Running Supremum, DrawDown Constraint, Azéma-Yor Processes, Max-Plus decomposition, and financial applications

jointed work with I.Karatzas (in the past), A.Meziou, J.Obloj

to Marc Yor
Geometrical Brownian motion and Running supremum

Third Lesson in the master program

Let $S_t$ be a geometrical Brownian motion, such that $S^\gamma$ is a martingale and $\bar{S} = \sup_{u \leq t} S_u$, its running supremum.

- By the symmetry principle, we have

$$\mathbb{P}(S_T \leq K, S^*_T \geq H) = \mathbb{P}(S_T \leq K, T_H \leq T)$$

$$= \left(\frac{x}{H}\right)^\gamma \mathbb{P}\left(\frac{H^2}{x^2} S_T \leq K\right) = \left(\frac{x}{H}\right)^\gamma \mathcal{N}\left(\delta_1 \left(\frac{Kx}{H^2}, \sigma, T\right)\right),$$

and

$$\mathbb{P}(S_T \leq K) = \mathcal{N}\left(\delta_1 \left(\frac{x}{K}, \sigma, T\right)\right) = \mathcal{N}\left(-\delta_0 \left(\frac{K}{x}, \sigma, T\right)\right).$$
Theorem
The tail function de $S_T$ given $\{S_T = K\}$ is given for $x, K \leq H$ by

$$\mathbb{P}( S_T^* \geq H \mid S_T = K) = \exp\left( -\frac{2}{\sigma^2 T} \ln\left( \frac{K}{H} \right) \ln\left( \frac{x}{H} \right) \right)$$

- Very useful for instance in Mont Carlo simulation of Barrier Option
- The proof is not completely immediate...
Azéma-Yor Processes
Azéma-Yor Processes (1979)

As usual, \((\Omega, \mathcal{F}_t, \mathbb{P})\) is a filtered probability space, satisfying usual assumptions.

**Notation and basic properties**

- The **running supremum** or maximum process of some adapted cadlag process \(X\) is defined as
  \[
  \bar{X}_t = \sup_{u \leq t} X_u.
  \]
  
  Between two dates, we write \(\bar{X}_{s,t} = \sup_{s < u \leq t} X_u\).

**Properties**

- \(\bar{X}_t\) is an increasing process, right-continuous, with the “max-additivity” property \(\bar{X}_t = \bar{X}_s \vee \bar{X}_{s,t}\).

- When \(\bar{X}_t\) is a continuous process, for instance when the process \(X\) has only negative jumps, the process \(\bar{X}_t\) only increases when \(\bar{X}_t = X_t\), that is
  \[
  \int_0^T (\bar{X}_t - X_t) d\bar{X}_t = 0
  \]
Let $u$ be a locally bounded Borel function. The primitive function 

$$U(x) = a^* + \int_{(a,x]} u(s) \, ds$$

is defined on $[a, \infty)$.

**Definition of AY Process**

Let $X$ be a cadlag semimartingale with **continuous** running supremum 

$$\bar{X}_t = \sup_{u \leq t} X_u,$$

and $u$ a locally bounded function.

The $(U, X)$-Azéma-Yor process is defined by one of these two equations

$$M_t^U (X) = U(\bar{X}_t) + u(\bar{X}_t)(X_t - \bar{X}_t)$$

or

$$= a^* + \int_0^t u(\bar{X}_t) \, dX_s$$

If $X$ is a local martingale, $M_t^U$ is also a local martingale.
Main properties

⇒ The equivalence between the two equations is straightforward when \( U \) is a regular function, since from Itô’s formula

\[
dM_t^U(X) = u(\overline{X}_t)d\overline{X}_t + u(\overline{X}_t)(dX_t - d\overline{X}_t) + (X_t - \overline{X}_t)u'(\overline{X}_t)d\overline{X}_t
\]

\[
= u(\overline{X}_t)dX_t
\]

⇒ The case of locally integrable function \( u \) can be attained for continuous local martingale \( X \) (Obloj,Yor 2004)
Non decreasing transformation

Let $\mathcal{U}_m$ be the set of primitive function $U$ of non negative locally bounded functions $u$, and $\mathcal{G}_m$ the subgroup of increasing functions $U$ s.t. the increasing inverse function $V$ of $U$, with first right-hand derivative $V' := v$ is in $\mathcal{U}_m$.

- Let $U$ be in $\mathcal{U}_m$, $X$ be a max-continuous semimartingale. The $(U,X)$-Azéma-Yor process $(M^U_t(X))$ is a max-continuous semimartingale since,

$$
M^U_t(X) = \overline{U(X_t)} = U(\overline{X_t}),
$$

- Pick $F$ in $\mathcal{U}_m$. Then, $M^U_t(M^F(X)) = M^{U \circ F}_t(X)$.

