Generalized stochastic target problems for pricing and partial hedging under loss constraints - Application in optimal book liquidation

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Modeling and managing financial risks
Paris, January 10 - 13, 2011

VWAP guaranteed contract

- Liquidation K stocks during $[0, T]$.
- Guarantee a better than δ basis point w.r.t market VWAP:

$$\text{Guaranteed VWAP} = \underbrace{\left(1 + \delta \cdot 10^{-4}\right)}_{\gamma} \text{VWAP}_{\text{mkt}} \,.$$

- Brokerage fee: ask for a *premium* such that, up to a functional:

$$\textit{premium} + \textit{realized gain} \geq 0 \text{ a.s.} \tag{1}$$

Mathematical modeling

- VWAP: Volume Weighted Average Price
- Cumulative trading volume L_t: continuous real-valued, non decreasing adapted process and

$$L_0 = 0 \, , L_T = K \; . \tag{2}$$

- Stock price $X^{L,1}(t)$
- Cumulative market volume: $\Theta(t) := \int_0^t \vartheta(s) ds$.

$$\begin{aligned} \text{Execution turnover} : dY^L(t) &= X^{L,1}(t) dL_t \\ \text{Market turnover} : dX^{L,2}(t) &= X^{L,1}(t) d\Theta_t = X^{L,1}(t)\vartheta(t) dt \\ Y^L(0) &= X^{L,2}(0) = 0 \; . \end{aligned}$$

Mathematical modeling cont.

- Due to *price impact* effect: stock price is influenced by trading activity
- In case of linear impact

$$\frac{dX^{L,1}(t)}{X^{L,1}(t)} = \mu(t, X^{L,1}(t))dt + \sigma(t, X^{L,1}(t))dW_t - \beta(t, X^{L,1}(t))dL_t \ .$$

- Realized gain in cash

$$\left(\frac{Y^L(T)}{K} - \gamma\frac{X^{L,2}(T)}{\Theta(T)}\right)K \ .$$

VWAP benchmark

- VWAP: involve (price, proportion of volume) jointly
- S: stock price, $(v_t)_{t\in[0,T]}$: density of trading volume
- Define

$$p_t = \frac{\int_0^t v_u du}{\int_0^T v_u du} \text{ then } p_0 = 0, p_T = 1 \text{ and VWAP } = \int_0^T S_t dp_t \ .$$

- VWAP benchmark, heuristically, minimize the following quantity (p^X, p^M: broker and market trading curve)

$$\int_0^T S_t dp_t^X - \int_0^T S_t dp_t^M$$

$\Longrightarrow$ Follow the market " $\equiv$ " minimize $\|p^X - p^M\|$.

Volume curve

- In reality, unable to determine $(p_t^M), t \in [0, T]$ before T!
- One of possible proxies:

$$p_t^X \in [l_t^M, u_t^M] \ \ : \ \text{so-called } \textit{trading envelope}.$$

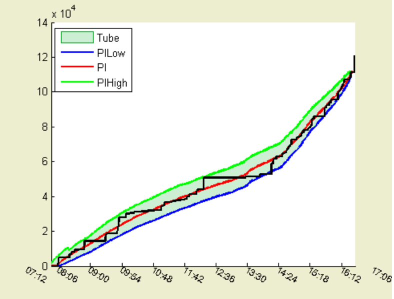

Figure: Trading envelopes

Mathematical modeling

- Introduce $X^{L,3}$:
$$X^{L,3}(t) \;=\; L_t - L_0 \;.$$
- Then require
$$X^{L,3}(s) \in [\underline{\Lambda}(s), \overline{\Lambda}(s)] \quad \text{for all } s \leq T \;, \tag{3}$$
where
$$\underline{\Lambda} \;<\; \overline{\Lambda} \text{ on } [0,T) \quad \text{and} \quad \underline{\Lambda}(T) = \overline{\Lambda}(T) = K \;.$$
- For future utilization, also require:
$$D\underline{\Lambda}, D\overline{\Lambda} \in (0, M] \text{ on } [0,T] \text{ for some } M > 0 \;.$$

Summarized problem

- For a control variable L, consider $(Y, X = (X^{L,1}, X^{L,2}, X^{L,3}), \Theta)$ whose dynamics are given as above.
- Given a function ℓ, finding y minimum such that (Equation (1))

$$\ell\left(y + \left(\frac{Y^L(T)}{K} - \gamma\frac{X^{L,2}(T)}{\Theta(T)}\right)K\right) \geq 0\,,$$

$\Longrightarrow$ Stochastic Target

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- and also

equation (2), (3) hold : $X^{L,3}(s) \in [\underline{\Lambda}(s), \bar{\Lambda}(s)] \quad$ for all $s \leq T$.

