# Running Supremum, DrawDown Constraint, Azéma-Yor Processes, Max-Plus decomposition, and financial applications

jointed work with I.Karatzas (in the past), A.Meziou, J.Obloj

to Marc Yor

# Geometrical Brownian motion and Running supremum

## Third Lesson in the master program

Let $S_t$ be a geometrical Brownian motion, such that $S^\gamma$ is a martingale and $\overline{S} = \sup_{u \le t} S_u$. it running supremum.

- By the symmetry principle, we have

$$\begin{aligned} &\mathbb{P}(S_T \le K, S_T^* \ge H) = \mathbb{P}(S_T \le K, T_H \le T) \\ &= \left(\frac{x}{H}\right)^\gamma \mathbb{P}\left(\frac{H^2}{x^2} S_T \le K\right) = \left(\frac{x}{H}\right)^\gamma \mathcal{N}\left(\delta_1\left(\frac{Kx}{H^2}, \sigma, T\right)\right), \end{aligned}$$

and

$$\mathbb{P}(S_T \le K) = \mathcal{N}\left(\delta_1\left(\frac{x}{K}, \sigma, T\right)\right) = \mathcal{N}\Big(-\delta_0\big(\frac{K}{x}, \sigma, T\big)\Big).$$

**Theorem**

The tail function de $\overline{S}_T$ given $\{S_T = K\}$ is given for $x, K \leq H$ by

$$\mathbb{P}(\mathbf{S^*_T} \geq \mathbf{H} \mid \mathbf{S_T} = \mathbf{K}) = \exp\left(-\frac{\mathbf{2}}{\sigma^{\mathbf{2}}\mathbf{T}}\mathrm{Ln}(\frac{\mathbf{K}}{\mathbf{H}})\mathrm{Ln}(\frac{\mathbf{x}}{\mathbf{H}})\right)$$

- Very useful for instance in Mont Carlo simulation of Barrier Option
- The proof is not completely immediate...

# Azéma-Yor Processes

# Azéma-Yor Processes (1979)

As usual, $(\Omega, \mathcal{F}_t, \mathbb{P})$ is a filtered probability space, satisfying usual assumptions.

## Notation and basic properties

- The **running supremum** or maximum process of some adapted cadlag process $X$ is defined as

$$\overline{X}_t = \sup_{u \leq t} X_u.$$

Between two dates, we write $\overline{X}_{s,t} = \sup_{s<u\leq t} X_u$.

**Properties**

$\Rightarrow$ $\overline{X}_t$ is an increasing process, right-continuous, with the "max-additivity" property $\overline{X}_t = \overline{X}_s \vee \overline{X}_{s,t}$.

$\Rightarrow$ When $\overline{X}_t$ is a continuous process, for instance when the process $X$ has only negative jumps, the process $\overline{X}_t$ only increases when $\overline{X}_t = X_t$, that is

$$\int_0^T (\overline{X}_t - X_t) d\overline{X}_t = 0$$

Let $u$ be a locally bounded Borel function. The primitive function $U(x) = a^* + \int_{(a,x]} u(s)\, ds$ is defined on $[a, \infty)$.

**Definition of AY Process**

Let $X$ be a cadlag semimartinale with **continuous** running supremum $\overline{X}_t = \sup_{u \leq t} X_u$, and $u$ a locally bounded function.
The $(U, X)$-Azéma-Yor process is defined by one of these two equations

$$M_t^U(X) = U(\overline{X}_t) + u(\overline{X}_t)(X_t - \overline{X}_t) \quad (1)$$

$$\text{or} = a^* + \int_0^t u(\overline{X}_t) dX_s \quad (2)$$

If $X$ is a local martingale, $M_t^U$ is also a local martingale.

## Main properties

⇒ The equivalence between the two equations is straightforward when $U$ is a regular function, since from Itô's formula

$$\begin{aligned} dM_t^U(X) &= u(\overline{X}_t)d\overline{X}_t + u(\overline{X}_t)(dX_t - d\overline{X}_t) + (X_t - \overline{X}_t)u'(\overline{X}_t)d\overline{X}_t \\ &= u(\overline{X}_t)dX_t \end{aligned}$$

⇒ The case of locally integrable function $u$ can be attained for continuous local martingale $X$ (Obloj,Yor 2004)

# Bachelier equation

## Non decreasing transformation

Let $\mathcal{U}_m$ be the set of primitive function $U$ of non negative locally bounded functions $u$, and $\mathcal{G}_m$ the subgroup of increasing functions $U$ s.t. the increasing inverse function $V$ of $U$, with first right-hand derivative $V' := v$ is in $\mathcal{U}_m$.