- Moreover, the processes $M^U(X)$ associated with $U \in \mathcal{G}_m$ is a group under the multiplication $\otimes$ defined by

$$
M^U \otimes M^F := M^{U \circ F}.
$$
• If $u$ is only defined on $[a, b)$, $M^U(X)$ may be defined up to the exit time $T_b$ of $[a, b)$ by $X$.

• If $u$ is non negative, $M^U(X)_{t \wedge T_b} = U(X_{t \wedge T_b})$

**Bachelier equation**

• By the property of the inverse, $u \circ V = 1/V' = 1/v$

• Since $M^U_t = U(X_t)$, $u(X_t) = u \circ V(U(X_t)) = (1/v)(M^U_t)$.

The AY-process is a solution of

$$dM^U_t = (1/v)(M^U_t) dX_t$$

Such equations were first introduced by Bachelier in 1906.

**Definition:** Let $\phi : [a^*, \infty)$ be a locally bounded away from 0 function and $X$ as below. The Bachelier equation is

$$dY_t = \phi(Y_t) dX_t, \quad Y_0 = a^*$$
Existence

⇒ \( M^U_t \) is a solution associated with \( \phi = 1/v \).

⇒ Conversely, given \( \phi : [a^*, \infty) \to (0, \infty) \) be a Borel function locally bounded away from zero, \( v = 1/\phi \) and \( V \) a primitive of \( v \). Then the inverse function \( U \) of \( V \) is defined on \((a^*, V(\infty))\).

\( Y_t = M^U_t(X) \) is a solution of the Bachelier equation on \((0, T_{V\infty})\).
Example

- $X$ is a geometrical Brownian motion with volatility $\sigma$,
- $U$ is the power function $U(x) = x^\gamma$, $\gamma < 1$

Then,

$\Rightarrow$ The AY Process $Y_t = M^U(X_t) = \bar{X}_t^\gamma (1 - \gamma) + \gamma (\bar{X}_t)^{\gamma - 1} X_t$ is also given by

$Y_t = \bar{Y}_t \left[ (1 - \gamma) + \gamma \left( \frac{Y_t}{Y_t} \right)^{1/\gamma} \right]$

$\Rightarrow$ The process $Z_t = X_t^\gamma$ is a supermartingale, with dynamic

$$dZ_t = \gamma Z_t \left( \frac{dX_t}{X_t} - \frac{1}{2} (1 - \gamma) \sigma^2 dt \right)$$

The martingale $Y_t$ is still above the supermartingale $Z$

$\Rightarrow$ The Bachelier equation becomes

$$dY_t = \gamma (\bar{Y}_t)^{1-1/\gamma} dX_t$$
Bachelier equation with power function

In green the AY process $Y$, in blue the path of $Z$, in red the running supremum of $Y$
Bachelier equation with power function

In red the AY process $Y$, in blue the path of $Z$, in green the martingale part of $Z$
Drawdown properties of the Bachelier equation

**Def:** Given a cadlag process $X$, and a (increasing) function $w$ such that $w(s) < s$, a DD constraint is a constraint of the type, $X_t \geq w(X_t)$.

**AY process and DD Constraints**

Let $X$ be a non negative max-continuous semimartingale and $u$ a non negative function, $U$ its primitive, and $V$ the inverse function of $U$.

⇒ The AY-process $M_t^U = U(\overline{X}_t) - u(\overline{X}_t)(\overline{X}_t - X_t)$ satisfies the DD Constraint $M_t^U \geq w(M_t^U)$, where the function $w$ is given by

$$w(y) = (U - \text{Id}.u) \circ V(y) = y - \frac{V(y)}{V'(y)} \leq y$$

⇒ $w$ is an increasing function if and only if $U(x)$ ($V(y)$) is a concave (convex) function.

⇒ Then $M_t^U \geq U(X_t) = Z_t = U(M^V(Y_t))$
DD and Bachelier equation

⇒ In terms of Bachelier equation associated with $\phi(y) = \frac{1}{V'(y)}$, we have:

The solution $Y$ satisfies the DD constraint with the function $w$ obtained by

- Taking a primitive $V$ of $V'(y) = 1/\phi(y)$ and
- Putting $w(y) = y - \frac{V(y)}{V'(y)}$
- Conversely, given a function $w$, put $\phi(y) = (V'(y))^{-1}$, where $V$ is a solution of the ODE equation

$$\frac{V'(y)}{V(y)} = \frac{1}{y - w(y)}$$
Dynamic strategy with drawdown constraints

Grossmann-Zhou(93), Cvitanic -Karatzas(95), Uryasev & alii(05), Elie& Touzi (2006-2008), Roche(06)......