$\Longrightarrow$ Stochastic Target under State Constraints problem

Outline

Problem formulation

- Given $\phi = (\nu, L) \in \mathcal{A} := \mathcal{U} \times \mathcal{L}$ the set of controls
- $Z^\phi = (X^\phi, Y^\phi) \in \mathbb{R}^d \times \mathbb{R}$ verifies:

$$dX^\phi = \mu_X(X^\phi, \nu_r)dr + \beta_X(X^\phi)dL + \sigma_X(X^\phi, \nu_r)dW_r$$
$$dY^\phi = \mu_Y(Z^\phi, \nu_r)dr + \beta_Y(Z^\phi)^\top dL + \sigma_Y(Z^\phi, \nu_r)^\top dW_r \ .$$

- Under Standing Assumption on constraint set:

$$\forall (t,x) : (x,y) \in O(t) \ , \ y' \geq y \Rightarrow (x, y') \in O(t) \ .$$

- Then

$$V(t) := \left\{ (x,y) : Z^\phi_{t,x,y}(s) \in O(s) \ \ \forall t \leq s \leq T \right\} \ .$$

is equivalent to

$$v(t,x) := \inf \{ y : (x,y) \in V(t) \} \ .$$

Super-hedging with cash delivery and proportional costs

- Let $d = 2$ and $\mu^i_X(x,u) = x^i\mu$, $\sigma^i_X(x,u) = x^i\sigma$, for $i \in \{1,2\}$,

$$\beta^{21}_X(x) = -1\ ,\ \beta^{22}_X(x) = 1\ ,\ \beta_Y(x,y) = (1-\lambda, -1-\lambda)\ .$$

- Then, $X^{L,1}$ follows a Black-Scholes dynamics, and

$$\begin{aligned} X^{2,L}_{t,x}(s) &= x^2 + \int_t^s \frac{X^{2,L}_{t,x}(r)}{X^1_{t,x}(r)} dX^1_{t,x}(r) - \int_t^s dL^1_r + \int_t^s dL^2_r \\ Y^L_{t,y}(s) &= y + \int_t^s (1-\lambda) dL^1_r - \int_t^s (1+\lambda) dL^2_r\ . \end{aligned}$$

- Define $O(t) = \mathbf{1}_{t<T}\mathbb{R}^*_+ \times \mathbb{R}^2 + \mathbf{1}_{t=T}\{(x,y) : \Lambda(y,x) \geq g(x)\}$ with $\Lambda(y,x) := y + (1-\lambda)x^2$.

Loss function pricing

- Super-hedging criteria is too strict in markets with proportional costs
- Given a non-decreasing function ℓ, price at time t:

$$\hat{v}(t,x;p) := \inf\left\{y : \mathbb{E}\left[\ell\left(\Lambda(Y^L(T),X^L(T)) - g(X^1(T))\right)\right] \geq p\right\}$$

- Bouchard-Elie-Touzi (2009) shows that

$$\hat{v}(t,x;p) := \inf\left\{y : G^L_{t,x,y}(T) \geq P^{\nu}_{t,p}(T)\ ,(\nu,L) \in \mathcal{U}\times\mathcal{L}\right\}$$

where

$$G^L_{t,x,y}(T) = \ell\left(\Lambda(Y^L_{t,y}(T),X^L_{t,x}(T)) - g\left(X^1_{t,x}(T)\right)\right) \in L^2$$

$$P^{\nu}_{t,p} := p + \int_t^{\cdot} \nu^1_s dW^1_s\ .$$

Geometric dynamic programming

- Firstly introduced by Soner and Touzi for super-hedging under Gamma constraints
- Extended to American type constraints: obstacle version of Bouchard-Vu (2010)