- Let $U$ be in $\mathcal{U}_m$, $X$ be a max-continuous semimartingale. The (U,X)-Azéma-Yor process $(M_t^U(X))$ is a max-continuous semimartingale since,

$$\overline{M_t^U(X)} = \overline{U(\overline{X}_t)} = U(\overline{X}_t),$$

- Pick $F$ in $\mathcal{U}_m$. Then, $\mathbf{M_t^U(M^F(X)) = M_t^{U \circ F}(X)}$.
- Moreover, the processes $M^U(X)$ associated with $U \in \mathcal{G}_m$ is a group under the multiplication $\otimes$ defined by

$$M^U \otimes M^F := M^{U \circ F}.$$

- If $u$ is only defined on $[a,b)$, $M^U(X)$ may be defined up to the exit time $T_b$ of $[a,b)$ by $X$.
- If $u$ is non negative, $\overline{M^U(X)}_{t\wedge T_b} = U(\overline{X}_{t\wedge T_b})$

## Bachelier equation

- By the property of the inverse, $u \circ V = 1/V' = 1/v$
- Since $\overline{M}_t^U = U(\overline{X}_t)$, $u(\overline{X}_t) = u \circ V(U(\overline{X}_t)) = (1/v)(\overline{M}_t^U)$.

The AY-process is a solution of

$$dM_t^U = (1/v)(\overline{\mathbf{M}}_{\mathbf{t}}^{\mathbf{U}})dX_t$$

Such equations were first introduced by Bachelier in 1906.

**Definition:** Let $\phi : [a^*, \infty)$ be a locally bounded away from 0 function and $X$ as below. The Bachelier equation is

$$dY_t = \phi(\overline{Y}_t)dX_t, \qquad Y_0 = a^*$$

**Existence**

$\Rightarrow$ $M_t^U$ is a solution associated with $\phi = \mathbf{1/v}$.

$\Rightarrow$ Conversely, given $\phi : [a^*, \infty) \to (0, \infty)$ be a Borel function locally bounded away from zero, $v = 1/\phi$ and $V$ a primitive of $v$. Then the inverse function $U$ of $V$ is defined on $(a^*, V(\infty))$.
$Y_t = M_t^U(X)$ is a solution of the Bachelier equation on $(0, T_{V_\infty})$.

# Example

- $X$ is a geometrical Brownian motion with volatility $\sigma$,
- $U$ is the power function $U(x) = x^\gamma, \gamma < 1$

Then,

$\Rightarrow$ The AY Process $\mathbf{Y_t = M^U(X_t)} = \overline{X}_t^\gamma(1-\gamma) + \gamma(\overline{X}_t)^{\gamma-1}X_t$ is also given by $Y_t = \overline{Y}_t\Big[(1-\gamma) + \gamma\left(\frac{Y_t}{\overline{Y}_t}\right)^{1/\gamma}\Big]$

$\Rightarrow$ The process $Z_t = X_t^\gamma$ is a supermartingale, with dynamic

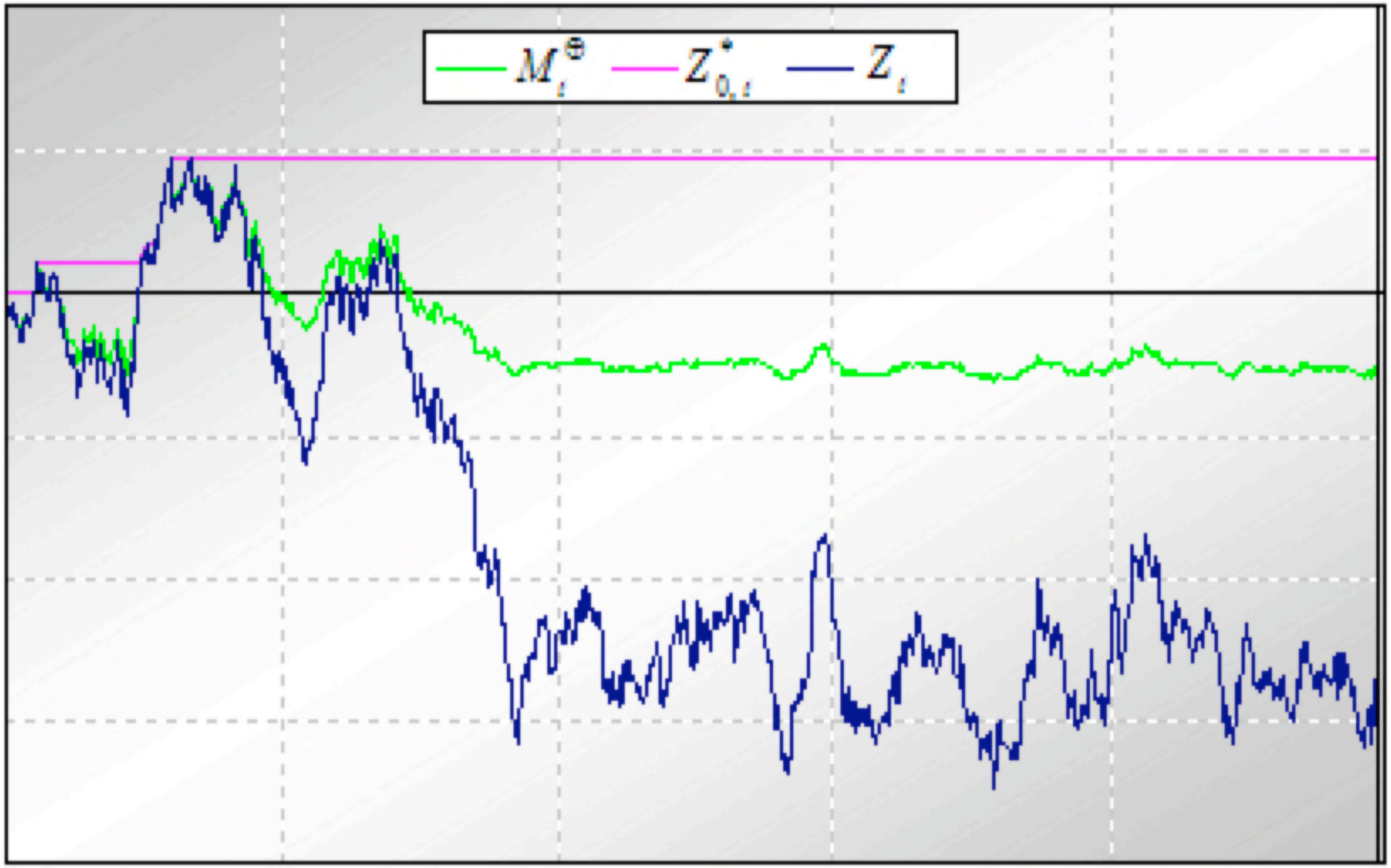
$$dZ_t = \gamma Z_t\left(\frac{dX_t}{X_t} - \frac{1}{2}(1-\gamma)\sigma^2 dt\right)$$

