Running Supremum, DrawDown Constraint, Azéma-Yor Processes, Max-Plus decomposition, and financial applications

jointed work with I.Karatzas (in the past), A.Meziou, J.Obloj

to Marc Yor

Geometrical Brownian motion and Running supremum

Third Lesson in the master program

Let S_t be a geometrical Brownian motion, such that S^γ is a martingale and $\overline{S} = \sup_{u \le t} S_u$. it running supremum.

- By the symmetry principle, we have

$$\mathbb{P}(S_T \le K, S_T^* \ge H) = \mathbb{P}(S_T \le K, T_H \le T)$$
$$= \left(\frac{x}{H}\right)^\gamma \mathbb{P}\left(\frac{H^2}{x^2} S_T \le K\right) = \left(\frac{x}{H}\right)^\gamma \mathcal{N}\left(\delta_1\left(\frac{Kx}{H^2}, \sigma, T\right)\right),$$

and

$$\mathbb{P}(S_T \le K) = \mathcal{N}\left(\delta_1\left(\frac{x}{K}, \sigma, T\right)\right) = \mathcal{N}\Big(-\delta_0\big(\frac{K}{x}, \sigma, T\big)\Big).$$

Theorem

The tail function de $\overline{S}_T$ given $\{S_T = K\}$ is given for $x, K \leq H$ by

$$\mathbb{P}(\mathbf{S^*_T} \geq \mathbf{H} \mid \mathbf{S_T} = \mathbf{K}) = \exp\left(-\frac{\mathbf{2}}{\sigma^{\mathbf{2}}\mathbf{T}}\mathrm{Ln}(\frac{\mathbf{K}}{\mathbf{H}})\mathrm{Ln}(\frac{\mathbf{x}}{\mathbf{H}})\right)$$

- Very useful for instance in Mont Carlo simulation of Barrier Option
- The proof is not completely immediate...

Azéma-Yor Processes

Azéma-Yor Processes (1979)

As usual, $(\Omega, \mathcal{F}_t, \mathbb{P})$ is a filtered probability space, satisfying usual assumptions.

Notation and basic properties

- The **running supremum** or maximum process of some adapted cadlag process X is defined as

$$\overline{X}_t = \sup_{u \leq t} X_u.$$

Between two dates, we write $\overline{X}_{s,t} = \sup_{s<u\leq t} X_u$.

Properties

⇒ $\overline{X}_t$ is an increasing process, right-continuous, with the "max-additivity" property $\overline{X}_t = \overline{X}_s \vee \overline{X}_{s,t}$.

⇒ When $\overline{X}_t$ is a continuous process, for instance when the process X has only negative jumps, the process $\overline{X}_t$ only increases when $\overline{X}_t = X_t$, that is

$$\int_0^T (\overline{X}_t - X_t) d\overline{X}_t = 0$$

Let u be a locally bounded Borel function. The primitive function $U(x) = a^* + \int_{(a,x]} u(s)\, ds$ is defined on $[a, \infty)$.

Definition of AY Process

Let X be a cadlag semimartinale with **continuous** running supremum $\overline{X}_t = \sup_{u \leq t} X_u$, and u a locally bounded function.
The (U, X)-Azéma-Yor process is defined by one of these two equations

$$M_t^U(X) = U(\overline{X}_t) + u(\overline{X}_t)(X_t - \overline{X}_t) \quad (1)$$

$$\text{or} = a^* + \int_0^t u(\overline{X}_t) dX_s \quad (2)$$

If X is a local martingale, M_t^U is also a local martingale.

Main properties

⇒ The equivalence between the two equations is straightforward when U is a regular function, since from Itô's formula

$$\begin{aligned} dM_t^U(X) &= u(\overline{X}_t)d\overline{X}_t + u(\overline{X}_t)(dX_t - d\overline{X}_t) + (X_t - \overline{X}_t)u'(\overline{X}_t)d\overline{X}_t \\ &= u(\overline{X}_t)dX_t \end{aligned}$$

⇒ The case of locally integrable function u can be attained for continuous local martingale X (Obloj,Yor 2004)

Bachelier equation

Non decreasing transformation

Let $\mathcal{U}_m$ be the set of primitive function U of non negative locally bounded functions u, and $\mathcal{G}_m$ the subgroup of increasing functions U s.t. the increasing inverse function V of U, with first right-hand derivative $V' := v$ is in $\mathcal{U}_m$.

- Let U be in $\mathcal{U}_m$, X be a max-continuous semimartingale. The (U,X)-Azéma-Yor process $(M_t^U(X))$ is a max-continuous semimartingale since,

$$\overline{M_t^U(X)} = \overline{U(\overline{X}_t)} = U(\overline{X}_t),$$

- Pick F in $\mathcal{U}_m$. Then, $\mathbf{M_t^U(M^F(X)) = M_t^{U\circ F}(X)}$.
- Moreover, the processes $M^U(X)$ associated with $U \in \mathcal{G}_m$ is a group under the multiplication $\otimes$ defined by

$$M^U \otimes M^F := M^{U\circ F}.$$

- If u is only defined on $[a, b)$, $M^U(X)$ may be defined up to the exit time T_b of $[a, b)$ by X.
- If u is non negative, $\overline{M^U(X)}_{t \wedge T_b} = U(\overline{X}_{t \wedge T_b})$

Bachelier equation

- By the property of the inverse, $u \circ V = 1/V' = 1/v$
- Since $\overline{M}_t^U = U(\overline{X}_t)$, $u(\overline{X}_t) = u \circ V(U(\overline{X}_t)) = (1/v)(\overline{M}_t^U)$.

The AY-process is a solution of

$$dM_t^U = (1/v)(\overline{\mathbf{M}}_{\mathbf{t}}^{\mathbf{U}})dX_t$$

Such equations were first introduced by Bachelier in 1906.

Definition: Let $\phi : [a^*, \infty)$ be a locally bounded away from 0 function and X as below. The Bachelier equation is

$$dY_t = \phi(\overline{Y}_t)dX_t, \qquad Y_0 = a^*$$

Existence

⇒ M_t^U is a solution associated with $\phi = \mathbf{1/v}$.

⇒ Conversely, given $\phi : [a^*, \infty) \to (0, \infty)$ be a Borel function locally bounded away from zero, $v = 1/\phi$ and V a primitive of v. Then the inverse function U of V is defined on $(a^*, V(\infty))$.
$Y_t = M_t^U(X)$ is a solution of the Bachelier equation on $(0, T_{V_\infty})$.

Example

- X is a geometrical Brownian motion with volatility σ,
- U is the power function $U(x) = x^\gamma, \gamma < 1$

Then,

$\Rightarrow$ The AY Process $\mathbf{Y_t = M^U(X_t)} = \overline{X}_t^\gamma(1-\gamma) + \gamma(\overline{X}_t)^{\gamma-1}X_t$ is also given by $Y_t = \overline{Y}_t\Big[(1-\gamma) + \gamma\,\big(\frac{Y_t}{\overline{Y}_t}\big)^{1/\gamma}\Big]$