Theorem:

$$V(t) = \left\{ z \ : \ \exists\, \phi \in \mathcal{A} \ \text{ s.t. } \ Z^{\phi}_{t,z}(\theta \wedge \tau) \in O \overset{\tau,\theta}{\bigoplus} V \ \text{ for all } \ \theta, \tau \in \mathcal{T}_{[t,T]} \right\}$$

$$O \overset{\tau,\theta}{\bigoplus} V := O(\tau)\, 1_{\tau \le \theta} + V(\theta)\, 1_{\tau > \theta} \ \text{ for } \theta, \tau \in \mathcal{T}_{[0,T]} \, .$$

Geometric dynamic programming

- Firstly introduced by Soner and Touzi for super-hedging under Gamma constraints
- Extended to American type constraints: obstacle version of Bouchard-Vu (2010)

Theorem: For all $\phi \in \mathcal{U} \times \mathcal{L}$ and $\theta \in \mathcal{T}_{[t,T]}$

1. GDP1: $y > v(t,x)$

$$Y^{\phi}_{t,x,y}(\theta) \geq v(\theta, X^{\phi}_{t,x}(\theta)) \text{ and } Z^{\phi}_{t,x,y} \in O \;\forall\, s \in [t,\, T]\,.$$

2. GDP2: $y < v(t,x)$, then $\forall\, (\phi,\theta) \in \mathcal{A} \times \mathcal{T}_{[t,T]}$

$$\mathbb{P}\left[Y^{\phi}_{t,x,y}(\theta) > v(\theta, X^{\phi}_{t,x}(\theta)) \text{ and } Z^{\phi}_{t,x,y} \in O \;\forall\, s \in [t,\, T]\right] < 1\,.$$

Interior of the domain

In the case where $\beta_X = \beta_Y = 0$:

- $y = v(t,x)$, GDP implies that $\exists \nu \in \mathcal{U}$ such that:

$$\text{for } \phi = (\nu, 0) \text{ then } dY^{\phi}_{t,x,y}(t) \geq dv(t, X^{\phi}_{t,x}(t)) \; . \tag{4}$$

- Formally:

$$dY^{\phi}(t) = \mu^{\phi}_Y(Z^{\phi}, \nu_t)dt + \sigma_Y(Z^{\phi}, \nu_t)dW_t$$
$$dv(t, X^{\phi}(t)) = \mathcal{L}^{\nu_t}_X v(t, X^{\phi})dt + Dv(t, X^{\phi}(t))^{\top}\sigma_X(X^{\phi}, \nu_t)dW_t$$

- Inequality (4) suggests:

$$\mu_Y(Z^{\phi}, \nu_t) \geq \mathcal{L}^{\nu_t}_X v(t, X^{\phi})$$
$$\text{and } \sigma_Y(Z^{\phi}, \nu_t) = \sigma_X(X^{\phi}, \nu_t)^{\top} Dv(t, X^{\phi}(t)) \; ,$$

Interior of the domain

In the case where $\beta_X, \beta_Y \neq 0$:

- $y = v(t,x)$, GDP implies that $\exists \nu \in \mathcal{U}$ such that:

$$\text{for } \phi = (\nu, L) \text{ then } dY^{\phi}_{t,x,y}(t) \geq dv(t, X^{\phi}_{t,x}(t)) \; . \tag{4}$$

- Formally:

$$dY^{\phi}(t) = \cdots + \beta_Y(Z^{\phi}) dL_t$$
$$dv(t, X^{\phi}(t)) = \cdots + Dv(t, X^{\phi}(t))^{\top} \beta_X(X^{\phi}) dL_t$$

- Inequality (4) suggests:

$$\mu_Y(Z^{\phi}, \nu_t) \geq \mathcal{L}_X^{\nu_t} v(t, X^{\phi})$$
$$\text{and } \sigma_Y(Z^{\phi}, \nu_t) = \sigma_X(X^{\phi}, \nu_t)^{\top} Dv(t, X^{\phi}(t)) \; ,$$

but in our case, also

$$\left(\beta_Y(Z^{\phi})^{\top} - Dv(t, X^{\phi}(t))^{\top} \beta_X(X^{\phi}) \right) dL_t \geq 0 \; .$$

Interior of the domain cont.