The martingale $\mathbf{Y_t}$ is still **above** the supermartingale $Z$

$\Rightarrow$ The Bachelier equation becomes

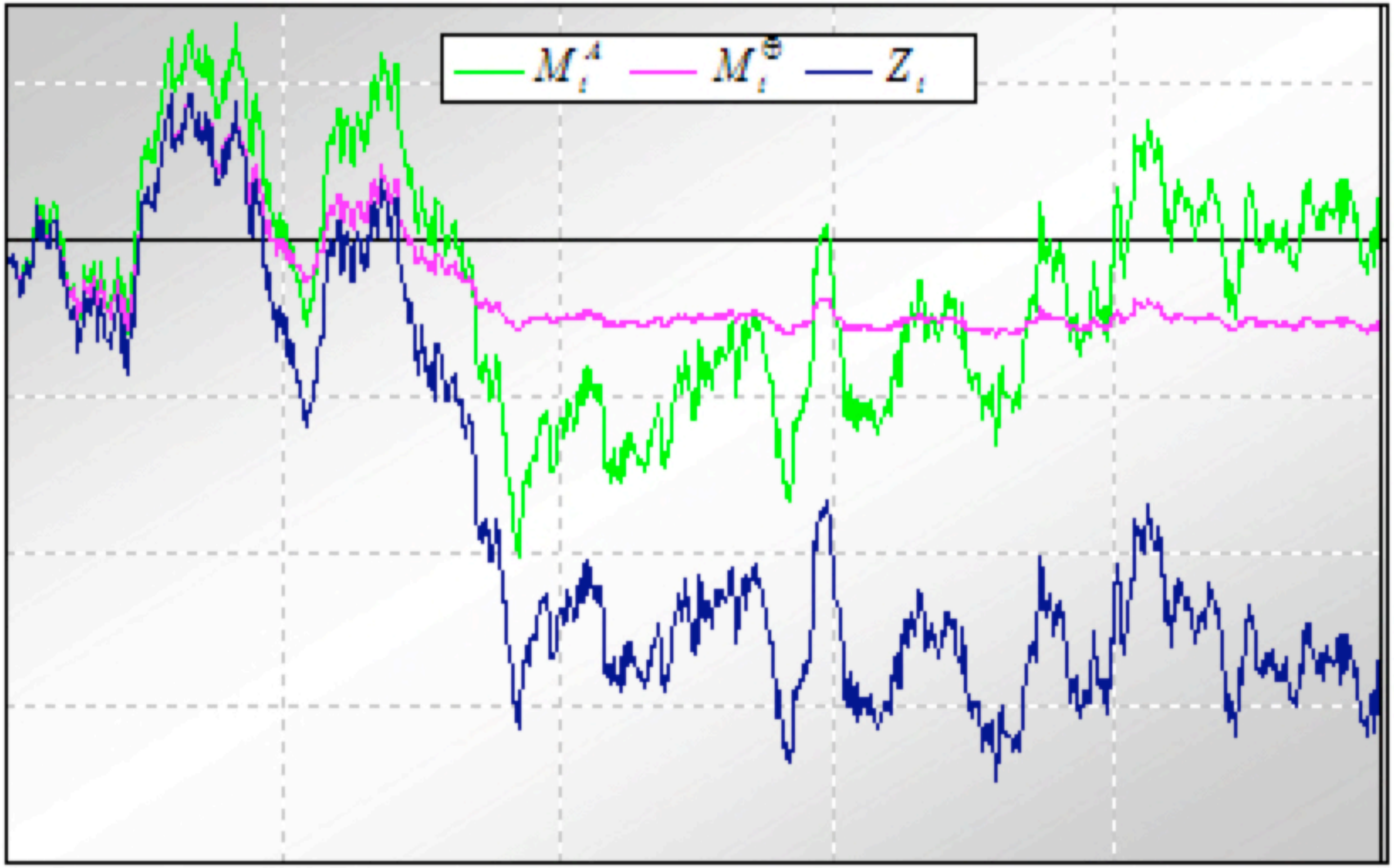
$$dY_t = \gamma(\overline{Y}_t)^{1-1/\gamma}dX_t$$