$\Rightarrow$ The process $Z_t = X_t^\gamma$ is a supermartingale, with dynamic

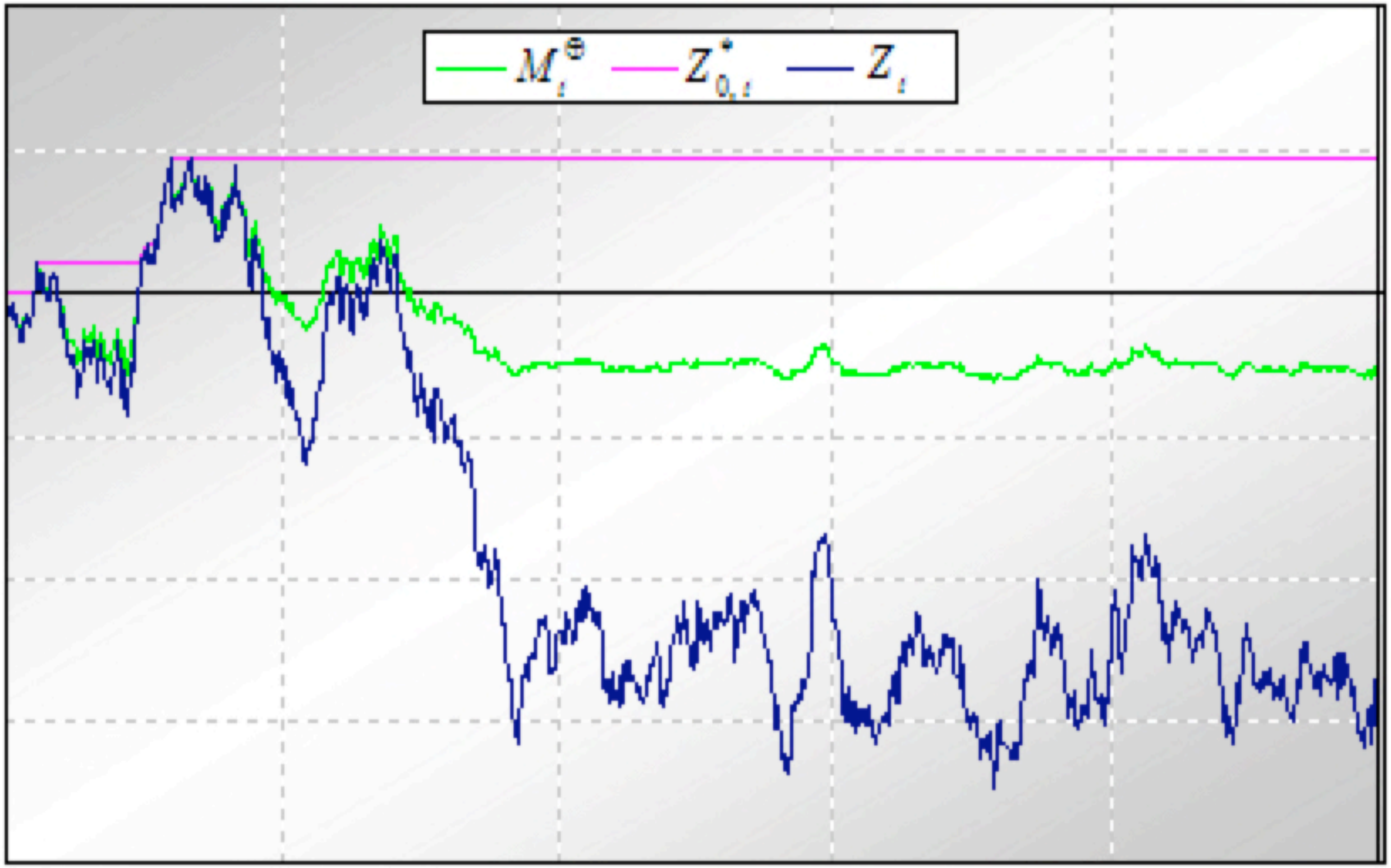
$$dZ_t = \gamma Z_t\,\Big(\frac{dX_t}{X_t} - \frac{1}{2}(1-\gamma)\sigma^2 dt\Big)$$

The martingale $\mathbf{Y_t}$ is still **above** the supermartingale Z

$\Rightarrow$ The Bachelier equation becomes

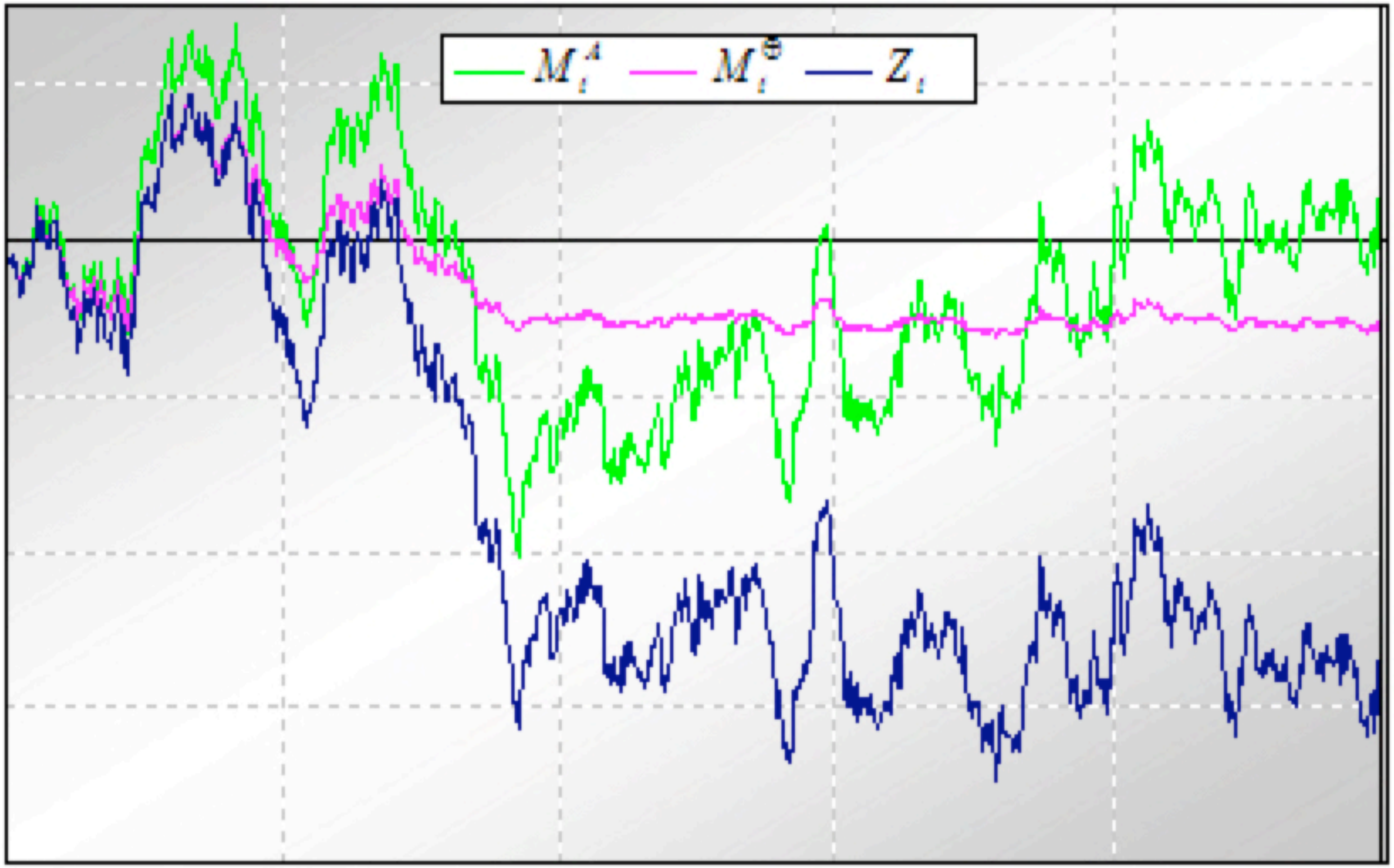
$$dY_t = \gamma(\overline{Y}_t)^{1-1/\gamma}dX_t$$

Bachelier equation with power function



In green the AY process Y, in blue the path of Z, in red the running supremum of Y

Bachelier equation with power function



In red the AY process Y, in blue the path of Z, in green the martingale part of Z

Drawdown properties of the Bachelier equation

Def : Given a cadlag process X, and a (increasing) function w such that $w(s) < s$, a **DD constraint** is a constraint of the type, $X_t \geq w(\overline{X}_t)$.

