Numerical methods for an optimal order execution

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Contents

1 Introduction

2 Model

3 Discretization

4 Algorithm

5 Numerical results
Market impact and liquidation costs

- Implementation shortfall: difference between a theoretical trading strategy and its implementation
  \[ \Rightarrow \text{Need of low-touch execution strategy} \]

- Market impact is key factor when executing large orders

- One famous worst case example: Kerviel’s portfolio liquidation

- Basic observation: market impact is reduced when the liquidation operation is extended in time

- Idea is to split a big order into several small orders
  \[ \Rightarrow \text{trade-off between rapid execution (big impact but reduced risk) and gradual trading (reduced impact but more risk)} \]

\[\Rightarrow \text{Our goal is to find optimal trading schedule and associated quantities}\]

Several authors propose price impact models based on stylized dynamics of order book: Schied et al. (2009)[10], Gatheral et al. (2010)[3] and Obizhaeva and Wang (2005) [6].


⇒ In this talk we use the model investigated in this last paper: impulse control formulation.
The model

We consider a financial market where an investor has to liquidate an initial position of \( y > 0 \) shares of risky asset by time \( T \).

- We set a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) equipped with a filtration \(\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) supporting a one-dimensional Brownian motion \( W \) on a finite horizon \([0, T], \ T < \infty\).
- \((P_t)_{t \in [0, T]}\) the market price of the risky asset
- \((X_t)_{t \in [0, T]}\) the cash holdings
- \((Y_t)_{t \in [0, T]}\) the number of stock shares held by the investor
- \((\Theta_t)_{t \in [0, T]}\) the time interval between \( t \) and the last trade before \( t \)
The model: trading strategies

- We represent a trading strategy by:

$$\alpha = (\tau_n, \xi_n)_{n \in \mathbb{N}}$$

where $(\tau_n)$ are $\mathcal{F}$-stopping times and $(\xi_n)$ are $\mathcal{F}_{\tau_n}$-measurable $\mathbb{R}$-valued variables.

- Dynamics for the shares and lag processes are under $\alpha$:

$$\Theta_t = t - \tau_n, \quad \tau_n \leq t < \tau_{n+1}$$

$$\Theta_{\tau_{n+1}} = 0, \quad n \geq 0.$$  

$$Y_s = Y_{\tau_n}, \quad \tau_n \leq s < \tau_{n+1}$$

$$Y_{\tau_{n+1}} = Y_{\tau_n} + \xi_{n+1}, \quad n \geq 0.$$
The model: price impact

- Market price of risky asset process follows a geometric Brownian motion:

\[ dP_t = P_t(bdt + \sigma dW_t) \]

- Suppose now that the investor decides to trade the quantity \( e \). If the current market price is \( p \), and the time lag from the last order is \( \theta \), then the price he actually get for the order \( e \) is:

\[ Q(e, p, \theta) = pf(e, \theta) \]

with

\[ f(e, \theta) = \exp \left( \lambda \frac{e}{\theta} \beta \text{sgn}(e) \right) \left( \kappa_a \mathbf{1}_{e>0} + \mathbf{1}_{e=0} + \kappa_b \mathbf{1}_{e<0} \right), \quad (1) \]

- Therefore cash holdings have the following dynamics

\[
X_t = X_{\tau_n}, \quad \tau_n \leq t < \tau_{n+1}, \quad n \geq 0.
\]

\[
X_{\tau_{n+1}} = X_{\tau_{n+1}} - \xi_{n+1} P_{\tau_{n+1}} f(\xi_{n+1}, \Theta_{\tau_{n+1}}) - \epsilon, \quad n \geq 0.
\]
Liquidation value and solvency constraints

Let us define the admissibility constraints on a trading strategy. We first define the liquidative value of the position \((x, y, p, \theta)\) by:

\[ L(x, y, p, \theta) = \max(x, x + ypf(-y, \theta) - \epsilon) \]

where \(\epsilon > 0\) is some fixed transaction fee. Constraints are:

- No short sale constraint:
  \[ Y_t \geq 0 , \forall t \in [0, T] \]

- Solvency constraint:
  \[ L(X_t, Y_t, P_t, \Theta_t) \geq 0 , \forall t \in [0, T] \]