Why DD constraints?

• **Hedge funds** : The final decision of a client into opening an account with a manager is most likely based on his account’s drawdown sizes and duration.

• Client would not tolerate drawdown for a long time period.

• In an investment bank setup, for proprietary trading, warming drawdown level are generally fixed to 20%
Strategy with Drawdown Constraints

Problem: To find a portfolio strategy based on a reference asset satisfying some drawdown constraints on the discounted prices at any time.

Framework

- the reference asset is the discounted value $S_t$ of some strategic portfolio. There exists a probability measure $Q$ such that $S_t$ is a $Q$ local martingale.

- the discounted value of any portfolio strategy $\pi$ evolves as:
  $$dX_\pi^t = \pi_t \frac{dS_t}{S_t}, \quad X_\pi^0 = x$$

- Drawdown constraints C.K (1995): $X_\pi^t > \alpha X^\pi_t, \quad \forall t, \quad 0 < \gamma < 1$.

- More generally, let $w$ be a positive increasing function such that $w(x) < x$. The DD-constraint becomes $X_\pi^t \geq w(X^\pi_t^*) \quad \forall t$. 

NEK, UPMC/CMAP
Portfolio Point of view

The AY-Martingale $M^U(S)_t$, associated with some well-chosen function $U$ is an admissible portfolio, if the budget constraint is satisfied.

⇒ Given a increasing DD-function $w$, with $w(x) < x$, let $V$ be a positive solution of the ODE

$$\frac{V'(y)}{V(y)} = \frac{1}{y - w(y)}$$

⇒ Then $V$ is convex and its inverse function $U$ is concave increasing.

⇒ Then $Y = M^U(S)$ is a self-financing strategy such that

$$dM^U_t = (M^U_t - w(M^U_t)) \frac{dS_t}{S_t}$$

• The portfolio strategy is very simple: at any time the amount invested in the risky asset is the distance to drawdown, and the amount invested in cash is $w(M^U_t)$. 
• There is a floor process $Z_t = U(S_t)$, which is a supermartingale.

• The existence of the floor implies a budget constraint that $x \geq U(S_0)$.

• The initial condition $M_0^U = x$ is satisfied if the function $V$ is chosen such that $V(x) = S_0$.

• When $w(y) = (1 - \gamma)y$, $U(x) = Cx^\gamma$
Bachelier solution of a power function

In black the AY process $Y$, in red the path of $Z$, in green the martingale part of $Z$, in blue the $Z$ running supremum
American Call options, and AY-martingales
Darling, Ligget, Taylor Point of View, (1972)

- $Z$ is a supermartingale on $[0, \zeta]$ and $\mathbb{E}[|Z_{0,\zeta}|] < +\infty$

- Assume $Z$ to be a conditional expectation of some running supremum process $\bar{L}_{s,t} = \sup_{s \leq u \leq t} L_u$, such that $\mathbb{E}[|\bar{L}_{0,\zeta}|] < +\infty$ and $Z_t = \mathbb{E}[\bar{L}_{t,\zeta} | \mathcal{F}_t]$

**American Call options** Let $C_t(Z, m)$ be the American Call option with strike $m$, $C_t(Z, m) = \text{ess sup}_{t \leq s \leq \zeta} \mathbb{E}[ (Z_s - m)^+ | \mathcal{F}_t]$. Then

\[
C_t(Z, m) = \mathbb{E}[ (\bar{L}_{t,\zeta} \vee Z_{\zeta} - m)^+ | \mathcal{F}_t]
\]

and the stopping time $D_t(m) = \inf\{s \in [t, \zeta]; L_s \geq m\}$ is optimal.
**Proof**

\[ \mathbb{E}[(\bar{L}_{t,\zeta} - m)^+ | \mathcal{F}_t] \] is a supermartingale dominating \( \mathbb{E}[\bar{L}_{t,\zeta} | \mathcal{F}_t] - m = Z_t - m \), and so \( C_t(Z, m) \)

\[ \Rightarrow \] Conversely, since on \( \{ \theta = D_t(m) < \infty \} \), \( \bar{L}_{\theta,\zeta} \geq m \), at time \( \theta = D_t(m) \), we can omit the sign +, and replace \( (\bar{L}_{\theta,\zeta} - m) \) by its conditional expectation \( Z_{D_t(m)} - m \), still nonnegative.
Perpetual American Call Options and Azéma Yor martingales

Framework

- \((N_t)\) is a positive local martingale, which tends to 0 as \(t\) goes to \(\infty\).
- \(g\) is a continuous increasing function on \(\mathbb{R}^+\) whose increasing concave envelope \(U\) is finite.
- The underlying process of the option is \(Y_t = g(N_t)\), and we assume that \(\mathbb{E}[\sup_{0, \infty} |g(N_t)|] < \infty\).