AY process and DD Constraints

Let X be a non negative max-continuous semimartingale and u a non negative function, U its primitive, and V the inverse function of U.

⇒ The AY-process $M_t^U = U(\overline{X}_t) - u(\overline{X}_t)(\overline{X}_t - X_t)$ satisfies the DD Constraint $M_t^U \geq w(\overline{M}_t^U)$, where the function w is given by

$$w(y) = (U - Id.u)oV(y) = y - \frac{V(y)}{V'(y)} \leq y$$

⇒ w is an **increasing** function if and only if $U(x)$ ($V(y)$) is a **concave**(convex) function.

⇒ Then $M_t^U \geq U(X_t) = Z_t = U(M^V(Y_t))$

DD and Bachelier equation

⇒ In terms of Bachelier equation associated with $\phi(y) = \frac{1}{V'(y)}$, we have:

The solution Y satisfies the DD constraint with the function w obtained by

- Taking a primitive V of $V'(y) = 1/\phi(y)$ and
- Putting $w(y) = y - \frac{V(y)}{V'(y)}$
- Conversely, given a function w, put $\phi(y) = (V'(y))^{-1}$, where V is a solution of the ODE equation

$$\frac{V'(y)}{V(y)} = \frac{1}{y - w(y)}$$

Dynamic strategy with drawdown constraints

Grossmann-Zhou(93), Cvitanic -Karatzas(95), Uryasev & alii(05), Elie& Touzi (2006-2008), Roche(06)......

Why DD constraints?

- **Hedge funds** : The final decision of a client into opening an account with a manager is most likely based on his account's drawdown sizes and duration.
- Client would not tolerate drawdown for a long time period.
- In an investment bank setup, for proprietary trading, warming drawdown level are generally fixed to 20%

Strategy with Drawdown Constraints

Problem : *To find a portfolio strategy based on a reference asset satisfying some drawdown constraints on the discounted prices at any time.*

Framework

- the reference asset is the **discounted value** S_t of some strategic portfolio. There exists a probability measure Q such that S_t is a Q local martingale.
- the discounted value of any portfolio strategy π evolves as:
$dX_t^\pi = \pi_t \frac{dS_t}{S_t}, \quad X_0^\pi = x$
- **Drawdown constraints** C.K (1995): $X_t^\pi > \alpha \overline{X}_t^\pi, \quad \forall t, \quad 0 < \gamma < 1.$
- More generally, let $\mathbf{w}$ be a positive **increasing** function such that $\mathbf{w}(\mathbf{x}) < \mathbf{x}$. The DD-constraint becomes $X_t^\pi \geq \mathbf{w}(\overline{X}_t^{\pi,*}) \quad \forall t.$

Portfolio Point of view

The AY-Martingale $M^U(S)_t$, associated with some well-chosen function U is an admissible portfolio, if the budget constraint is satisfied.

⇒ Given a increasing DD-function w, with $w(x) < x$, let V be a positive solution of the ODE

$$\frac{V'(y)}{V(y)} = \frac{1}{y - w(y)}$$

⇒ Then V is **convex** and its inverse function U is **concave** increasing.

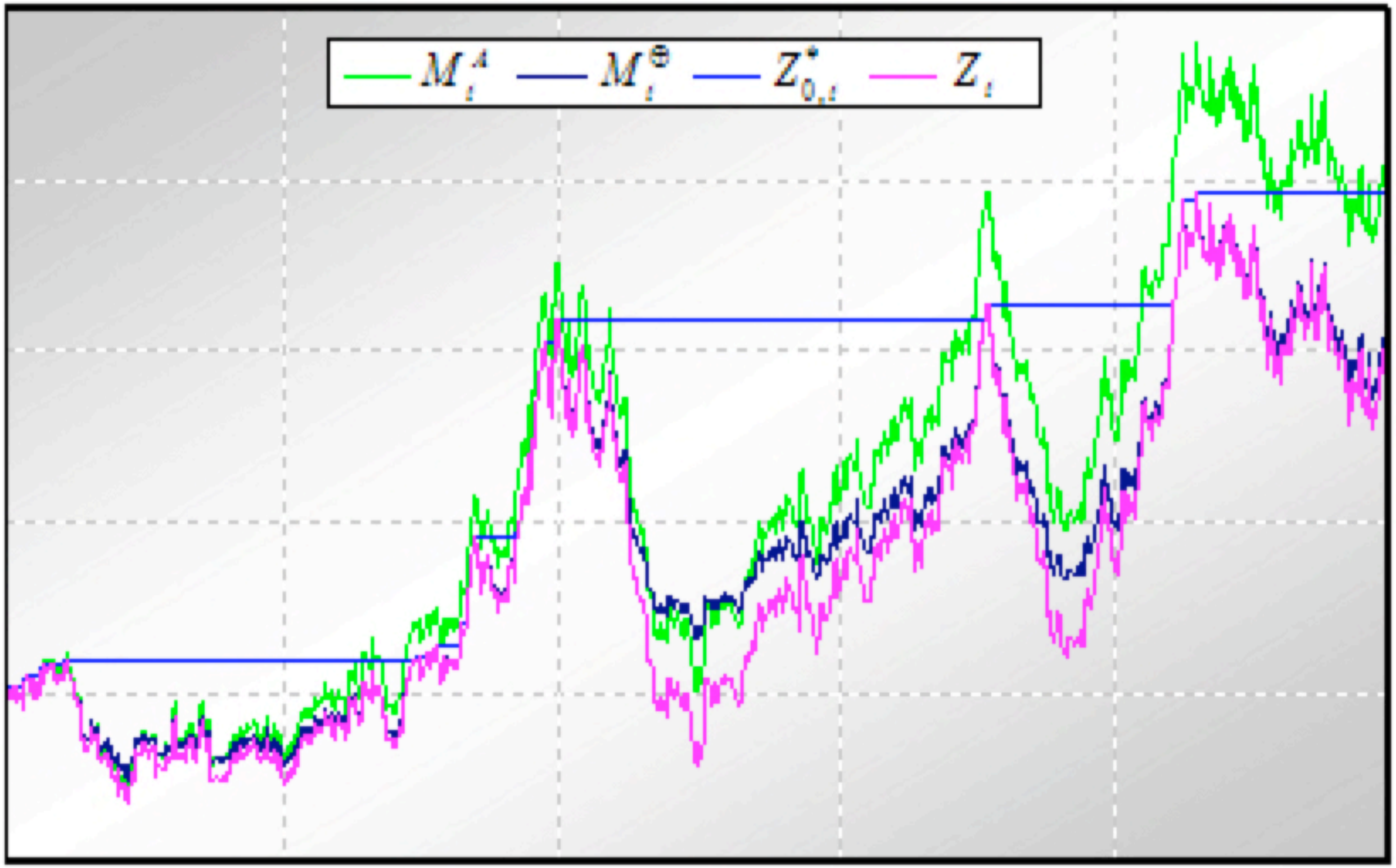
⇒ Then $Y = M^U(S)$ is a self-financing strategy such that

$$dM_t^U = \left(\mathbf{M_t^U} - \mathbf{w}(\overline{\mathbf{M}}_\mathbf{t}^\mathbf{U})\right)\frac{dS_t}{S_t}$$

- The portfolio strategy is very simple: at any time the amount invested in the risky asset is the distance to drawdown, and the amount invested in cash is $w(\overline{M}_t^U)$.

