information set makes reduced form and structural modeling almost identical, Jarrow, Protter, Yildirim (2004).

Correlated PDF 1

We need to analyze behavior of two correlated Brownian motions in a quarter-plane.
Alternatively, we can map a quarter-plane into an angle, $\alpha = \arctan\left(-\bar{\rho}/\rho\right)$.
There are two methods to solve the problem: (A) method of eigenfunction expansion; (B) method of images. In case (A) we have

$$G_{\alpha}\left(t, r, \phi | 0, r', \phi'\right) = \frac{2e^{-\left(r^2+r'^2\right)/2t}}{\alpha t} \sum_{n=1}^{\infty} I_{n\pi/\alpha}\left(\frac{rr'}{t}\right) \sin\left(\frac{n\pi\phi'}{\alpha}\right) \sin\left(\frac{n\pi\phi}{\alpha}\right)$$

In case (B) we introduce the following function $f\left(p, q\right)$ with $p \geq 0$, $-\infty < q < \infty$, (which is a close relation of α-stable Levy distributions):

$$f\left(p, q\right) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2}e^{-p(\cos(q)+1)} - \frac{1}{\pi}\int_0^{\infty} \frac{e^{-p(\cosh(q\zeta)+1)}}{\zeta^2+1} d\zeta\right)$$

Correlated PDF 2

We also define the following auxiliary function $h(p, \phi)$:

$$h(p, \phi) = s_+ f(p, \pi + \phi) + s_- f(p, \pi - \phi)$$

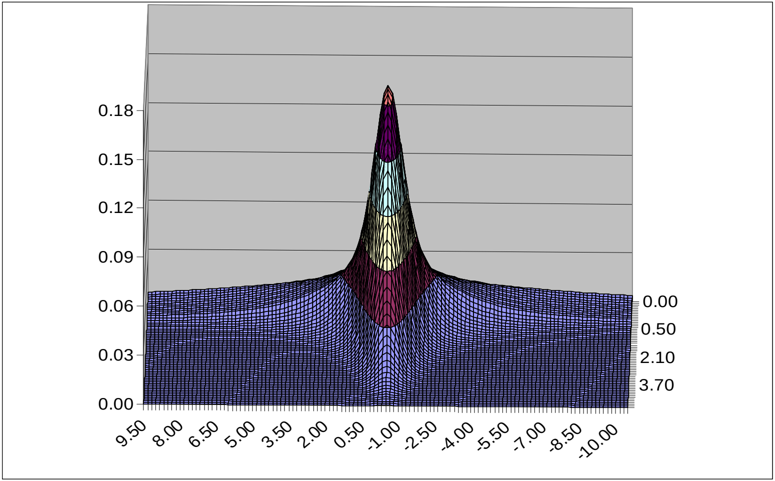
Then we can represent unbounded $G(t, r, \phi | 0, r', \phi')$ in the following remarkable form, which can be viewed as a direct generalization of the planar 2D case:

$$G(t, r, \phi | 0, r', \phi') = \frac{1}{\sqrt{t}} g\left(\frac{r - r'}{\sqrt{t}}\right) \frac{1}{\sqrt{t}} h\left(\frac{rr'}{t}, \phi - \phi'\right)$$

The corresponding G_α can be written as follows:

$$G_\alpha(* | 0, r', \phi') = \sum_{n=-\infty}^{\infty} \left[G(* | 0, r', \phi' + 2n\alpha) - G(* | 0, r', -\phi' + 2n\alpha) \right]$$

Non-periodic pdf figure



Three- and Multi-name case

Consider N companies and assume that their asset values are driven by the following SDEs

$$\frac{dV_i(t)}{V_i(t)} = (r - d_i - \kappa_i \lambda_i(t))\, dt + \sigma_i(t)\, dW_i(t) + \left(e^{J_i} - 1\right) dN_i(t) \qquad V_i(0$$

$$\kappa_i = \mathbb{E}\left\{e^{J_i} - 1\right\}$$

Default boundaries are driven by the ODEs of the form

$$\frac{dB_i(t)}{B_i(t)} = (r - d_i)\, dt \qquad B_i(0) = b_i$$

In log coordinates $x_i(t) = \ln(V_i(t)/B_i(t))$ we have

$$dx_i(t) = \mu_i(t)\, dt + \sigma_i(t)\, dW_i(t) + J_i dN_i(t) \qquad x_i(0) = \ln\left(\frac{v_i}{b_i}\right) = \xi_i$$

$$\mu_i = -\tfrac{1}{2}\sigma_i^2 - \kappa_i \lambda_i$$

The corresponding default boundaries are now flat, $x_i = 0$.