# Bachelier equation with power function




In green the AY process $Y$, in blue the path of $Z$, in red the running supremum of $Y$

# Bachelier equation with power function




In red the AY process $Y$, in blue the path of $Z$, in green the martingale part of $Z$

# Drawdown properties of the Bachelier equation

**Def :** Given a cadlag process $X$, and a (increasing) function $w$ such that $w(s) < s$, a **DD constraint** is a constraint of the type, $X_t \geq w(\overline{X}_t)$.

**AY process and DD Constraints**

Let $X$ be a non negative max-continuous semimartingale and $u$ a non negative function, $U$ its primitive, and $V$ the inverse function of $U$.

⇒ The AY-process $M_t^U = U(\overline{X}_t) - u(\overline{X}_t)(\overline{X}_t - X_t)$ satisfies the DD Constraint $M_t^U \geq w(\overline{M}_t^U)$, where the function $w$ is given by

$$w(y) = (U - Id.u)oV(y) = y - \frac{V(y)}{V'(y)} \leq y$$

⇒ $w$ is an **increasing** function if and only if $U(x)$ $(V(y)\ )$ is a **concave**(convex) function.

⇒ Then $M_t^U \geq U(X_t) = Z_t = U(M^V(Y_t))$

## DD and Bachelier equation

$\Rightarrow$ In terms of Bachelier equation associated with $\phi(y) = \frac{1}{V'(y)}$, we have:
*The solution $Y$ satisfies the DD constraint with the function $w$ obtained by*

- Taking a primitive $V$ of $V'(y) = 1/\phi(y)$ and
- Putting $w(y) = y - \frac{V(y)}{V'(y)}$
- Conversely, given a function $w$, put $\phi(y) = (V'(y))^{-1}$, where $V$ is a solution of the ODE equation

$$\frac{V'(y)}{V(y)} = \frac{1}{y - w(y)}$$

# Dynamic strategy with drawdown constraints

Grossmann-Zhou(93), Cvitanic -Karatzas(95), Uryasev & alii(05), Elie& Touzi (2006-2008), Roche(06)......

**Why DD constraints?**

- **Hedge funds** : The final decision of a client into opening an account with a manager is most likely based on his account's drawdown sizes and duration.
- Client would not tolerate drawdown for a long time period.
- In an investment bank setup, for proprietary trading, warming drawdown level are generally fixed to 20%

# Strategy with Drawdown Constraints

**Problem :** *To find a portfolio strategy based on a reference asset satisfying some drawdown constraints on the discounted prices at any time.*

**Framework**

- the reference asset is the **discounted value** $S_t$ of some strategic portfolio. There exists a probability measure $\mathsf{Q}$ such that $S_t$ is a $\mathsf{Q}$ local martingale.
- the discounted value of any portfolio strategy $\pi$ evolves as: $dX_t^\pi = \pi_t \frac{dS_t}{S_t}, \quad X_0^\pi = x$
- **Drawdown constraints** C.K (1995): $X_t^\pi > \alpha \overline{X}_t^\pi, \quad \forall t, \quad 0 < \gamma < 1.$
- More generally, let $\mathbf{w}$ be a positive **increasing** function such that $\mathbf{w}(\mathbf{x}) < \mathbf{x}$. The DD-constraint becomes $X_t^\pi \geq \mathbf{w}(\overline{X}_t^{\pi,*}) \quad \forall t.$

# Portfolio Point of view

The AY-Martingale $M^U(S)_t$, associated with some well-chosen function $U$ is an admissible portfolio, if the budget constraint is satisfied.

$\Rightarrow$ Given a increasing DD-function $w$, with $w(x) < x$, let $V$ be a positive solution of the ODE

$$\frac{V'(y)}{V(y)} = \frac{1}{y - w(y)}$$

$\Rightarrow$ Then $V$ is **convex** and its inverse function $U$ is **concave** increasing.