We define the admissibility region:

\[ S = \{(z, \theta) = (x, y, p, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times (0, \infty) \times [0, T] : L(z, \theta) \geq 0 \text{ and } y \geq 0\} \]

Finally, we will call admissible the strategies \(\alpha\) in the following set:

\[ \mathcal{A} = \{\alpha = (\tau_n, \xi_n)_n : \forall t \in [0, T] \ (X_t^\alpha, Y_t^\alpha, P_t^\alpha, \Theta_t^\alpha) \in S\} \]
We choose a CRRA utility function $U(x) = x^{\gamma}$ with $\gamma \in (0, 1)$ and denote $U_L(.) = U(L(.))$

The value function is defined by (we denoted $z = (x, y, p)$):

$$v(t, z, \theta) = \sup_{\alpha \in A(t, z, \theta)} \mathbb{E}[U_L(Z_T)], \quad (t, z, \theta) \in [0, T] \times S$$
• From [4] \( v \) is a unique viscosity solution to a quasi-variational inequality (QVI) written as:

\[
\min \left[ -\frac{\partial}{\partial t} v - L v , v - H v \right] = 0, \quad \text{on} \quad [0, T) \times S,
\]

\[
\min [v - U_L, v - H v] = 0, \quad \text{on} \quad \{ T \} \times S.
\]

• \( L \) is the infinitesimal generator associated to the process \((X, Y, P, \Theta)\) in a no trading period:

\[
L \varphi = \frac{\partial}{\partial \theta} \varphi + b p \frac{\partial}{\partial p} \varphi + \frac{1}{2} \sigma^2 p^2 \frac{\partial^2}{\partial p^2} \varphi
\]

• \( H \) is the impulse operator:

\[
H \varphi(t, z, \theta) = \sup_{e \in C(t, z, \theta)} \varphi(t, \Gamma(z, \theta, e), 0)
\]

with \( \Gamma(z, \theta, e) = (x - epf(e, \theta) - \epsilon, y + e, p) \), \( z = (x, y, p) \in S, \ e \in \mathbb{R} \).

From now, our goal is to solve numerically this HJBQVI.
We propose the following discretization scheme:

\[
S^h(t, z, \theta, v^h(t, z, \theta), v^h) = 0, \quad (t, z, \theta) \in [0, T] \times \tilde{S},
\]

with

\[
S^h(t, z, \theta, r, \varphi)
\begin{cases}
\min \left[ r - \mathbb{E}[\varphi(t + h, Z_{t+h}^{0,t,z}, \Theta_{t+h}^{0,t,\theta})], \ r - \mathcal{H}\varphi(t, z, \theta) \right] & \text{if } t \in [0, T - h] \\
\min \left[ r - \mathbb{E}[\varphi(T, Z_T^{0,t,z}, \Theta_T^{0,t,\theta})], \ r - \mathcal{H}\varphi(t, z, \theta) \right] & \text{if } t \in (T - h, T) \\
\min \left[ r - U_L(z, \theta), \ r - \mathcal{H}\varphi(t, z, \theta) \right] & \text{if } t = T.
\end{cases}
\]

which can be formulated equivalently as an implicit backward scheme:

\[
v^h(T, z, \theta) = \max [U_L(z, \theta), \mathcal{H}v^h(T, z, \theta)],
\]

\[
v^h(t, z, \theta) = \max \left[ \mathbb{E}[v^h(t + h, Z_{t+h}^{0,t,z}, \theta + h)], \mathcal{H}v^h(t, z, \theta) \right], \quad 0 \leq t \leq T - h,
\]

and \(v^h(t, z, \theta) = v^h(T - h, z, \theta)\) for \(T - h < t < T\).
Explicit backward scheme

The usual way to treat implicit backward scheme is to solve by iterations a sequence of optimal stopping problems:

\[ v^{h,n+1}(T, z, \theta) = \max \left[ U_L(z, \theta), \mathcal{H}v^{h,n}(T, z, \theta) \right], \]
\[ v^{h,n+1}(t, z, \theta) = \max \left[ \mathbb{E}[v^{h,n+1}(t+h, Z^{0,t,z}_{t+h}, \theta + h)], \mathcal{H}v^{h,n}(t, z, \theta) \right], \]

starting from \( v^{h,0} = \mathbb{E}[U_L(Z^0_{T}, \Theta^0_{T})] \). Due to the effect of the lag variable \( \Theta_t \) in the market impact function, it is not optimal to trade immediately after a trade. Therefore we are able to write equivalently this scheme as an explicit backward scheme:

\[ v^h(T, z, \theta) = \max \left[ U_L(z, \theta), \mathcal{H}U_L(z, \theta) \right], \]
\[ v^h(t, z, \theta) = \max \left[ \mathbb{E}[v^h(t+h, Z^{0,t,z}_{t+h}, \theta + h)], \sup_{e \in C_\varepsilon(z, \theta)} \mathbb{E}[v^h(t + h, Z^{0,t,z^e}_{t+h}, h)] \right], \]

where \( z^e_{\theta} = \Gamma(z, \theta, e) \).
Convergence analysis

**Monotonicity**

The numerical scheme $S^h$ is monotone.

**Stability**

The numerical scheme $S^h$ is stable.

**Consistency**

The numerical scheme $S^h$ is consistent.

**Theorem: convergence**

The solution $v^h$ of the numerical scheme $S^h$ converges locally uniformly to $v$ on $[0, T) \times S$. 
We compute the conditional expectation arising in the numerical scheme $S^h$ using an optimal quantization method:

$$
\mathcal{E}^h(t_i, z, \theta_j) := \mathbb{E}[v^h(t_i + h, Z_{t_i+h}^{0:t_i,z}, \theta_j + h)] \\
= \mathbb{E}[v^h(t_i + h, x, y, p \exp ((b - \frac{\sigma^2}{2})h + \sigma \sqrt{h}U), \theta_j + h)],
$$

that we approximate by

$$
\mathcal{E}^h(t_i, z, \theta_j) \simeq \frac{1}{N} \sum_{k=1}^{N} \pi_k v^h(t_i + h, x, y, p \exp ((b - \frac{\sigma^2}{2})h + \sigma \sqrt{h}u_k), \theta_j + h).
$$

And we use weights ($\pi_k$) and values ($u_k$) from the optimal quantization of the normal law. From a computational point of view, this is reduced to a inner product computation, which is quite fast.
Algorithm

$$v^h(t, z, \theta) = \max \left[ \mathbb{E} \left[ v^h(t + h, Z_{t+h}^0, z, \theta + h) \right], H v^h(t, z, \theta) \right]$$
Policy shape in plane (Price × Shares)

Policy sliced in the (price, shares) plane
Performance analysis

We provide some numerical results that we obtain from our implementation. We tested the optimal strategy against a benchmark of two other strategies.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Utility $\hat{V}$</th>
<th>Mean $\hat{L}$</th>
<th>Standard Dev. $\hat{\sigma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Naive</td>
<td>0.99993</td>
<td>0.99986</td>
<td>0.00429</td>
</tr>
<tr>
<td>Uniform</td>
<td>0.99994</td>
<td>0.99988</td>
<td>0.00240</td>
</tr>
<tr>
<td>Optimal</td>
<td>1.00116</td>
<td>1.00233</td>
<td>0.00436</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Formula</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Winning percentage</td>
<td>$\frac{1}{Q} \sum_{i=1}^{Q} 1{L_{opt}^{(i)} &gt; \max(L_{naive}^{(i)}, L_{uniform}^{(i)})}$</td>
<td>58.8%</td>
</tr>
<tr>
<td>Utility Sharpe Ratio</td>
<td>$\frac{\hat{V}<em>{opt} - \max(\hat{V}</em>{naive}, \hat{V}<em>{uniform})}{\hat{\sigma}</em>{opt}}$</td>
<td>0.28017</td>
</tr>
<tr>
<td>Performance Sharpe Ratio</td>
<td>$\frac{\hat{L}<em>{opt} - \max(\hat{L}</em>{naive}, \hat{L}<em>{uniform})}{\hat{\sigma}</em>{opt}}$</td>
<td>0.56140</td>
</tr>
</tbody>
</table>
Performance analysis

Empirical distribution of performance

- Naive
- Uniform
- Optimal
Number of trades in one day liquidation

Empirical distribution of number of trades

![Empirical distribution of number of trades](image.png)
Behavior on historical data: price dependency

Figure: Strategy realization on the BNP.PA stock the 04/19/2010.
Behavior on historical data: price dependency

Figure: Strategy realization on the BNP.PA stock the 04/22/2010.
Bibliography


Any questions?