- There is a floor process $Z_t = U(S_t)$, which is a supermartingale.
- The existence of the floor implies a budget constraint that $\mathbf{x} \geq \mathbf{U(S_0)}$.
- The initial condition $M_0^U = x$ is satisfied if the function V is chosen such that $V(x) = S_0$.
- When $w(y) = (1-\gamma)y$, $\mathbf{U(x) = Cx}^{\gamma}$

Bachelier solution of a power function



In black the AY process Y, in red the path of Z, in green the martingale part of Z, in blue the Z running supremum

American Call options, and AY-martingales

Darling, Ligget, Taylor Point of View,(1972)

- Z is a supermartingale on $[0,\zeta]$ and $\mathbb{E}\big[|\overline{Z}_{0,\zeta}|\big] < +\infty$
- Assume Z to be **a conditional expectation of some running supremum** process $\overline{L}_{s,t} = \sup_{\{s\le u\le t\}} L_u$, such that $\mathbb{E}\big[|\overline{L}_{0,\zeta}|\big] < +\infty$ and $\mathbf{Z_t} = \mathbb{E}\big[\overline{L}_{t,\zeta}|\mathcal{F}_t\big]$

American Call options Let $C_t(Z,m)$ be the American Call option with strike m, $\mathbf{C_t(Z,m)} = \operatorname{ess\,sup}_{\mathrm{t}\le\mathrm{S}\le\zeta}\mathbb{E}\big[(\mathbf{Z_S}-\mathrm{m})^+|\mathcal{F}_\mathrm{t}\big]$. Then

$$\mathbf{C_t(Z,m)} = \mathbb{E}\Big[\big(\overline{\mathbf{L}}_{\mathbf{t},\zeta}\vee\mathbf{Z}_\zeta-\mathbf{m}\big)^+|\mathcal{F}_\mathbf{t}\Big]$$

and the stopping time $\mathbf{D_t(m)} = \inf\{s\in[t,\zeta];\ L_s\ge m\}$ is optimal.

Proof

⇒ $\mathbb{E}\big[\big(\overline{L}_{t,\zeta} - m\big)^+ |\mathcal{F}_t\big]$ is a supermartingale dominating $\mathbb{E}\big[\overline{L}_{t,\zeta}|\mathcal{F}_t\big] - m = Z_t - m$, and so $C_t(Z, m)$

⇒ Conversely, since on $\{\theta = D_t(m) < \infty\}$, $\overline{L}_{\theta,\zeta} \geq m$, at time $\theta = D_t(m)$, we can omit the sign $+$, and replace $(\overline{L}_{\theta,\zeta} - m)$ by its conditional expectation $Z_{D_t(m)} - m$, still nonnegative.

Perpetual American Call Options and Azéma Yor martingales

Framework

- (N_t) is a positive local martingale, which tends to 0 as t goes to ∞.
- $\mathbf{g}$ is a continuous increasing function on $\mathbb{R}^+$ whose increasing concave envelope $\mathbf{U}$ is finite.
- the underlying process of the option is $\mathbf{Y_t} = g(N_t)$, and we assume that $\mathbb{E}[\sup_{0,\infty} |g(N_t)|] < \infty$.

Galtchouk, Mirochnitchenko Result (1994): The process $\mathbf{Z_t} = U(N_t)$ is the Snell envelope of Y,

- $\overline{Z}_t = U(\overline{N}_t)$ is the running supremum of Z, and $\overline{\mathbf{Z}}_{\mathbf{s,t}} = \sup_{s \le u \le t} Z_u$ is the running supremum between s and t.
- $M_t^U = U(\overline{N}_t) - u(\overline{N}_t)(\overline{N}_t - N_t)$ is the Azéma Yor martingale associated with U. Observe that the concavity of U implies that at any time t, $M_t^{AY} \geq Z_t$.

Main Result

Theorem Under the previous assumption, Z is the conditional expectation of the running supremum $h(\overline{N}_{t,\infty})$ where $\mathbf{h(y) = U(y) - yu(y)}$ is a nondecreasing function on $\mathbb{R}^+$.

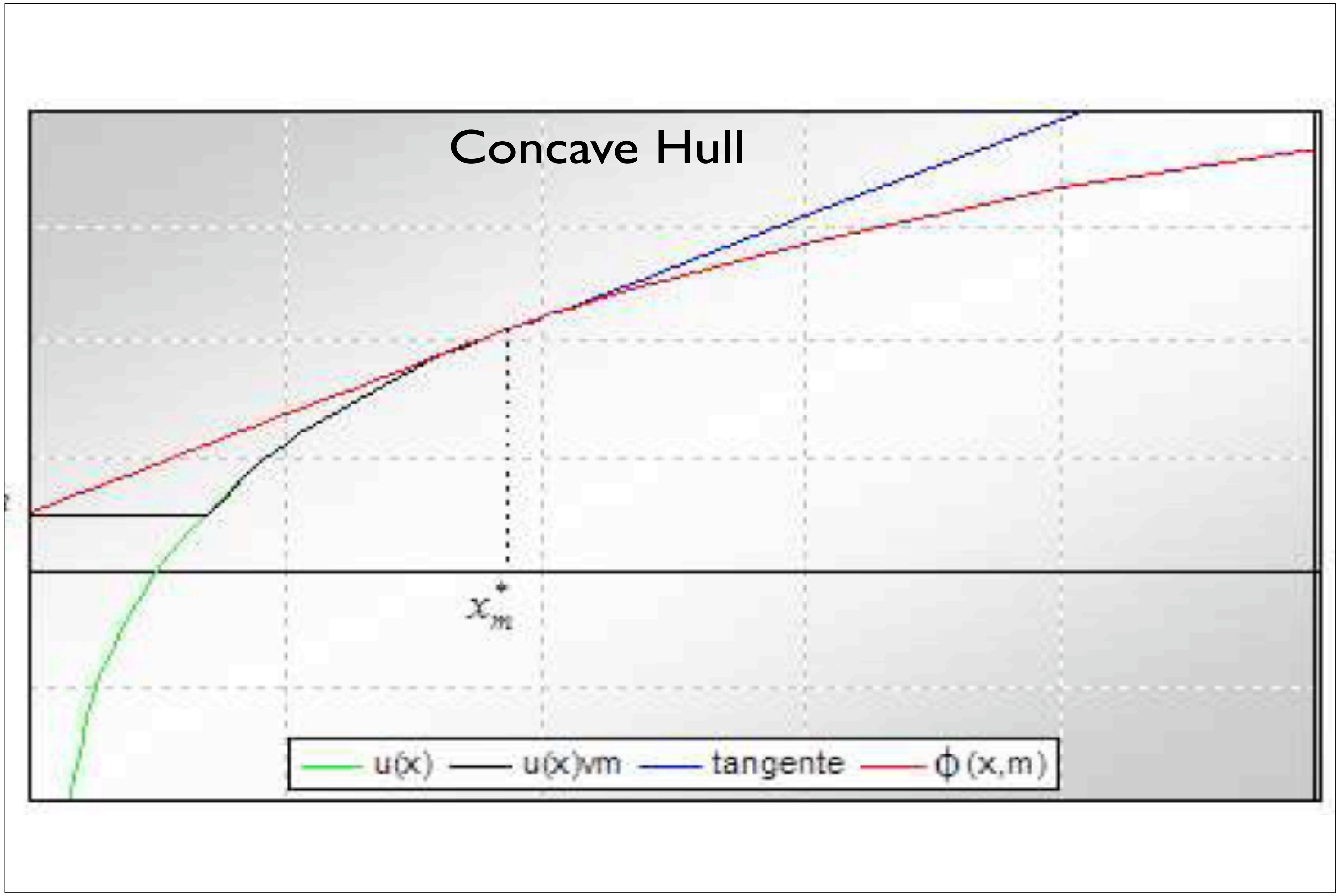
- The American Call option $C_t(Z, m)$ is optimally stopped at the time $D_t(m) = \inf\{s \in [t, \infty];\ h(N_t) \geq m\}$.
- The Call price at time t is given by

$$C_t(Y, m) = \mathbb{E}[(h(\overline{N}_{t,\infty}) - m)^+ | \mathcal{F}_t] = \mathbf{V}(N_t, m) = \phi(N_t) - m$$

  where $\mathbf{V}(z, m)$ is the concave envelope of $(g(z) - m)^+$.