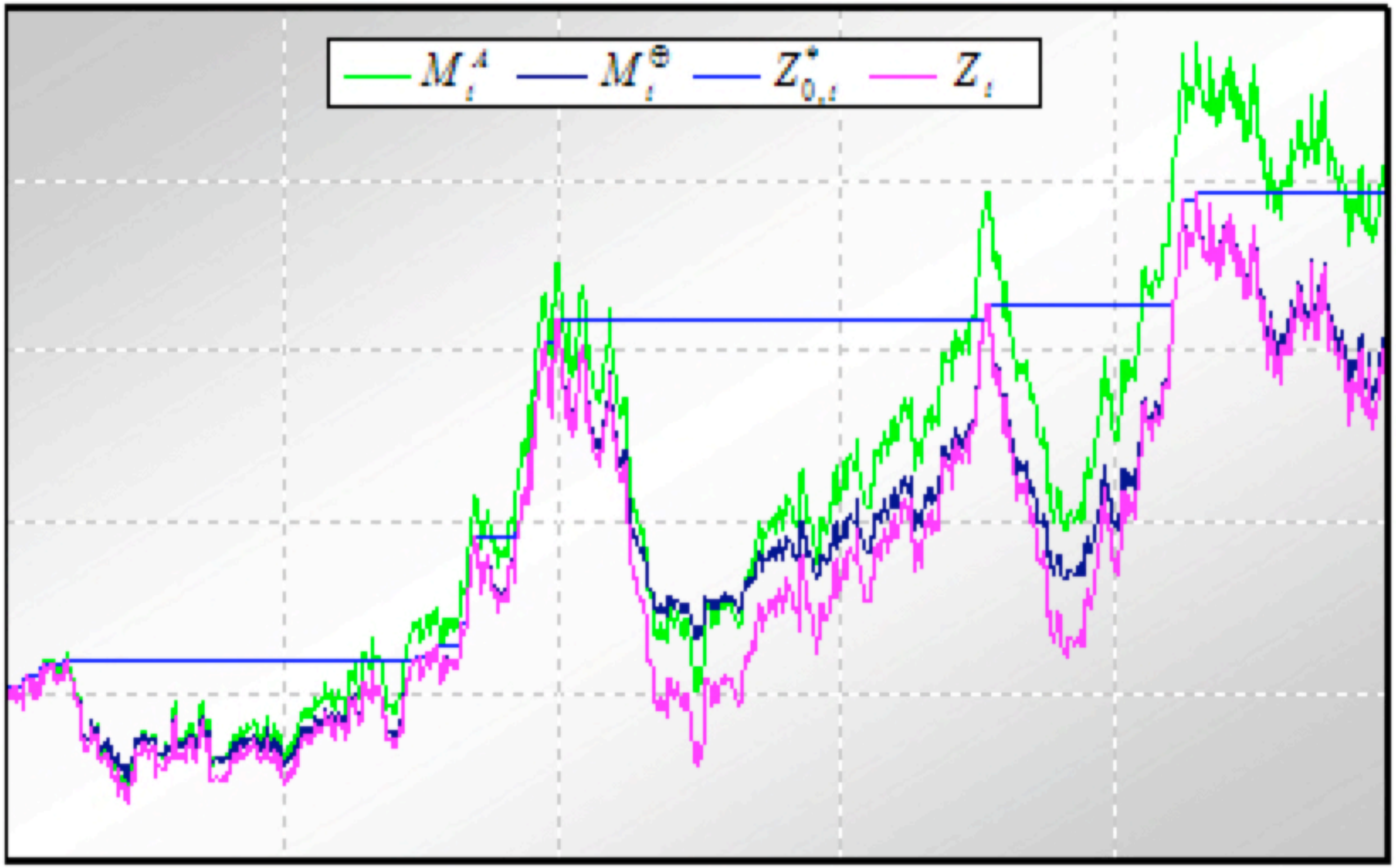
$\Rightarrow$ Then $Y = M^U(S)$ is a self-financing strategy such that

$$dM_t^U = \left(\mathbf{M_t^U} - \mathbf{w}(\overline{\mathbf{M}}_\mathbf{t}^\mathbf{U})\right)\frac{dS_t}{S_t}$$

- The portfolio strategy is very simple: at any time the amount invested in the risky asset is the distance to drawdown, and the amount invested in cash is $w(\overline{M}_t^U)$.

- There is a floor process $Z_t = U(S_t)$, which is a supermartingale.
- The existence of the floor implies a budget constraint that $\mathbf{x} \geq \mathbf{U}(\mathbf{S_0})$.
- The initial condition $M_0^U = x$ is satisfied if the function $V$ is chosen such that $V(x) = S_0$.
- When $w(y) = (1-\gamma)y$, $\mathbf{U}(\mathbf{x}) = \mathbf{C}\mathbf{x}^{\gamma}$

# Bachelier solution of a power function




In black the AY process $Y$, in red the path of $Z$, in green the martingale part of $Z$, in blue the $Z$ running supremum

# American Call options, and AY-martingales

# Darling, Ligget, Taylor Point of View,(1972)

- $Z$ is a supermartingale on $[0,\zeta]$ and $\mathbb{E}\big[|\overline{Z}_{0,\zeta}|\big] < +\infty$
- Assume $Z$ to be **a conditional expectation of some running supremum** process $\overline{L}_{s,t} = \sup_{\{s\le u\le t\}} L_u$, such that $\mathbb{E}\big[|\overline{L}_{0,\zeta}|\big] < +\infty$ and $\mathbf{Z_t} = \mathbb{E}\big[\overline{L}_{t,\zeta}|\mathcal{F}_t\big]$

**American Call options** Let $C_t(Z,m)$ be the American Call option with strike $m$, $\mathbf{C_t(Z,m)} = \operatorname{ess\,sup}_{t\le S\le\zeta} \mathbb{E}\big[(\mathbf{Z_S} - \mathrm{m})^+|\mathcal{F}_\mathrm{t}\big]$. Then

$$\boxed{\mathbf{C_t(Z,m)} = \mathbb{E}\Big[\big(\overline{\mathbf{L}}_{\mathbf{t},\zeta} \vee \mathbf{Z}_\zeta - \mathbf{m}\big)^+|\mathcal{F}_\mathbf{t}\Big]}$$

and the stopping time $\mathbf{D_t(m)} = \inf\{s \in [t,\zeta];\ L_s \ge m\}$ is optimal.