Galtchouk, Mirochnichenko Result (1994): The process \(Z_t = U(N_t)\) is the Snell envelope of \(Y\),

- \(\overline{Z}_t = U(\overline{N}_t)\) is the running supremum of \(Z\), and \(\overline{Z}_{s, t} = \sup_{s \leq u \leq t} Z_u\) is the running supremum between \(s\) and \(t\).
- \(M_t^U = U(\overline{N}_t) - u(\overline{N}_t)(\overline{N}_t - N_t)\) is the Azéma Yor martingale associated with \(U\). Observe that the concavity of \(U\) implies that at any time \(t\), \(M_t^{AY} \geq Z_t\).
Main Result

**Theorem** Under the previous assumption, $Z$ is the conditional expectation of the running supremum $h(N_{t,\infty})$ where $h(y) = U(y) - yu(y)$ is a nondecreasing function on $\mathbb{R}^+$.

- The American Call option $C_t(Z, m)$ is optimally stopped at the time $D_t(m) = \inf\{s \in [t, \infty]; h(N_t) \geq m\}$.
- The Call price at time $t$ is given by
  $$C_t(Y, m) = \mathbb{E}[(h(N_{t,\infty}) - m)^+ | \mathcal{F}_t] = \mathbf{V}(N_t, m) = \phi(N_t) - m$$
  where $\mathbf{V}(z, m)$ is the concave envelope of $(g(z) - m)^+$.

**Proof:** We only have to observe that $Z_t = U(N_t) = \mathbb{E}[h(N_{t,\infty}) | \mathcal{F}_t]$. 

The concave envelop of $u(y) \lor m$
American Call Options for Supermartingales with Independent Increments

Continuous case Let $N$ be a geometric Brownian motion with return $= 0$ and volatility to be specified. Let $Z$ be a supermartingale defined on $[0, \infty]$ such that

- a geometric Brownian motion with negative drift,
  \[
  \frac{dZ}{Z_t} = -rdt + \sigma dW_t, \quad Z_0 = z > 0.
  \]
- Setting $\gamma = 1 + \frac{2r}{\sigma^2}$, $N_t = Z_t^\gamma$ is a local martingale, with volatility $\gamma \sigma$
- $Z_t = U(N_t)$ where $U$ is the increasing concave function $U(x) = x^{1/\gamma}$.
- $h(x) = U(x) - x u(x) = \frac{\gamma - 1}{\gamma} x^{1/\gamma} = \frac{\gamma - 1}{\gamma} z$,
- the optimal boundary for American Call options, is given by $y^*(m) = \frac{\gamma}{\gamma - 1} m$, where $\frac{\gamma}{\gamma - 1} = \mathbb{E}[Z_\infty/Z_0]$. 
• Let $Z$ be a Brownian motion with negative drift $-(r + \frac{1}{2}\sigma^2) \geq 0$

$$dZ_t = -(r + \frac{1}{2}\sigma^2)dt + \sigma dW_t, \quad Z_0 = z.$$ 

Then $Z_t = \frac{1}{\gamma} \ln(N_t), \; h(z) = z - \frac{1}{\gamma}$ and the Call American boundary is $y^*(m) = m + \frac{1}{\gamma}$.

• the exponentional of a Lévy process with jumps

Assume $Z$ to be a supermartingale with a continuous and integrable supremum. Then the same result holds with a modified coefficient $\gamma_{Levy}$, such that $Z_{t,Levy}^\gamma$ defines a local martingale that goes to 0 at $\infty$.

• Finite horizon $T$ without Azéma-Yor martingale

Same kind of solution: we have to find a function $b(.)$ such that at any time $t$

$$Z_t = \mathbb{E} \left[ \sup_{t \leq u \leq T} b(T - u) Z_u | \mathcal{F}_t \right]$$
Universal Boundary and Pricing Rule

**Framework:** Let \( Z = U(N) \) be an increasing concave function of the cadlag local martingale \( N \) going to 0 at infinity, with continuous running supremum. Assume \( \mathbb{E}[\| Z_{0,\infty} \|] < +\infty \).