Proof: We only have to observe that $Z_t = U(N_t) = \mathbb{E}[h(\overline{N}_{t,\infty}) | \mathcal{F}_t]$.

The concave envelop of $u(y) \vee m$



American Call Options for Supermartingales with Independent Increments

Continuous case Let N be a geometric Brownian motion with return=0 and volatility to be specified. Let Z be a supermartingale defined on $[0, \infty]$ such that

- a **geometric** Brownian motion with **negative drift** ,
  $\frac{dZ_t}{Z_t} = -rdt + \sigma dW_t, \quad Z_0 = z > 0.$
- Setting $\gamma = 1 + \frac{2r}{\sigma^2}$, $N_t = Z_t^{\gamma}$ is a local martingale, with volatility $\gamma\sigma$
- $Z_t = U(N_t)$ where U is the increasing concave function $U(x) = x^{1/\gamma}$.
- $h(x) = U(x) - x\,u(x) = \frac{\gamma-1}{\gamma} x^{1/\gamma} = \frac{\gamma-1}{\gamma} z,$
- the optimal boundary for American Call options, is given by $\mathbf{y^*(m) = \frac{\gamma}{\gamma-1}\,m}$, where $\frac{\gamma}{\gamma-1} = \mathbb{E}[\overline{Z}_\infty / Z_0]$.

- Let Z be a **Brownian motion** with negative drift $-(r+\frac{1}{2}\sigma^2) \geq 0$

  $dZ_t = -(r+\frac{1}{2}\sigma^2)dt + \sigma dW_t, \quad Z_0 = z.$

  Then $Z_t = \frac{1}{\gamma}\ln(N_t)$, $h(z) = z - \frac{1}{\gamma}$ and the Call American boundary is $y^*(m) = m + \frac{1}{\gamma}$.

- **the exponentional of a Lévy process with jumps**

  Assume Z to be a supermartingale with a continuous and integrable supremum. Then the same result holds with a modified coefficient γ_{Levy}, such that $Z_t^{\gamma_{Levy}}$ defines a local martingale that goes to 0 at ∞.

- **Finite horizon** T without Azéma-Yor martingale

  Same kind of solution: we have to find a function b(.) such that at any time t

$$Z_t = \mathbb{E}\Big[\sup_{t\leq u\leq T} \mathbf{b(T-u)} Z_u \Big| \mathcal{F}_t\Big]$$

Universal Boundary and Pricing Rule

Framework: Let $Z = U(N_.)$ be a increasing concave function of the cadlag local martingale N going to 0 at infinity, with continuous running supremum. Assume $\mathbb{E}[|\overline{Z}_{0,\infty}|] < +\infty$.

- Let V be the increasing convex, inverse function of U, such that $V(Z) = N$ is a local martingale and $w(z) = h\,o\,V(z) = z - \dfrac{V(z)}{V'(z)}$. Then

$$Z_t = \mathbb{E}[w(\overline{Z}_{t,\infty})|\mathcal{F}_t], \qquad C_t^Z(m) = \mathbb{E}[(w(\overline{Z}_{t,\infty}) - m)^+|\mathcal{F}_t]$$

- Optimal boundary and price of the American Call options are given by the universal rule

$$\begin{aligned} y^*(m) &= w^{-1}(m) = m + \frac{V(y^*(m))}{V'(y^*(m))} \\ C_t^Z(m) &= \begin{cases} (Z_t - m) & \text{if} \quad Z_t \geq y^*(m) \\ \frac{y^*(m)-m}{V(y^*(m))}\,\varphi(Z_t) & \text{if} \quad Z_t \leq y^*(m) \end{cases}. \end{aligned}$$

Max-Plus decomposition

Azéma-Yor martingales are well adapted to get very easily explicit formulae for optimal strategies in portfolio insurance.

The same ideas may be used in ageneral case, based on a new decomposition of general supermartingale.

Max-Plus Supermartingale Decomposition

Let Z be a càdlàg supermartingale in the class $(\mathcal{D})$ defined on $[,\zeta]$.

- There exists $L = \big(L_t\big)_{\leq t \leq \zeta}$ adapted, with upper-right continuous paths with **running supremum** $L^*_{t,s} = \sup_{t \leq u \leq s} L_u$, s.t.

$$\mathbf{Z_t} = \mathbb{E}\big[(\sup_{t \leq u \leq \zeta} L_u) \vee Z_\zeta | \mathcal{F}_t\big] = \mathbb{E}\big[L^*_{t,\zeta} \oplus Z_\zeta | \mathcal{F}_t\big] = \mathbb{E}\Big[\oint_{\mathbf{t}}^{\zeta} \mathbf{L_u} \oplus \mathbf{Z}_\zeta | \mathcal{F}_{\mathbf{t}}\Big]$$

- Let $M^\oplus$ be the martingale: $\mathbf{M_t^\oplus} := \mathbb{E}\big[L^*_{\mathbf{0},\zeta} \oplus Z_\zeta \big| \mathcal{F}_t)\big]$.Then,

$$M_t^\oplus \geq \max(Z_t, L^*_{\mathbf{0},t}) = Z_t \oplus L^*_{\mathbf{0},t} \quad \leq t \leq \zeta$$

and the equality holds at times when L^* **increases** or at **maturity** ζ:

$$M_S^\oplus = \max(Z_S, L^*_{\mathbf{0},S}) = Z_S \oplus L^*_{0,S} \quad \text{for all stopping times } S \in \mathcal{A}_{L^\star} \cup \{\zeta\}.$$

Martingale optimization problem

The optimization problem

Set $\mathcal{M}(x) = \Big\{ (M_t)_{t \geq 0} \, \text{u.i.martingale} | M_0 = x \ \text{and} \ \mathbf{M_t \geq Y_t} \ \ \forall t \in [0, \zeta] \Big\}$

- We aim at finding a martingale (M_t^*) in $\mathcal{M}(x)$ such that for all martingales (M_t) in $\mathcal{M}(x)$

$$\mathbf{M}_\zeta^* \leq_{\mathbf{cx}} \mathbf{M}_\zeta$$

- The initial value of any martingale dominating Y must be **at least** equal to the one of the Snell envelope $Z_0^Y = \sup_{\tau \in \mathcal{T}_{0,\zeta}} \mathbb{E}\left[Y_\tau\right],$

Z^Y- Max-Plus Martingale is optimal

The martingale $\mathbf{M^{Y,\oplus}}$ *of the* Z^Y *Max Plus decomposition is the smallest martingale in* $\mathcal{M}^Y(Z_0^Y)$*, with respect to the convex stochastic order on the terminal value. In particular,* $M_\zeta^{Y,\oplus}$ *is less variable than* $M_\zeta^A(Y)$*.*

Sketch of proof: Let M be in $\mathcal{M}^Y(Z_0^Y)$. Since M dominates Z^Y, the American Call option $C_t(M,m)$ also dominates $C_t(Z^Y,m)$. By convexity,

$$C_t(M,m) = \mathbb{E}\big[(M_\zeta - m)^+|\mathcal{F}_S\big] \geq \mathbb{E}\big[(L_{S,\zeta}^{Y,*} \vee Y_\zeta - m)^+|\mathcal{F}_S\big] \quad \forall S \in \mathcal{T}.$$

More generally, this inequality holds true for any convex function g, and

$$\mathbb{E}\big[g\big(M_\zeta\big)\big] \geq \mathbb{E}\big[g\big(L_{0,\zeta}^{Y,*} \vee Y_\zeta\big)\big] = \mathbb{E}\big[g(M_\zeta^{Y,\oplus})\big]$$

Initial condition $x \geq Z_0^Y$ Same result by using $L^{Y,*}S,\zeta \vee m$ in place of $L_{S,\zeta}^{Y,*}$.