## Proof

$\Rightarrow$ $\mathbb{E}\big[\big(\overline{L}_{t,\zeta} - m\big)^{+}|\mathcal{F}_t\big]$ is a supermartingale dominating $\mathbb{E}\big[\overline{L}_{t,\zeta}|\mathcal{F}_t\big] - m = Z_t - m$, and so $C_t(Z, m)$

$\Rightarrow$ Conversely, since on $\{\theta = D_t(m) < \infty\}$, $\overline{L}_{\theta,\zeta} \geq m$, at time $\theta = D_t(m)$, we can omit the sign $+$, and replace $(\overline{L}_{\theta,\zeta} - m)$ by its conditional expectation $Z_{D_t(m)} - m$, still nonnegative.

# Perpetual American Call Options and Azéma Yor martingales

## Framework

- $(N_t)$ is a positive local martingale, which tends to 0 as $t$ goes to $\infty$.
- $\mathbf{g}$ is a continuous increasing function on $\mathbb{R}^+$ whose increasing concave envelope $\mathbf{U}$ is finite.
- the underlying process of the option is $\mathbf{Y_t} = g(N_t)$, and we assume that $\mathbb{E}[\sup_{0,\infty} |g(N_t)|] < \infty$.

**Galtchouk, Mirochnitchenko Result (1994)**: The process $\mathbf{Z_t} = U(N_t)$ is the Snell envelope of $Y$,

- $\overline{Z}_t = U(\overline{N}_t)$ is the running supremum of $Z$, and $\mathbf{\overline{Z}_{s,t}} = \sup_{s \leq u \leq t} Z_u$ is the running supremum between $s$ and $t$.
- $M_t^U = U(\overline{N}_t) - u(\overline{N}_t)(\overline{N}_t - N_t)$ is the Azéma Yor martingale associated with $U$. Observe that the concavity of $U$ implies that at any time $t$, $M_t^{AY} \geq Z_t$.

# Main Result

**Theorem** Under the previous assumption, $Z$ is the conditional expectation of the running supremum $h(\overline{N}_{t,\infty})$ where $\mathbf{h(y) = U(y) - yu(y)}$ is a nondecreasing function on $\mathbb{R}^+$.

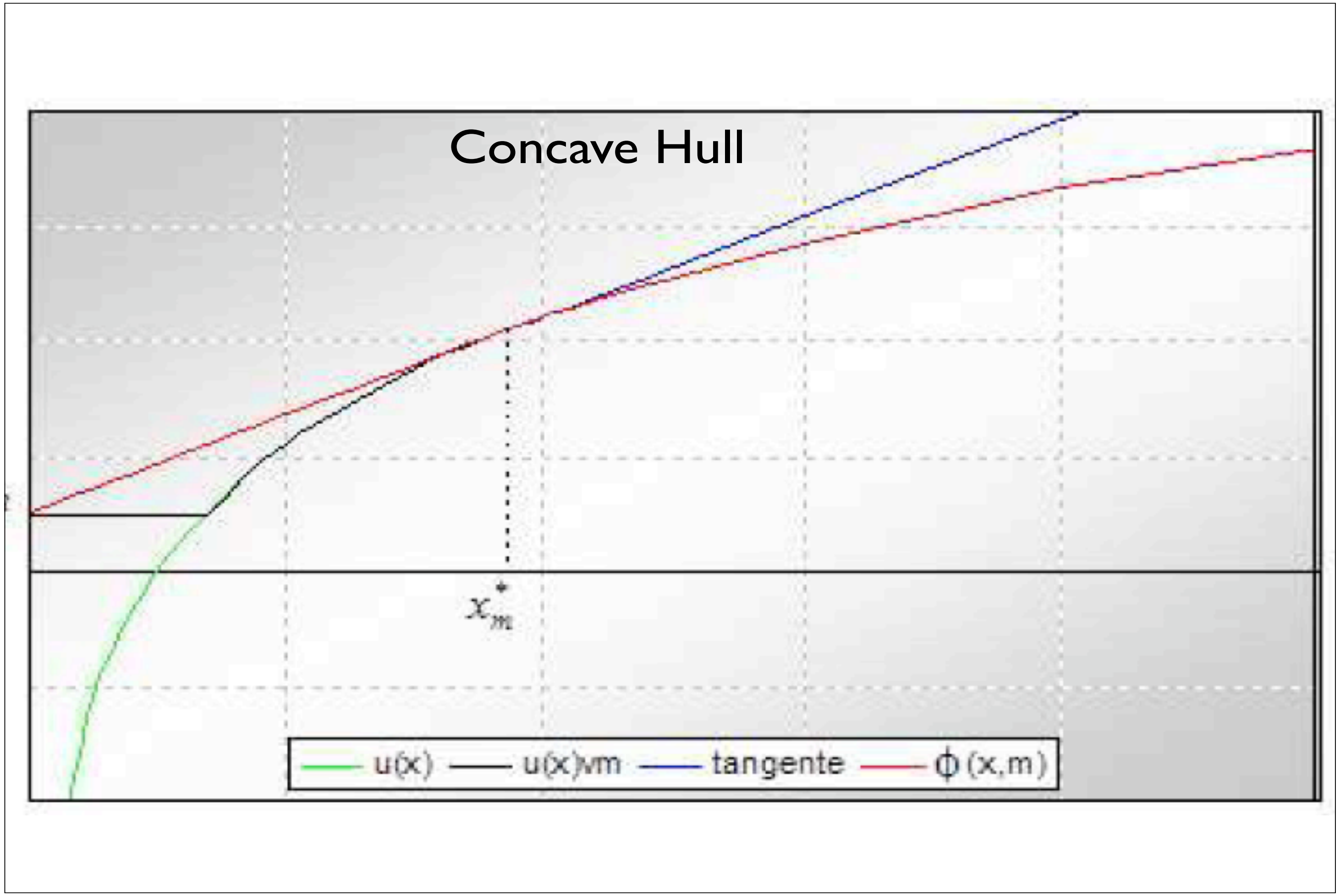
- The American Call option $C_t(Z, m)$ is optimally stopped at the time $D_t(m) = \inf\{s \in [t, \infty];\ h(N_t) \geq m\}$.
- The Call price at time $t$ is given by