- Let \( V \) be the increasing convex, inverse function of \( U \), such that \( V(Z) = N \) is a local martingale and \( w(z) = h o V(z) = z - \frac{V'(z)}{V(z)} \). Then

  \[
  Z_t = \mathbb{E}[w(Z_{t,\infty})|\mathcal{F}_t], \quad C_t^Z(m) = \mathbb{E}[(w(Z_{t,\infty}) - m)^+|\mathcal{F}_t]
  \]

- Optimal boundary and price of the American Call options are given by the universal rule
\[ y^*(m) = w^{-1}(m) = m + \frac{V(y^*(m))}{V'(y^*(m))} \]

\[ C_t^Z(m) = \begin{cases} 
(Z_t - m) & \text{if } Z_t \geq y^*(m) \\
\frac{y^*(m) - m}{V(y^*(m))} \varphi(Z_t) & \text{if } Z_t \leq y^*(m)
\end{cases} \]
Max-Plus decomposition

Azéma-Yor martingales are well adapted to get very easily explicit formulae for optimal strategies in portfolio insurance.

The same ideas may be used in a general case, based on a new decomposition of general supermartingale.
Max-Plus Supermartingale Decomposition

Let $Z$ be a càdlàg supermartingale in the class $(\mathcal{D})$ defined on $[,\zeta]$.

- There exists $L = (L_t)_{t \leq \zeta}$ adapted, with upper-right continuous paths with running supremum $L^*_{t,s} = \sup_{t \leq u \leq s} L_u$, s.t.

\[
Z_t = \mathbb{E}\left[(\sup_{t \leq u \leq \zeta} L_u) \vee Z_\zeta | \mathcal{F}_t\right] = \mathbb{E}\left[L^*_{t,\zeta} \oplus Z_\zeta | \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^\zeta L_u \oplus Z_\zeta | \mathcal{F}_t\right]
\]

- Let $M^\oplus$ be the martingale: $M^\oplus_t := \mathbb{E}[L^*_{0,\zeta} \oplus Z_\zeta | \mathcal{F}_t]$. Then,

\[
M^\oplus_t \geq \max(Z_t, L^*_{0,t}) = Z_t \oplus L^*_{0,t} \quad \leq t \leq \zeta
\]

and the equality holds at times when $L^*$ increases or at maturity $\zeta$:

$M^\oplus_S = \max(Z_S, L^*_{0,S}) = Z_S \oplus L^*_{0,S}$ for all stopping times $S \in \mathcal{A}_{L^*} \cup \{\zeta\}$. 

NEK, UPMC/CMAP
Martingale optimization problem

The optimization problem

Set $\mathcal{M}(x) = \left\{ (M_t)_{t \geq 0} \text{ u.i. martingale} \middle| M_0 = x \text{ and } M_t \geq Y_t \quad \forall t \in [0, \zeta] \right\}$

- We aim at finding a martingale $(M^*_t)$ in $\mathcal{M}(x)$ such that for all martingales $(M_t)$ in $\mathcal{M}(x)$

$$M^*_\zeta \leq_{cx} M_\zeta$$

- The initial value of any martingale dominating $Y$ must be \textbf{at least} equal to the one of the Snell envelope $Z_0^Y = \sup_{\tau \in \mathcal{T}_{0,\zeta}} \mathbb{E}[Y_\tau]$,
**Z^Y - Max-Plus Martingale is optimal**

The martingale $M^Y, \oplus$ of the $Z^Y$ Max Plus decomposition is the smallest martingale in $M^Y(Z_0^Y)$, with respect to the convex stochastic order on the terminal value. In particular, $M^Y, \oplus_\zeta$ is less variable than $M^A_\zeta(Y)$.

**Sketch of proof:** Let $M$ be in $M^Y(Z_0^Y)$. Since $M$ dominates $Z^Y$, the American Call option $C_t(M, m)$ also dominates $C_t(Z^Y, m)$. By convexity,

$$C_t(M, m) = \mathbb{E}[(M_\zeta - m)^+ | \mathcal{F}_S] \geq \mathbb{E}[(L^Y, S, \zeta \vee Y_\zeta - m)^+ | \mathcal{F}_S] \quad \forall S \in \mathcal{T}.$$  

More generally, this inequality holds true for any convex function $g$, and

$$\mathbb{E}[g(M_\zeta)] \geq \mathbb{E}[g(L^Y, S, \zeta \vee m)] = \mathbb{E}[g(M^Y, \oplus_\zeta)]$$

**Initial condition** $x \geq Z_0^Y$ Same result by using $L^Y, S, \zeta \vee m$ in place of $L^Y, S, \zeta \vee m$. 