Second lecture

Maximum distribution of the non negative martingale

and Skorohod embedding problems

AY Process, Definition

Let u be a locally bounded Borel function. The primitive function $U(x) = a^* + \int_{(a,x]} u(s)\, ds$ is defined on $[a, \infty)$.

Definition of AY Process

Let X be a cadlag semimartinale with **continuous** running supremum $\overline{X}_t = \sup_{u \leq t} X_u$, and u a locally bounded function.
The (U, X)-Azéma-Yor process is defined by one of these two equations

$$M_t^U(X) = U(\overline{X}_t) + u(\overline{X}_t)(X_t - \overline{X}_t) \tag{3}$$

$$\text{or} = a^* + \int_0^t u(\overline{X}_t) dX_s \tag{4}$$

If X is a local martingale, M_t^U is also a local martingale.

Maximum distribution

Let us come back to AY Martingale, written on a process (N_t), $N_0 = 1$ which is a max-continuous non-negative local martingale such that $N_t \to 0$ a.s. when $t \to \infty$. The typical example is the Geometrical Brownian Motion (GBM).

- **Well-known result.** Assume that $N_0 = 1$.The running supremum ${}^{o}vlN_\infty$ is distibuted as the inverse of uniform r.v.:

$$1/N^*_\infty \text{ has } \textbf{a uniform distribution on } [0,1].$$

- Moreover if there exists a constant $b \geq 1$ and a stopping time ζ s.t $N_\zeta \in \{0, b\}$, then given the event $\{\overline{N}_\zeta < b\} = \{\overline{N}_\zeta = 0\}$ $\mathbf{1/\overline{N}_\zeta}$ is uniformly distributed on $(1/b, 1]$ and $\mathbb{P}(\overline{N}_\zeta = b) = 1/b$

- True also for $\mathcal{F}_t$-conditional distribution, $1/\overline{N}\infty \sim (1/m) \wedge (1/x)U$, where U is uniform, and m and x hold for $m = \overline{N}_t$ and $x = N_t \leq m$

Proof: Let $u(x) = (K - x)+$ the “Put “function. Then, $M^U(N)$ is bounded and u.i. martingale, such that

$$\mathbb{E}\big((K - \overline{N}_\infty)^+ + \mathbf{1}_{\{K > \overline{N}_\infty\}} \overline{N}_\infty\big) = K\mathbb{P}(K \geq \overline{N}_\infty) = K - 1$$

Analytic result

Given a U function we define the function h as $h(x) = U(x) - x\,u(x)$.

Analytic lemma Let h be a function defined on $(0,\infty)$, such that $\frac{|h(x)|}{x^2}$ is integrable away from 0, then

- the solution of equation

$$U(x) - xU'(x) = h(x), \quad \text{is}\, U(x) = x\int_x^{\infty} \frac{h(u)}{u^2}du = \int_0^1 h(\frac{x}{u})du$$

- When h is increasing, then U is concave.

- If h_m is the function $h(.\vee m)$, constant on $(0,m)$, then the associated function $U_\infty(m,x)$ is affine on $(0,m)$,

$$U_\infty(m,x) = U_\infty(m) - xU'_\infty(m)(m-x),\ if\ x < m$$

and $U_\infty(m,x) = U_\infty(x)$ if $x \leq m$.

This analytical lemma allows us to characterize Azema-Yor martingales from their terminal values.

Characterization from terminal value

Let h such that $h(x)/x^2$ is integrable away from 0, and U_∞ the solution of the previous ODE.
Let N be a max-continuous non negative local martingale, going to 0 at ∞ and $\zeta = T_0(N)$.

- Then, $h(\overline{N}_\zeta)$ is an integrable random variable and the closed martingale $H_{t\wedge\zeta} = \mathbb{E}\big(h(\overline{N}_\zeta)|\mathcal{F}_{t\wedge\zeta}\big)$ is the Azema-Yor martingale $M^{U_\infty}(N)$.
- The semimartingale $U_\infty(N_{t\wedge\zeta}) = \mathbb{E}\big(h(\overline{N}_{t,\zeta})|\mathcal{F}_{t\wedge\zeta}\big)$

Skohorod Embedding problem

Analytical Result

Let μ be a centered probability measure on $\mathbb{R}$.

- $\overline{\mu}(x) = \mu\big([x,\infty)\big)$ is the right continuous tail distribution function.
- Let $\overline{q} : [0,1] \to \mathbb{R}$ is the tail quantile function that is the left-continuous inverse of $\overline{\mu}$, $\overline{\mu}(x) < y$ iff $\overline{q}(y) < x$.

- If $q(0^+) = \infty$, the solution U_μ of the previous equation with $h(x) = \overline{\mu}(1/x)$ verifies
$$U_\mu(1/x) = \int_0^1 \overline{q}(ux)\mathrm{d}u = 1/x \int_0^x \overline{q}(x)\mathrm{d}u := \mathrm{AVaR(x)}$$
$U_\mu(1/x)$ is the average value at risk (AVaR) of μ.

- The barycentre function $\Psi_\mu(.)$ is defined as

$$\Psi_\mu(x) = \frac{1}{\overline{\mu}(x)} \int_{[x,\infty)} s\, \mu(\mathrm{d}s).$$

For a.e x, $\mathrm{AVaR(x)} = \psi_\mu(\overline{\mathrm{q}}(\mathrm{x}))$

- Let w_μ be the increasing draw-down function associated with μ by $w_\mu(U_\mu(x)) = \overline{q}(1/x)$ or equivalently $w_\mu(\mathrm{AVaR(x)}) = \overline{\mathrm{q}}(\mathrm{x})$.
The inverse function of w is a.e. equal to the barycentre function ψ_μ.

Corollary

Let U be the solution of ODE associated with $h(x) = \overline{q}(1/x)$, and $Y_\mu = M^{U_\mu}(N)$ be the Azema-Yor martingale associated.

- Then $Y_\zeta = \overline{q}\big(1/\overline{N}_\zeta\big)$ is distributed according to μ.
- Since $\overline{Y}_\zeta = U(\overline{N}_\zeta)$, $Y_\zeta = w(\overline{Y}_\zeta)$ and ζ is the first time where the DD constraint $Y_t \leq w(\overline{Y}_t)$ does not hold.
- Since w^{-1} is the barycentre function Ψ_μ, ζ is the first time where $\Psi_\mu(Y_t) \leq \overline{Y}_\zeta$, which is the definition of the Azema-Yor stopping time.
- $\overline{Y}_\zeta = U(\overline{N}_\zeta) = AVaR_\mu(1/\overline{N}_\zeta)$ is a Hardy and Littlewood maximal r.v. associated with μ. (Gilat and Meilijson), that is a r.v. $X^* = AVaR_\mu(\xi)$ where ξ is uniformly distributed on $[0,1]$.