$$C_t(Y, m) = \mathbb{E}[(h(\overline{N}_{t,\infty}) - m)^+ | \mathcal{F}_t] = \mathbf{V}(N_t, m) = \phi(N_t) - m$$

  where $\mathbf{V}(z, m)$ is the concave envelope of $(g(z) - m)^+$.

**Proof:** We only have to observe that $Z_t = U(N_t) = \mathbb{E}[h(\overline{N}_{t,\infty}) | \mathcal{F}_t]$.

# The concave envelop of $u(y) \vee m$



# American Call Options for Supermartingales with Independent Increments

**Continuous case** Let $N$ be a geometric Brownian motion with return=0 and volatility to be specified. Let Z be a supermartingale defined on $[0, \infty]$ such that

- a **geometric** Brownian motion with **negative drift** ,
  $\frac{dZ_t}{Z_t} = -rdt + \sigma dW_t, \quad Z_0 = z > 0.$
- Setting $\gamma = 1 + \frac{2r}{\sigma^2}$, $N_t = Z_t^\gamma$ is a local martingale, with volatility $\gamma\sigma$

- $Z_t = U(N_t)$ where $U$ is the increasing concave function $U(x) = x^{1/\gamma}$.
- $h(x) = U(x) - x\,u(x) = \frac{\gamma-1}{\gamma} x^{1/\gamma} = \frac{\gamma-1}{\gamma} z,$
- the optimal boundary for American Call options, is given by $\mathbf{y^*(m) = \frac{\gamma}{\gamma-1}\, m}$, where $\frac{\gamma}{\gamma-1} = \mathbb{E}[\overline{Z}_\infty / Z_0]$.

- Let $Z$ be a **Brownian motion** with negative drift $-(r+\frac{1}{2}\sigma^2) \geq 0$
  $dZ_t = -(r+\frac{1}{2}\sigma^2)dt + \sigma dW_t, \quad Z_0 = z.$
  Then $Z_t = \frac{1}{\gamma}\ln(N_t)$, $h(z) = z - \frac{1}{\gamma}$ and the Call American boundary is $y^*(m) = m + \frac{1}{\gamma}$.

- **the exponentional of a Lévy process with jumps**
  Assume $Z$ to be a supermartingale with a continuous and integrable supremum. Then the same result holds with a modified coefficient $\gamma_{Levy}$, such that $Z_t^{\gamma_{Levy}}$ defines a local martingale that goes to 0 at $\infty$.

- **Finite horizon** $T$ without Azéma-Yor martingale
  Same kind of solution: we have to find a function b(.) such that at any time $t$

$$Z_t = \mathbb{E}\Big[\sup_{t\leq u\leq T} \mathbf{b(T-u)} Z_u \Big| \mathcal{F}_t\Big]$$

# Universal Boundary and Pricing Rule

**Framework**: Let $Z = U(N_.)$ be a increasing concave function of the cadlag local martingale $N$ going to 0 at infinity, with continuous running supremum. Assume $\mathbb{E}[|\overline{Z}_{0,\infty}|] < +\infty$.