Correlation structure

We introduce correlation between diffusions in the usual way and assume that

$$dW_i(t)\, dW_j(t) = \rho_{ij}(t)\, dt$$

We introduce correlation between jumps following the Marshall-Olkin idea. Let $\Pi^{(N)}$ be the set of all subsets of N names except for the empty subset $\{\varnothing\}$, and π its typical member. With every π we associate a Poisson process $N_\pi(t)$ with intensity $\lambda_\pi(t)$, and represent $N_i(t)$ as follows

$$N_i(t) = \sum_{\pi \in \Pi^{(N)}} 1_{\{i \in \pi\}} N_\pi(t)$$

$$\lambda_i(t) = \sum_{\pi \in \Pi^{(N)}} 1_{\{i \in \pi\}} \lambda_\pi(t)$$

Thus, we assume that there are both common and idiosyncratic jump sources.

Multi-name pricing problem

We now formulate a typical pricing equation in the positive cone $R_{+}^{(N)}$. We have

$$\partial_t U(t,\vec{x}) + \mathcal{L}^{(N)} U(t,\vec{x}) = \chi(t,\vec{x})$$
$$U(t,\vec{x}_{0,k}) = \phi_{0,k}(t,\vec{y}), \qquad U(t,\vec{x}_{\infty,k}) = \phi_{\infty,k}(t,\vec{y})$$
$$U(T,\vec{x}) = \psi(\vec{x})$$

where $\vec{x}_{0,k}$, $\vec{x}_{\infty,k}$, $\vec{y}_k$ are N and $N-1$ dimensional vectors, respectively,

$$\vec{x}_{0,k} = \left(x_1, \ldots, \underset{k}{0}, \ldots x_N \right), \qquad \vec{x}_{\infty,k} = \left(x_1, \ldots, \underset{k}{\infty}, \ldots x_N \right), \qquad \vec{y}_k = (x_1, \ldots, x_N)$$

The integro-differential operator $\mathcal{L}^{(N)}$ can be written as

$$\begin{aligned} \mathcal{L}^{(N)} f(\vec{x}) &= \frac{1}{2}\sum_i \sigma_i^2 \partial_i^2 f(\vec{x}) + \sum_{i,j,j>i} a_{ij}\partial_i\partial_j f(\vec{x}) + \sum_i \beta_i \partial_i f(\vec{x}) \\ &\quad -\tilde{\gamma} f(\vec{x}) + \sum_{\pi\in\Pi^{(N)}} \lambda_\pi \prod_{i\in\pi} \mathcal{J}_i f(\vec{x}) \end{aligned}$$

Here $a_{ij} = \sigma_i\sigma_j\rho_{ij}$, $\tilde{\gamma} = \gamma + \sum_{\pi\in\Pi^{(N)}} \lambda_\pi$.

Jump terms

In the case of negative exponential jumps,

$$J_i f\left(\vec{x}\right) = \phi_i \int_{-x_i}^{0} f\left(x_1, ..., x_i + j, ...x_N\right) e^{-\phi_i j} dj$$

while in the case of discrete negative jumps

$$J_i f\left(\vec{x}\right) = \mathbf{H}\left(x_i + J_i\right) f\left(x_1, ..., x_i + J_i, ...x_N\right)$$

and **H** is the Heaviside function.

We naturally split the operator $\mathcal{L}^{(N)}$ into the local (differential) and non-local (integral) parts:

$$\mathcal{L}^{(N)} f\left(\vec{x}\right) = \mathcal{D}^{(N)} f\left(\vec{x}\right) + \mathcal{I}^{(N)} f\left(\vec{x}\right)$$

Adjoint operator

The corresponding adjoint operator is

$$\begin{aligned}\mathcal{L}^{(N)\dagger} g\left(\vec{x}\right) &= \frac{1}{2}\sum_{i}\sigma_i^2\partial_i^2 g\left(\vec{x}\right) + \sum_{i,j,j>i} a_{ij}\partial_i\partial_j g\left(\vec{x}\right) - \sum_{i}\beta_i\partial_i g\left(\vec{x}\right) \\ &\quad -\tilde{\gamma} g\left(\vec{x}\right) + \sum_{\pi\in\Pi^{(N)}}\lambda^{\pi}\prod_{i\in\pi}\mathcal{J}_i^{\dagger} g\left(\vec{x}\right)\end{aligned}$$

where

$$J_i^{\dagger} g\left(\vec{x}\right) = \int_0^{\infty} g\left(x_1, ..., x_i - j, ...x_N\right)\varpi_i\left(j\right) dj$$

$$J_i^{\dagger} g\left(\vec{x}\right) = g\left(x_1, ..., x_i - J_i, ...x_N\right)$$

