Numerical methods for an optimal order execution

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Joint work with H.Pham[1] and M.Mnif[2]

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Contents

Market impact and liquidation costs

- Implementation shortfall: difference between a theoretical trading strategy and its implementation
  $\Longrightarrow$ Need of low-touch execution strategy
- Market impact is key factor when executing large orders
- One famous worst case example: Kerviel's portfolio liquidation
- Basic observation: market impact is reduced when the liquidation operation is extended in time
- Idea is to split a big order into several small orders
  $\Longrightarrow$ trade-off between rapid execution (big impact but reduced risk) and gradual trading (reduced impact but more risk)

$\Longrightarrow$ Our goal is to find optimal trading schedule and associated quantities

Bibliographical review

- Almgren and Chriss (2001) [1] first provided a framework to manage market impact: mean-variance criterion, static strategy
- Several authors propose price impact models based on stylized dynamics of order book: Schied *et al.* (2009)[10], Gatheral *et al.* (2010)[3] and Obizhaeva and Wang (2005) [6].
- Some approaches using optimal control: Rogers and Singh (2008) [9], Forsyth (2009)[2], Ly Vath, Mnif and Pham (2007)[5], Predoiu, Shaikhet and Shreve (2010) [8], Kharroubi and Pham (2009)[4]

$\Longrightarrow$ In this talk we use the model investigated in this last paper: impulse control formulation.

The model

We consider a financial market where an investor has to liquidate an initial position of $y > 0$ shares of risky asset by time T.

- We set a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ supporting a one-dimensional Brownian motion W on a finite horizon $[0, T]$, $T < \infty$.
- $(P_t)_{t \in [0,T]}$ the market price of the risky asset
- $(X_t)_{t \in [0,T]}$ the cash holdings
- $(Y_t)_{t \in [0,T]}$ the number of stock shares held by the investor
- $(\Theta_t)_{t \in [0,T]}$ the time interval between t and the last trade before t

The model: trading strategies

- We represent a trading strategy by:

$$\alpha = (\tau_n, \xi_n)_{n\in\mathbb{N}}$$

  where (τ_n) are $\mathbb{F}$-stopping times and (ξ_n) are $\mathcal{F}_{\tau_n}$-measurable $\mathbb{R}$-valued variables.

- Dynamics for the shares and lag processes are under α:

$$\begin{aligned} \Theta_t &= t - \tau_n, \;\; \tau_n \leq t < \tau_{n+1} \\ \Theta_{\tau_{n+1}} &= 0, \;\; n \geq 0. \\ Y_s &= Y_{\tau_n}, \quad \tau_n \leq s < \tau_{n+1} \\ Y_{\tau_{n+1}} &= Y_{\tau_n} + \xi_{n+1}, \quad n \geq 0. \end{aligned}$$

The model: price impact

- Market price of risky asset process follows a geometric Brownian motion:

$$dP_t = P_t(bdt + \sigma dW_t)$$

- Suppose now that the investor decides to trade the quantity e. If the current market price is p, and the time lag from the last order is θ, then the price he actually get for the order e is:

$$Q(e, p, \theta) = pf(e, \theta)$$

with

$$f(e, \theta) = \exp\left(\lambda \left|\frac{e}{\theta}\right|^{\beta} \mathrm{sgn}(e)\right).\left(\kappa_a \mathbf{1}_{e>0} + \mathbf{1}_{e=0} + \kappa_b \mathbf{1}_{e<0}\right), \qquad (1)$$

- Therefore cash holdings have the following dynamics

$$\begin{aligned} X_t &= X_{\tau_n}, \quad \tau_n \le t < \tau_{n+1}, \quad n \ge 0. \\ X_{\tau_{n+1}} &= X_{\tau_{n+1}^-} - \xi_{n+1} P_{\tau_{n+1}} f(\xi_{n+1}, \Theta_{\tau_{n+1}^-}) - \epsilon, \quad n \ge 0. \end{aligned}$$

Liquidation value and solvency constraints

Let us define the admissibility constraints on a trading strategy. We first define the liquidative value of the position (x, y, p, θ) by:

$$L(x, y, p, \theta) = \max(x, x + ypf(-y, \theta) - \epsilon)$$

where $\epsilon > 0$ is some fixed transaction fee. Constraints are:

- No short sale constraint:

$$Y_t \geq 0 \ , \ \forall t \in [0, T]$$

- Solvency constraint:

$$L(X_t, Y_t, P_t, \Theta_t) \geq 0 \ , \ \forall t \in [0, T]$$

We define the admissibility region:

$$\mathcal{S} = \{(z, \theta) = (x, y, p, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times (0, \infty) \times [0, T] \ : L(z, \theta) \geq 0 \text{ and } y \geq 0\}$$

Finally, we will call admissible the strategies α in the following set:

$$\mathcal{A} = \{\alpha = (\tau_n, \xi_n)_n \ : \ \forall t \in [0, T] \ (X_t^\alpha, Y_t^\alpha, P_t^\alpha, \Theta_t^\alpha) \in \mathcal{S}\}$$

Optimal execution criterion

- We choose a CRRA utility function $U(x) = x^\gamma$ with $\gamma \in (0, 1)$ and denote $U_L(.) = U(L(.))$
- The value function is defined by (we denoted $z = (x, y, p)$):

$$v(t, z, \theta) = \sup_{\alpha \in \mathcal{A}(t,z,\theta)} \mathbb{E}\Big[U_L(Z_T)\Big], \quad (t, z, \theta) \in [0, T] \times \mathcal{S}$$

- From [4] v is a unique viscosity solution to a quasi-variational inequality (QVI) written as:

$$\begin{aligned} \min\left[-\frac{\partial}{\partial t}v - \mathcal{L}v\ ,\ v - \mathcal{H}v\right] &= 0, \quad \text{on} \quad [0,T)\times\mathcal{S}, \\ \min\left[v - U_L, v - \mathcal{H}v\right] &= 0, \quad \text{on} \quad \{T\}\times\mathcal{S}. \end{aligned}$$

- $\mathcal{L}$ is the infinitesimal generator associated to the process (X, Y, P, Θ) in a no trading period:

$$\mathcal{L}\varphi = \frac{\partial}{\partial\theta}\varphi + bp\frac{\partial}{\partial p}\varphi + \frac{1}{2}\sigma^2 p^2\frac{\partial^2}{\partial p^2}\varphi$$

- $\mathcal{H}$ is the impulse operator:

$$\mathcal{H}\varphi(t,z,\theta) = \sup_{e\in\mathcal{C}(t,z,\theta)} \varphi(t,\Gamma(z,\theta,e),0)$$

with $\Gamma(z,\theta,e) = (x - epf(e,\theta) - \epsilon, y + e, p),\ \ z = (x,y,p)\in\mathcal{S},\ \ e\in\mathbb{R}$

From now, our goal is to solve numerically this HJBQVI.

Discrete scheme

We propose the following discretization scheme:

$$S^h(t,z,\theta,v^h(t,z,\theta),v^h) \quad = \quad 0, \quad (t,z,\theta) \in [0,T] \times \bar{\mathcal{S}},$$

with

$$\begin{aligned} & S^h(t,z,\theta,r,\varphi) \\ := & \begin{cases} \min\left[r - \mathbb{E}\left[\varphi(t+h, Z^{0,t,z}_{t+h}, \Theta^{0,t,\theta}_{t+h})\right] \, , \, r - \mathcal{H}\varphi(t,z,\theta)\right] & \text{if } t \in [0, T-h] \\ \min\left[r - \mathbb{E}\left[\varphi(T, Z^{0,t,z}_{T}, \Theta^{0,t,\theta}_{T})\right] \, , \, r - \mathcal{H}\varphi(t,z,\theta)\right] & \text{if } t \in (T-h, T) \\ \min\left[r - U_L(z,\theta) \, , \, r - \mathcal{H}\varphi(t,z,\theta)\right] & \text{if } t = T. \end{cases} \end{aligned}$$

which can be formulated equivalently as an implicit backward scheme:

$$\begin{aligned} v^h(T,z,\theta) &= \max\left[U_L(z,\theta) \, , \, \mathcal{H}v^h(T,z,\theta)\right], \\ v^h(t,z,\theta) &= \max\left[\mathbb{E}\left[v^h(t+h, Z^{0,t,z}_{t+h}, \theta+h)\right] , \mathcal{H}v^h(t,z,\theta)\right], \quad 0 \le t \le T-h, \end{aligned}$$

and $v^h(t,z,\theta) = v^h(T-h,z,\theta)$ for $T-h < t < T$.