Define

$$F_\varepsilon^u := \sup\{\mu_Y(x,y,u) - \mathcal{L}_X^u v\ ,\ u \in N_\varepsilon v\}$$
$$G := \max\left\{[\beta_Y(z)^\top - Dv(t,x)\beta_X(x)]\ell\ ,\ \ell \in \Delta_+\right\}$$

where

$$N_\varepsilon v := \{u \in U\ :\ |\sigma_Y(\cdot,v,u) - Dv\sigma_X(\cdot,u)| \leq \varepsilon\}$$
$$\Delta_+ := \mathbb{R}_+^d \cup \partial B_1(0)\ .$$

then PDE characterization in the interior of the domain:

$$\max\{F_0 v, Gv\} = 0 \text{ on } (t,x,v(t,x)) \in \operatorname{int}(D)$$
$$\text{where } D := \{(t,x,y)\ :\ (x,y) \in O(t)\}\ .$$

On the boundaries of the domain

- Suppose $\exists \delta \in C^{1,2}$ such that $\delta = 0$ uniquely in ∂D, takes opposite sign inside and outside of D
- Then the state constraints require:

$$d\delta(Z^{\phi}_{t,z}(t)) \geq 0 \text{ if } (t,z) \in \partial D \ .$$

- Hence,

$$\mathcal{L}^{u}_{Z}\delta(t,x,y) \geq 0 \text{ and } D\delta(t,x,y)\sigma_Z(x,y,u) = 0 \ .$$

- Define F^{in}_0 and G^{in} similarly as above, then PDE:

$$\max\left\{F^{\text{in}}_0 v, G^{\text{in}} v\right\} = 0 \text{ on } (t,x,v(t,x)) \in \partial D \ .$$

- Terminal condition + Relax all operators in $(\varepsilon, t, x, v, Dv, D^2 v) \ldots$

Mathematical modeling of the VWAP liquidation problem

Value function:

$$v(t,x,p) := \inf\left\{ y : \exists L \in \mathcal{L} \;\middle|\; X^{3,L}_{t,x} \in [\underline{\Lambda}, \overline{\Lambda}] \text{ and } \mathbb{E}\left[\Psi(Z^L_{t,x,y}(T))\right] \geq p \right\}$$

$$\text{where } \Psi(x,y) = \ell(y - \gamma x^2) \; .$$

Theorem:

$$v(t,x,p) := \inf\{y \geq 0 \;:\; \mathcal{A}_{t,x,y,p} \neq \emptyset\} \; ,$$

where $\mathcal{A}_{t,x,y,p} := \left\{(\nu, L) \in \mathcal{A} \;\middle|\; (Z^L_{t,x,y}, P^\nu_{t,p}) \in V \text{ on } [t,T]\right\}$ with

$$V := \left\{(x,y,p) \;:\; x^3 \in [\underline{\Lambda}, \overline{\Lambda}]\right\}\mathbf{1}_{[0,T)} + \left\{(x,y,p) \;:\; x^3 = K \text{ and } \ell(y-\gamma x^2) \geq p\right\}\mathbf{1}_{\{T\}} \; ,$$

and $P^\nu_{t,p} := p + \int_t^\cdot \nu_s dW_s$.

PDE characterization

Proposition: Under "good assumption", v_* is a viscosity supersolution of

$$\max\left\{F_0\varphi\ ,\ x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi\right\} = 0\ .$$

And v^* is a subsolution of

$$\begin{array}{rll} \min\left\{\varphi\ ,\ \max\left\{F_0\varphi\ ,\ x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi\right\}\right\} = 0 & \text{if} & \underline{\Lambda} < x^3 < \overline{\Lambda} \\ \min\left\{\varphi\ ,\ x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi\right\} = 0 & \text{if} & \underline{\Lambda} = x^3 \\ \min\left\{\varphi\ ,\ F_0\varphi\right\} = 0 & \text{if} & x^3 = \overline{\Lambda}\ . \end{array}$$