Green's formula

We introduce Green's function $\Phi(t, \vec{x})$, or, more explicitly, $\Phi\left(t', \vec{x}; t, \vec{\xi}\right)$, such that

$$\partial_{t'} \Phi\left(t', \vec{x}\right) - \mathcal{L}^{(N)\dagger} \Phi\left(t', \vec{x}\right) = 0$$

$$\Phi\left(t', \vec{x}_{0k}\right) = 0, \qquad \Phi\left(t', \vec{x}_{\infty k}\right) = 0, \qquad \Phi(t, \vec{x}) = \delta\left(\vec{x} - \vec{\xi}\right)$$

It can be shown that

$$\begin{aligned} U\left(t, \vec{\xi}\right) &= \int_{R_+^{(N)}} \psi(\vec{x})\, \Phi\left(T, \vec{x}; t, \vec{\xi}\right) d\vec{x} \\ &+ \sum_k \int_t^T \int_{R_+^{(N-1)}} \phi_k\left(t', \vec{y}\right) \left\{ \frac{1}{2} \sigma_k^2 \Phi_k\left(t', \vec{y}; t, \vec{\xi}\right) \right\} dt' d\vec{y} \\ &- \int_t^T \int_{R_+^{(N)}} \chi\left(t', \vec{x}\right) \Phi\left(t', \vec{x}; t, \vec{\xi}\right) dt' d\vec{x} \end{aligned}$$

In other words, instead of solving the backward pricing problem with nonhomogeneous rhs and boundary conditions, we can solve the forward propagation problem for Green's function with homogeneous rhs and boundary conditions.

Input Data 1

	$s(0)$	$L(0)$	R	$I(0)$	$a(0)$	ξ	$\nu^{\{\text{dis}\}}$	$\nu^{\{\text{exp}\}}$	[illegible]
JPM	41.77	508	40%	203	245	0.1869	0.1869	5.35	2
MS	32.12	534	40%	214	246	0.1401	0.1401	7.14	2

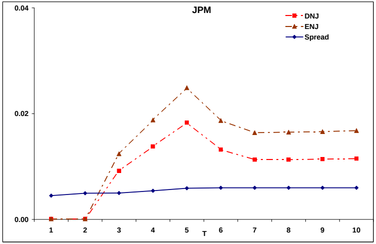
Table 1. Market data, as of October 2, 2009, for JPM and MS.

Input Data

	DF	CDS Sprd (%)		Survival Prob		Default Leg		Annuity L	
T		JPM	MS	JPM	MS	JPM	MS	JPM	
1y	0.9941	0.45	1.10	0.9925	0.9818	0.0045	0.0109	0.9924	0.
2y	0.9755	0.49	1.19	0.9836	0.9611	0.0097	0.0231	1.9632	1.
3y	0.9466	0.50	1.24	0.9753	0.9397	0.0145	0.0354	2.9006	2.
4y	0.9121	0.54	1.29	0.9644	0.9173	0.0206	0.0479	3.7971	3.
5y	0.8749	0.59	1.34	0.9515	0.8937	0.0274	0.0604	4.6474	4.
6y	0.8374	0.60	1.35	0.9414	0.8732	0.0326	0.0709	5.4519	5.
7y	0.8002	0.60	1.35	0.9320	0.8539	0.0372	0.0803	6.2135	5.
8y	0.7642	0.60	1.33	0.9228	0.8373	0.0415	0.0881	6.9338	6.
9y	0.7297	0.60	1.31	0.9137	0.8217	0.0455	0.0950	7.6148	7.
10y	0.6961	0.60	1.30	0.9046	0.8064	0.0494	0.1015	8.2583	7.

Table 2. CDS spreads and other relevant outputs for JPM and MS.

Intensities



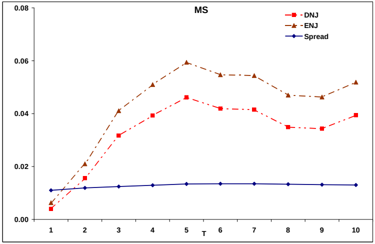


Figure: Calibrated intensity rates for JMP (lhs) and MS(rhs) in the models with ENJs and DNJs, respectively.

PDFs JPM

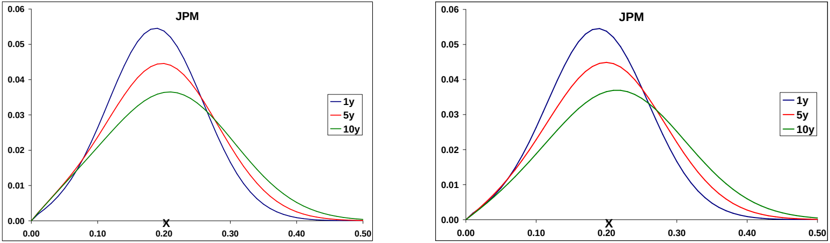


Figure: PDF of the driver $x(T)$ for JMP in the model with DNJs (lhs) and ENJs (rhs) for $T = 1y$, $5y$, and $10\ y$.