Explicit backward scheme

The usual way to treat implicit backward scheme is to solve by iterations a sequence of optimal stopping problems:

$$\begin{aligned} v^{h,n+1}(T,z,\theta) &= \max\left[U_L(z,\theta)\ ,\ \mathcal{H}v^{h,n}(T,z,\theta)\right], \\ v^{h,n+1}(t,z,\theta) &= \max\left[\mathbb{E}\left[v^{h,n+1}(t+h, Z^{0,t,z}_{t+h}, \theta+h)\right]\ , \mathcal{H}v^{h,n}(t,z,\theta)\right], \end{aligned}$$

starting from $v^{h,0} = \mathbb{E}[U_L(Z_T^{0,t,z}, \Theta_T^{0,t,\theta})]$. Due to the effect of the lag variable Θ_t in the market impact function, it is not optimal to trade immediately after a trade. Therefore we are able to write equivalently this scheme as an explicit backward scheme:

$$\begin{aligned} v^{h}(T,z,\theta) &= \max\left[U_L(z,\theta)\ ,\ \mathcal{H}U_L(z,\theta)\right], \\ v^{h}(t,z,\theta) &= \max\left[\mathbb{E}\left[v^{h}(t+h, Z^{0,t,z}_{t+h}, \theta+h)\right]\ ,\ \sup_{e\in\mathcal{C}_\varepsilon(z,\theta)} \mathbb{E}\left[v^{h}(t+h, Z^{0,t,z^e_\theta}_{t+h}, h)\right]\right], \end{aligned}$$

where $z^e_\theta = \Gamma(z,\theta,e)$

Convergence analysis

Monotonicity

The numerical scheme S^h is monotone.

Stability

The numerical scheme S^h is stable.

Consistency

The numerical scheme S^h is consistent.

Theorem: convergence

The solution v^h of the numerical scheme S^h converges locally uniformly to v on $[0, T) \times \mathcal{S}$.

Implementation

We compute the conditional expectation arising in the numerical scheme S^h using an optimal quantization method:

$$\begin{aligned}\mathcal{E}^h(t_i, z, \theta_j) &:= \mathbb{E}\Big[v^h(t_i + h, Z^{0,t_i,z}_{t_i+h}, \theta_j + h)\Big] \\ &= \mathbb{E}\Big[v^h(t_i + h, x, y, p\exp\big((b - \frac{\sigma^2}{2})h + \sigma\sqrt{h}U\big), \theta_j + h)\Big],\end{aligned}$$

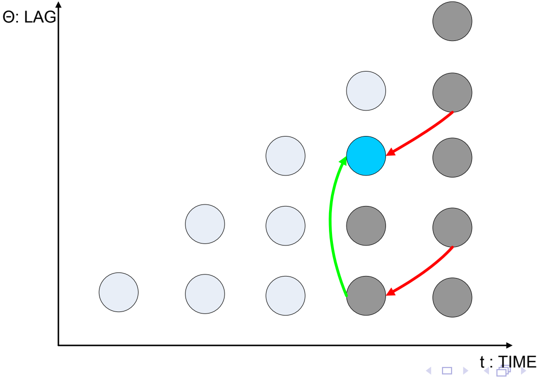
that we approximate by

$$\mathcal{E}^h(t_i, z, \theta_j) \simeq \frac{1}{N}\sum_{k=1}^{N} \pi_k \; v^h(t_i + h, x, y, p\exp\big((b - \frac{\sigma^2}{2})h + \sigma\sqrt{h}u_k\big), \theta_j + h).$$

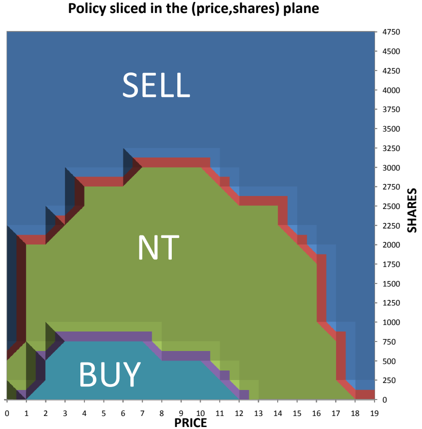
And we use weights (π_k) and values (u_k) from the optimal quantization of the normal law. From a computational point of view, this is reduced to a inner product computation, which is quite fast.