Moreover,

$$v_*(T, x, p) = v^*(T, x, p) = \Psi^{-1}(x, p) \text{ for all } (x, p) \in [0, \infty)^2 \times \{K\} \times \mathbb{R}\ .$$

Additional assumption

- Operator F_0 defined as:

$$F_0\varphi := -\mathcal{L}_X\varphi - \frac{(x^1\sigma)^2}{2}\left(|D_{x^1}\varphi/D_p\varphi|^2 D_p^2\varphi - 2(D_{x^1}\varphi/D_p\varphi)D^2_{(x^1,p)}\varphi\right)$$

with

$$\mathcal{L}_X\varphi := \partial_t\varphi + x^1\mu D_{x^1}\varphi + x^1\vartheta D_{x^2}\varphi + \frac{1}{2}(x^1\sigma)^2 D^2_{x^1}\varphi \ .$$

- "Good assumption" on loss function ℓ

$$\exists\epsilon > 0 \text{ s.t. } \epsilon \le D^-\ell\ ,\ D^+\ell \le \epsilon^{-1}\ ,$$
$$\text{and } \lim_{r\to\infty} D^+\ell(r) = \lim_{r\to\infty} D^-\ell(r) =: D\ell(\infty)\ .$$

- Also other conditions on boundaries $\underline{\Lambda}, \overline{\Lambda}$.

A priori estimates

- Proposition: $h \in (-(x^1 \wedge 1), 1)$

$$v(t,x,p) \geq \quad v(t,x,p-\epsilon^{-1}|h|) + |h|$$

- Corollary:

v_* is a viscosity supersolution of

$$D_p\varphi - \epsilon \qquad = 0, (*)$$

and v^* is a viscosity subsolution of

$$-D_p\varphi + \epsilon \qquad = 0 (**)$$

A priori estimates

- Proposition: $h \in (-(x^1 \wedge 1), 1)$, $\exists C$ depends only in x s.t.

$$v(t,x,p) \geq v(t, x + he_1, p - C(x)|h|)$$

- Corollary:

v_* is a viscosity supersolution of

$$\min\{(D_{x^1}\varphi - C(x)D_p\varphi)\mathbf{1}_{x^1>0}, -D_{x^1}\varphi + C(x)D_p\varphi\} = 0, (*)$$

and v^* is a viscosity subsolution of

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A priori estimates

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and v^* is a viscosity subsolution of

$$\max\left\{ -D_p\varphi + \epsilon, (D_{x^1}\varphi - C(x)D_p\varphi)\mathbf{1}_{x^1>0}, -D_{x^1}\varphi + C(x)D_p\varphi \right\} = 0(**)$$

Comparison principle and uniqueness

Assumption: $\exists\ \hat{x}^1 > 0$ s.t. $\mu(\cdot, \hat{x}^1) = \sigma(\cdot, \hat{x}^1) = 0$.

Theorem: U (resp. V) non-negative lower-semicontinuous supersolution(resp. upper-semicontinuous subsolution) and continuous in x^3. Assume that

$$U(t,x,p) \geq V(t,x,p) \text{ if } t = T \text{ or } x^1 \in \{0, 2\hat{x}^1\},$$

and $\exists c_+ > 0, c_- \in \mathbb{R}$ such that

$$\limsup_{(t',x',p')\to(t,x,\infty)} V(t',x',p')/p' \leq c_+ \leq \liminf_{(t',y',p')\to(t,y,\infty)} U(t',y',p')/p' \ ,$$

$$\limsup_{(t',x',p')\to(t,x,-\infty)} V(t',x',p') \leq c_- \leq \liminf_{(t',y',p')\to(t,y,-\infty)} U(t',y',p') \ .$$

If either U is a viscosity supersolution of $(*)$ and continuous in p, or V is a viscosity subsolution of $(**)$ and continuous in p, then

$$U \geq V \ .$$

Conclusion

- Propose generalized stochastic target problem:
  - controls in the form of bounded variation process
  - under state constraint.
- Suitable framework for:
  - pricing derivatives under loss constraint
  - models involving liquidity costs.
- Application in optimal liquidation: Pricing guaranteed VWAP contract with trading envelopes.
- Under "good assumptions", comparison holds.
- Work on numerical resolution is in progress.

Thank you for your attention

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