PDFs MS

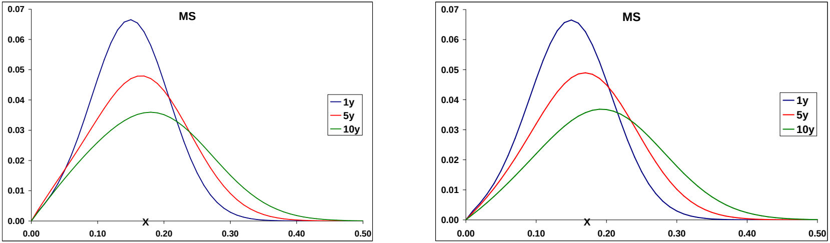


Figure: Same graphs as in Figure 3 but for MS.

Swaption Vols

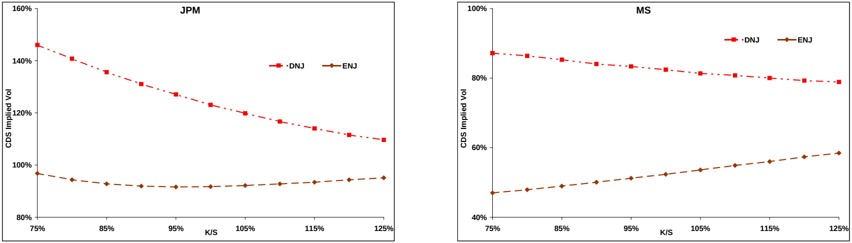
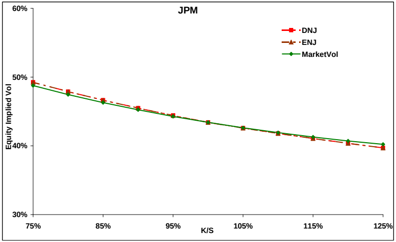


Figure: Log-normal credit default swaption volatility implied from model with $T_e = 1y$, $T_t = 5y$ as a function of the inverse moneyness $K/\bar{c}\left(T_e, T_e + T_t\right)$.

Equity Vols



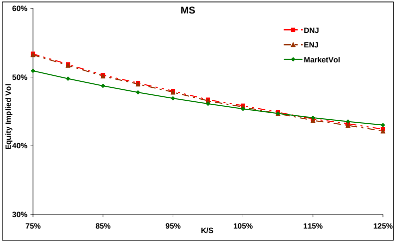
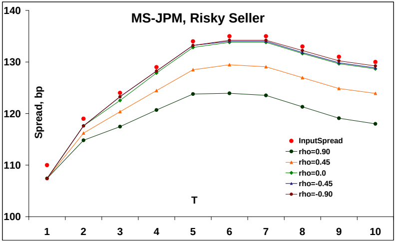


Figure: Log-normal equity volatility implied by the model as a function of inverse moneyness K/s for put options with $T = 0.5y$.

Spreads JPM



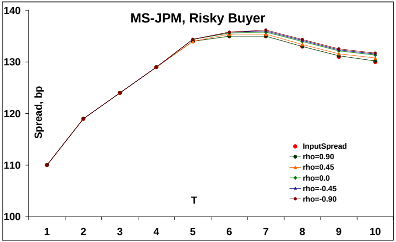


Figure: Equilibrium spread for protection buyer and protection seller of a CDS on MS with JPM as the counterparty.

Spreads MS

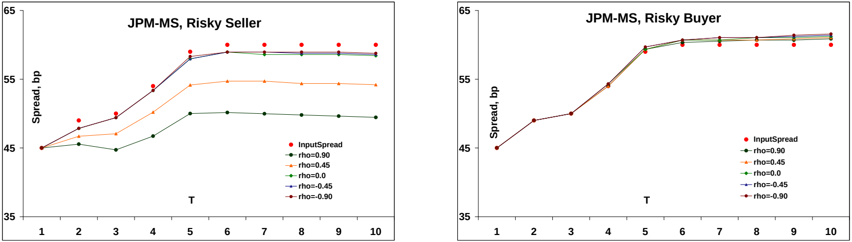


Figure: Same for a CDS on JPM with MS as the counterparty.

Conclusions

We have given a broad overview of the state of credit modeling.

We have shown how to build meaningful reduced-form and structural default models and connected them to each other.

We have shown that even the simplest elementary building blocks of credit universe are difficult to describe properly and require a lot of effort.

The opinions expressed in this talk are those of the speaker and do not necessarily reflect the views or opinions of Bank of America Merrill Lynch.