Algorithm

$$v^h(t,z,\theta) = \max\left[\mathbb{E}\left[v^h(t+h, Z^{0,t,z}_{t+h}, \theta+h)\right], \mathcal{H}v^h(t,z,\theta)\right]$$



Policy shape in plane (Price $\times$ Shares)



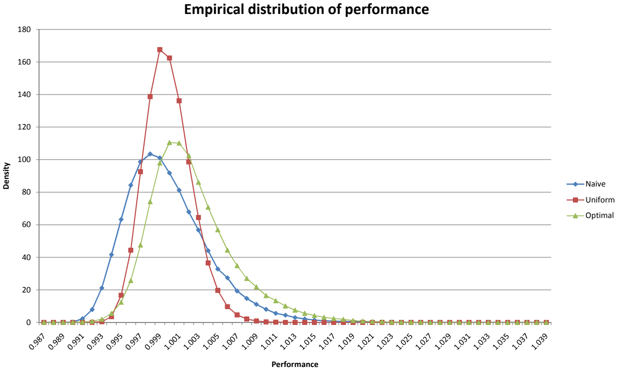
Performance analysis

We provide some numerical results that we obtain from our implementation. We tested the optimal strategy against a benchmark of two other strategies.

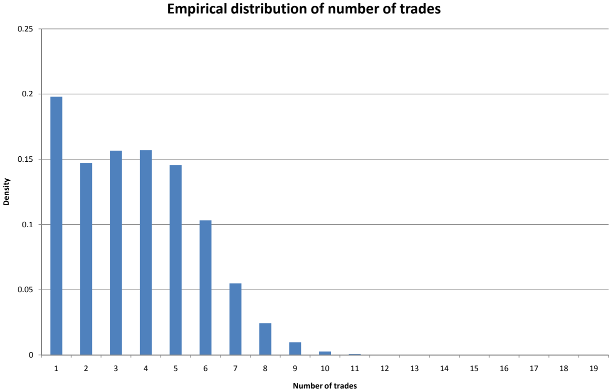
Strategy	Utility $\hat{V}$	Mean $\hat{L}$	Standard Dev. $\hat{\sigma}$
Naive	0.99993	0.99986	0.00429
Uniform	0.99994	0.99988	0.00240
Optimal	1.00116	1.00233	0.00436

Quantity	Formula	Value
Winning percentage	$\frac{1}{Q}\sum_{i=1}^{Q} 1_{\{L_{opt}^{(i)} > \max(L_{naive}^{(i)}, L_{uniform}^{(i)})\}}$	58.8%
Utility Sharpe Ratio	$\frac{\hat{V}_{opt} - \max(\hat{V}_{naive}, \hat{V}_{uniform})}{\hat{\sigma}_{opt}}$	0.28017
Performance Sharpe Ratio	$\frac{\hat{L}_{opt} - \max(\hat{L}_{naive}, \hat{L}_{uniform})}{\hat{\sigma}_{opt}}$	0.56140

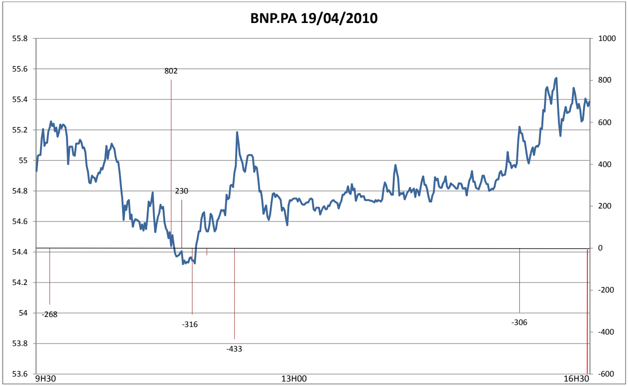
Performance analysis



Number of trades in one day liquidation



Behavior on historical data: price dependency



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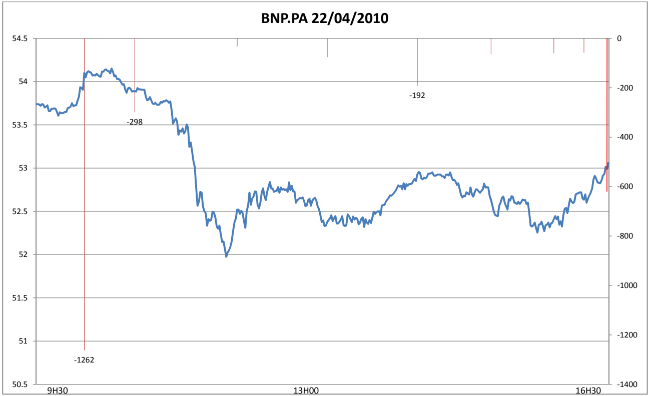

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Thank you for your attention

Any questions?