- Let $V$ be the increasing convex, inverse function of $U$, such that $V(Z) = N$ is a local martingale and $w(z) = h\, o\, V(z) = z - \dfrac{V(z)}{V'(z)}$. Then

$$Z_t = \mathbb{E}[w(\overline{Z}_{t,\infty})|\mathcal{F}_t], \qquad C_t^Z(m) = \mathbb{E}[(w(\overline{Z}_{t,\infty}) - m)^+|\mathcal{F}_t]$$

- Optimal boundary and price of the American Call options are given by the universal rule

$$
\boxed{
\begin{aligned}
y^*(m) &= w^{-1}(m) = m + \frac{V(y^*(m))}{V'(y^*(m))} \\
C_t^Z(m) &= \begin{cases} (Z_t - m) & \text{if} \quad Z_t \geq y^*(m) \\ \frac{y^*(m)-m}{V(y^*(m))}\,\varphi(Z_t) & \text{if} \quad Z_t \leq y^*(m) \end{cases} .
\end{aligned}
}
$$

# Max-Plus decomposition

Azéma-Yor martingales are well adapted to get very easily explicit formulae for optimal strategies in portfolio insurance.

The same ideas may be used in ageneral case, based on a new decomposition of general supermartingale.

# Max-Plus Supermartingale Decomposition

Let $Z$ be a càdlàg supermartingale in the class $(\mathcal{D})$ defined on $[,\zeta]$.

- There exists $L = \big(L_t\big)_{\leq t \leq \zeta}$ adapted, with upper-right continuous paths with **running supremum** $L^*_{t,s} = \sup_{t \leq u \leq s} L_u$, s.t.

$$\mathbf{Z_t} = \mathbb{E}\big[\big( \sup_{t \leq u \leq \zeta} L_u\big) \vee Z_\zeta \big| \mathcal{F}_t\big] = \mathbb{E}\big[L^*_{t,\zeta} \oplus Z_\zeta \big| \mathcal{F}_t\big] = \mathbb{E}\Big[\oint_{\mathbf{t}}^{\zeta} \mathbf{L_u} \oplus \mathbf{Z}_\zeta \big| \mathcal{F}_{\mathbf{t}}\Big]$$

- Let $M^\oplus$ be the martingale: $\mathbf{M}^\oplus_\mathbf{t} := \mathbb{E}\big[L^*_{\mathbf{0},\zeta} \oplus Z_\zeta \big| \mathcal{F}_t\big)\big]$.Then,

$$M^\oplus_t \geq \max(Z_t, L^*_{\mathbf{0},t}) = Z_t \oplus L^*_{\mathbf{0},t} \quad \leq t \leq \zeta$$

and the equality holds at times when $L^*$ **increases** or at **maturity** $\zeta$:

$$M^\oplus_S = \max(Z_S, L^*_{\mathbf{0},S}) = Z_S \oplus L^*_{0,S} \quad \text{for all stopping times } S \in \mathcal{A}_{L^\star} \cup \{\zeta\}.$$

# Martingale optimization problem

## The optimization problem

Set $\mathcal{M}(x) = \Big\{ (M_t)_{t\geq 0}\,\text{u.i.martingale}|M_0 = x \text{ and } \mathbf{M_t} \geq \mathbf{Y_t} \;\; \forall t \in [0,\zeta]\Big\}$

- We aim at finding a martingale $(M_t^*)$ in $\mathcal{M}(x)$ such that for all martingales $(M_t)$ in $\mathcal{M}(x)$

$$\mathbf{M}_\zeta^* \leq_{\mathbf{cx}} \mathbf{M}_\zeta$$

- The initial value of any martingale dominating $Y$ must be **at least** equal to the one of the Snell envelope $Z_0^Y = \sup_{\tau\in\mathcal{T}_{0,\zeta}} \mathbb{E}\left[Y_\tau\right]$,

# $Z^Y$- Max-Plus Martingale is optimal

*The martingale* $\mathbf{M}^{\mathbf{Y},\oplus}$ *of the* $Z^Y$ *Max Plus decomposition is the smallest martingale in* $\mathcal{M}^Y(Z_0^Y)$*, with respect to the convex stochastic order on the terminal value. In particular,* $M_\zeta^{Y,\oplus}$ *is less variable than* $M_\zeta^A(Y)$*.*

**Sketch of proof**: Let $M$ be in $\mathcal{M}^Y(Z_0^Y)$. Since $M$ dominates $Z^Y$, the American Call option $C_t(M,m)$ also dominates $C_t(Z^Y,m)$. By convexity,

$$C_t(M,m) = \mathbb{E}\big[(M_\zeta - m)^+|\mathcal{F}_S\big] \geq \mathbb{E}\big[(L^{Y,*}_{S,\zeta} \vee Y_\zeta - m)^+|\mathcal{F}_S\big] \quad \forall S \in \mathcal{T}.$$

More generally, this inequality holds true for any convex function $g$, and

$$\mathbb{E}\big[g\big(M_\zeta\big)\big] \geq \mathbb{E}\big[g\big(L^{Y,*}_{0,\zeta} \vee Y_\zeta\big)\big] = \mathbb{E}\big[g(M_\zeta^{Y,\oplus})\big]$$

**Initial condition $x \geq Z_0^Y$** Same result by using $L^{Y,*}S,\zeta \vee m$ in place of $L^{Y,*}_{S,\zeta}$.