Skohorod embedding:AY Solution

Let (X_t) be a continuous local martingale, $X_0 = 0$, $\langle X\rangle_\infty = \infty$ a.s. and μ a centered probability measure on $\mathbb{R}$: $\int |x|\mu(\mathrm{d}x) < \infty$, $\int x\mu(\mathrm{d}x) = 0$. Then $(X_{t\wedge T_\psi})$

is a UI martingale and $X_{T_\psi} \sim \mu$, where T_ψ is defined via (**??**)-(**??**). Moreover, $\overline{Y}_{T_\psi}$ is distibuted as $V(1/U)$

Skorokhod embedding problem : Other formulation

- Given a strictly increasing function g, such that $\forall\ s\ \ g(s) < s$, our goal is to study the distribution of M_{τ_g} where

$$\tau_g = \inf\{t \geq 0 | M_t \leq g(S_t)\}.$$

Proposition. Assume that $(M_{t\wedge\tau_g})$ is a u.i. martingale.

a) Denote by μ^S the law of S_{τ_g}, and by $G^S(x) = \mathbb{P}(S_{\tau_g} \geq x)$ the hazard function.

$$\mu^S(dy) = \frac{G^S(y)}{y - g(y)} dy$$

b) Denote by μ^M the law of M_{τ_g}, and by $G^M(x) = G^S(g^{-1}(x))$ its tail function. Then

$$g^{-1}(x) = \frac{1}{\mu^M([x, +\infty))} \int_{[x,+\infty)} y\mu^M(dy)$$

is the barycenter function of the measure μ.

Local Volatility

B.Dupire (95), E.Derman& Kani(95)

Implied Diffusion

Which Model

⇒ How to,extend Black-Scholes model to make it compatible with market option prices?

⇒ To price and hedge with vanilla options exotics options, as barrier, start forward options, basket, asian, with early exercice....

⇒ For easy implementation, we are looking for a Markovian diffusion,

$$\frac{dS_t}{S_t} = rdt + \sigma^{\mathrm{Dup}}(t, S_t)dW_t$$

fitting market data

$$\mathbb{E}[e^{rT(S_T - K)^+}] = C^{\mathrm{Mar}}(T, K)$$

⇒ Are there several solutions?

Dupire ANSWER: One and only one way to do it.

PDE forward and Dupire formula

No interest rate, no dividend

Dupire formula $C(0,K) = (S_0 - K)^+$, and

$$\partial_T C(T,K) = \frac{1}{2} K^2 (\sigma^{\text{Dup}})^2 (T,K) C''_{KK}(T,K)$$

That is the dual PDE integrated twice. From probabilistic point of view, the simplest proof is the following

1. Assume that S is driven by a stochastic volatility γ_t
2. Apply Itô's formula to $((S_T - K)^+)^2$, take the expectation, and consider the first derivative with respect to T.

$$\partial_T Call^{\text{square}}(T,K) = \mathbb{E}\Big(\gamma_t^2 \mathbf{1}_{\{S_T \geq K\}} S_T^2\Big) = \mathbb{E}\Big(\mathbb{E}(\gamma_T^2 | S_T) \mathbf{1}_{\{S_T \geq K\}} S_T^2\Big)$$

3. Then, take the derivative w.r to K

$$2\partial_T Call(T,K) = \sigma^2(T,K) K^2 C''_{KK}(T,K),\ \boldsymbol{\sigma^2(\mathbf{T},\mathbf{K}) = \mathbb{E}(\gamma_\mathbf{T}^2 | \mathbf{S_T} = \mathbf{K})}$$

Drawbacks, and performances

- Very sensitive to the process used to interpolate
- The local volatility surface is not very regular and process to regularize the surface are very times consuming.

If the Dupire formula is difficult to implement, the dual PDEs is a useful tool to generate a large number of Call prices from a given local volatility. It may be use to generate local volatility by fixed point argument. In particular

Other Markovian projections

Obviously we have to relax some assumptions

⇒ Dynamic "copula method":

- Choose a BS diffusion, X. At any time, calibrate a strictly increasing function $\phi(t, x)$ s.t $\phi(t, X_t)$ has the marginal distribution of S_t.
- Study the Markovian diffusion $Y_t = \phi(t, X_t)$, fitting the market, but not risk-neutral

⇒ Skorohod Embedding problem

See below

Calibration via Skorohod embedding problem

Ref: D.Madan,and M.Yor :Making Martingales meet marginals: with explicit construction.(Bernouilli 2002)

Assumptions As Madan & Yor, we use Brownian Motion in place of Geometrical BM.

- We assume marginal density $g(y,t)$, ($y \in \mathbb{R}$) for the centered underlying, $S_t - S_0$, and assume that

$$\int |y| g(y,t) dy < \infty, \quad \int y g(y,t) dy = 0$$

- By no arbitrage assumption, Call prices are increasing in maturity, property equivalent to say that $g(s,y)$ is smaller than $g(t,y)$, $\quad \forall s \leq$ for the concave order.

- Moreover, we assume that the family of barycentre functions defined by

$$\psi(x,t) = \frac{\int_x yg(y,t)dy}{\int_x g(y,t)dy}$$

are increasing in t for any x.

Necessary condition implied by the martingale property

Main result

Theorem

Under the previous assumptions on g(y,t), and the baycentre functions $\psi(x,t)$, for a standard BM $B(u)$, there exists an increasing family of stopping times T_t, defined via the embedding theorem by

$$T_t = \inf\{u \mid \overline{B}_u \geq \psi(B_u, t)$$

such that

1. $Y_t = B(T_t)$ is a martingale
2. $(Y_t; t \geq 0)$ is an inhomogeneous Markov process
3. for any t, the density of Y_t is $g(t,y)$

The semigroup only depend on B, since the change of time T_t only increase when $\overline{B}_u = \psi(B_u, t)$, and so $\overline{B}_u$ is know as function of B at this date.

A one side pure jump process

The Q_t semigroup of the Markov process may be compute from y and $m_s = \psi(x, s)$

$$
\begin{aligned}
Q_t f(y, s) &= \alpha f(\psi^{-1}(m_s, t)) + (1 - \alpha)\Psi^f(x, t) \\
\alpha &= \frac{m_s - x}{m_s - \psi^{-1}(x, t)} \\
\Psi^f(x, t) &= \frac{\int_{\psi^{-1}(m_s, t)} g(y, t) dy}{\int_{\psi^{-1}(m_s, t)} g(y, t) dy}
\end{aligned}
$$

Optimal Stopping of the Maximum Process

Optimal Stopping problem of Maximum Processes

Framework

On the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, we consider a Brownian motion (B_t), and the **maximum process** $S_t = \sup_{\{0 \le u \le t\}} B_u$.

Let ϕ be a non-negative, **increasing** and continuous function and c a continuous, **positive** function.

The problem (in short OSMP) is to **maximize** $\mathbb{E}(\mathbf{\Psi}_\tau)$

$$\Psi_\tau = \phi(S_\tau) - \int_0^\tau c(B_s)ds \tag{5}$$

over all integrable **stopping times** such that

$$\mathbb{E}\Big(\phi(S_\tau) + \int_0^\tau c(B_s)ds\Big) < +\infty \tag{6}$$

Related Works

1. 1987 with $\phi(x) = x$ and $c(x) = c$: Dubins and Schwarz were the first to introduce this problem in order to obtain Doob-like inequalities.

2. Peskir(1995-2004) studied in many papers different versions of this problem, in general when $\phi(x) = x$.

3. Meilijson (1997) with a general function ϕ and $c(x) = c$.

4. Peskir(2000) and Obloj(2004) have related this problem to the embedding Skorohod problem, and Azema-Yor stopping times

5. Espinoza-Touzi (2010) based on the running maximum of OU process.

Main Theorem

Theorem (Peskir) Assume $\phi(x) = x$.

The OSMP problem has an optimal solution with finite value function iff there exists a **maximal** solution $\mathbf{g}_*$ of

$$\mathbf{g}'(\mathbf{s}) = \frac{\mathbf{1}}{\mathbf{2c(g(s))(s - g(s))}}$$

which stays strictly below the **diagonal** in $\mathbb{R}^2$ $(g_*(s) < s)$.

The **Azéma-Yor stopping time**

$$\tau_* = \inf\{t \leq 0 \,|\, B_t \leq g_*(S_t)\}$$

is then optimal whenever it satisfies the integrability constraint.

The theorem will be proved for the geometrical Brownian motion.

Some extensions

1. If $\phi \equiv 1$, τ_* satisfies $\mathbb{E}\big(\int_0^{\tau_*} c(B_s)ds\big) < +\infty$ whenever there exists a stopping time which satisfies this constraint.

2. (**Meilijson**). Let us assume $c(x) = c$, ϕ constant on some interval $[x_0, \infty)$ and $H(x) = \sup_\tau \mathbb{E}\big(\phi(x + S_\tau) - c\tau\big)$.
   Then $g^*(x) = x - \frac{H'(x)}{2c}$, and $H(x)$ is the unique solution that equals ϕ on $[x_0, \infty)$ of the differential equation,
   $$H(x) - \frac{1}{4c}(H'(x))^2 = \phi(x) \tag{7}$$

3. In the general case, $V_* = \phi(0) - 2\int_{\phi^{-1}(g_*^Y(0))}^{\phi^{-1}(0)} uc(u)du$,
   where g_*^Y is a function explicitly given in Peskir2.
   Furthemore, if there exists a solution σ_* of the optimal stopping problem, then $\mathbb{P}(\tau_* \leq \sigma_*) = 1$ and τ_* satisfies the constraint.

4. If there is no maximal solution, then $V_* = \infty$ and tno optimal stopping time.

Skorokhod problem, and OSMP

Consider the following converse problem:
Given a centered probability measure μ, find a pair of functions (ϕ, c) such that the optimal stopping problem τ_* solves the PMOSM (ϕ, c)-problem and embeds μ, i.e. $B_{\tau_*} \sim \mu$.

- (**Peskir**). If $\phi(x) = x$, then

$$c(x) = \frac{G'_\mu(x)}{G_\mu(x)},$$

  with $G_\mu(x) := \mu([x, +\infty))$.

- (**Meilijson**). Conversely if c is fixed, we can determine ϕ by

$$H'(x) = 2c(x - \psi_\mu^{-1}(x)),$$

  where ψ_μ is the barycenter function of the measure μ.

Back to AY Framework in portfolio insurance

Portfolio Insurance in AY Framework

Same framework than for DD-Constraints.

Theorem:

- U is a concave **increasing** function and φ its **inverse** function;
- the floor process $Z_t = U(S_t)$ is a function of the reference asset. This specific assumption makes sense in benchmarked management
- The floor process is a supertingale with martingale part

$$dM^Z = S_t u(S_t) \frac{dS_t}{S_t}$$

M^Z satisfies the floor constraint.

- The AY-martingale $M_t^U = U(\overline{S}_t) + u((\overline{S}_t)(S_t - \overline{S}_t)$, $M_0 = u(S_0)$ is an admissible strategy satisfying also the floor constraint,

$$M_t^U \geq U(S_t)$$

,

- Since $\overline{\mathbf{M}}^{\mathbf{U}}_{\mathbf{t}} = \mathbf{U}(\overline{\mathbf{S}}_{\mathbf{t}}) = \overline{\mathbf{Z}}_{\mathbf{t}}$, the running supremum of the martingale $\mathbf{M}^{\mathbf{U}}$ is less than the running supremum of any martingale U_t dominating Z_t, and with the same initial value.

- M^U_∞ is optimal is optimal for the concave order of the terminal value of any martingale X_t dominating Z_t :

$$\text{given an increasing concave function } g,\ \mathbb{E}[g(M^U_\infty)] \leq \mathbb{E}[g(X_\infty)]$$

Proof Since g is concave, we only have to study

$$\mathbb{E}[g'(M^U_\infty)(M^U_\infty - X_\infty)] = \mathbb{E}[g'(h(\overline{N}_\infty)(M^U_\infty - X_\infty)]$$

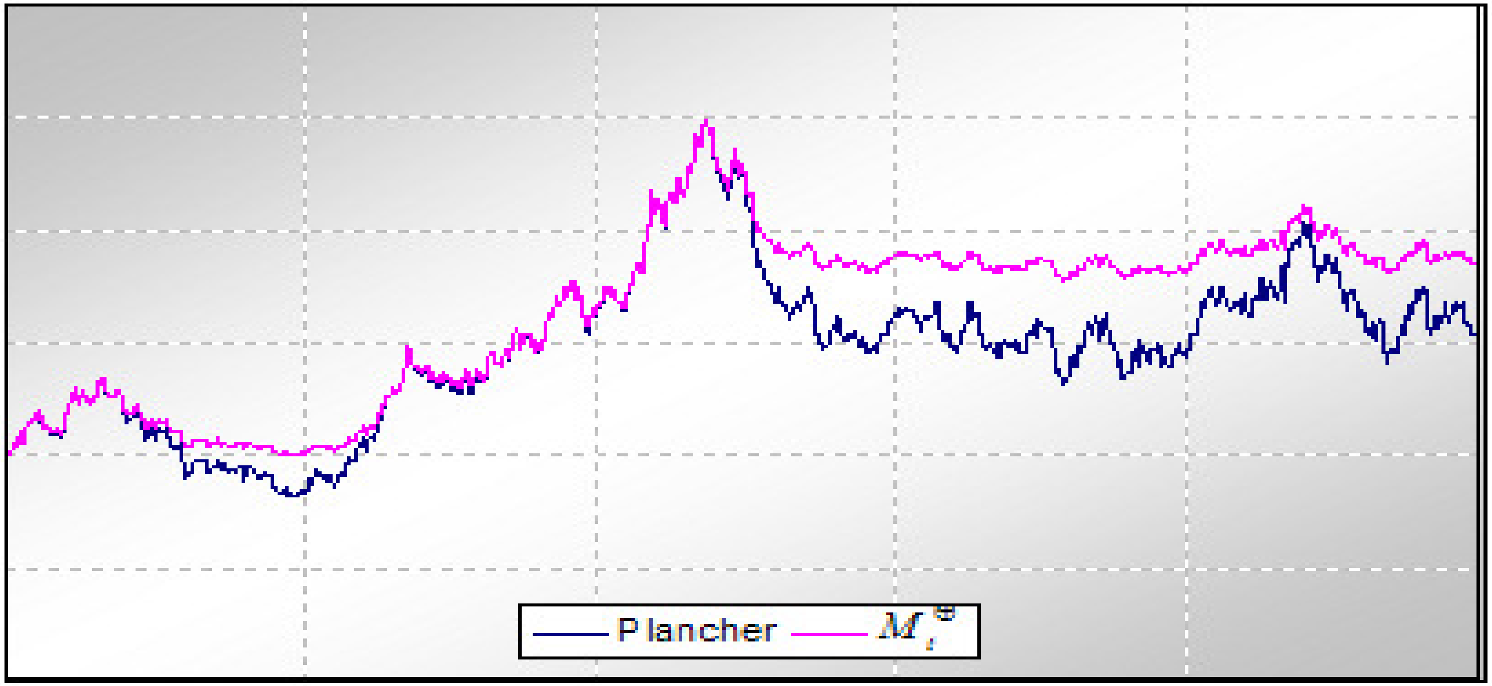
$$\mathbb{E}[\int_0^\infty g'(h(\overline{N}_t)d(M^U_t - X_t)] + \mathbb{E}[\int_0^\infty (M^U_t - X_t)g''(h(\overline{N}_t))dh(\overline{N}_t)$$

- The first term is the difference of two martingales, and so has a null expectation

- For the second integral, $\overline{N}_t$ only increases when $\overline{N}_t = N_t$, on which $M^U = \overline{M}^U = Z_t \leq X_t$

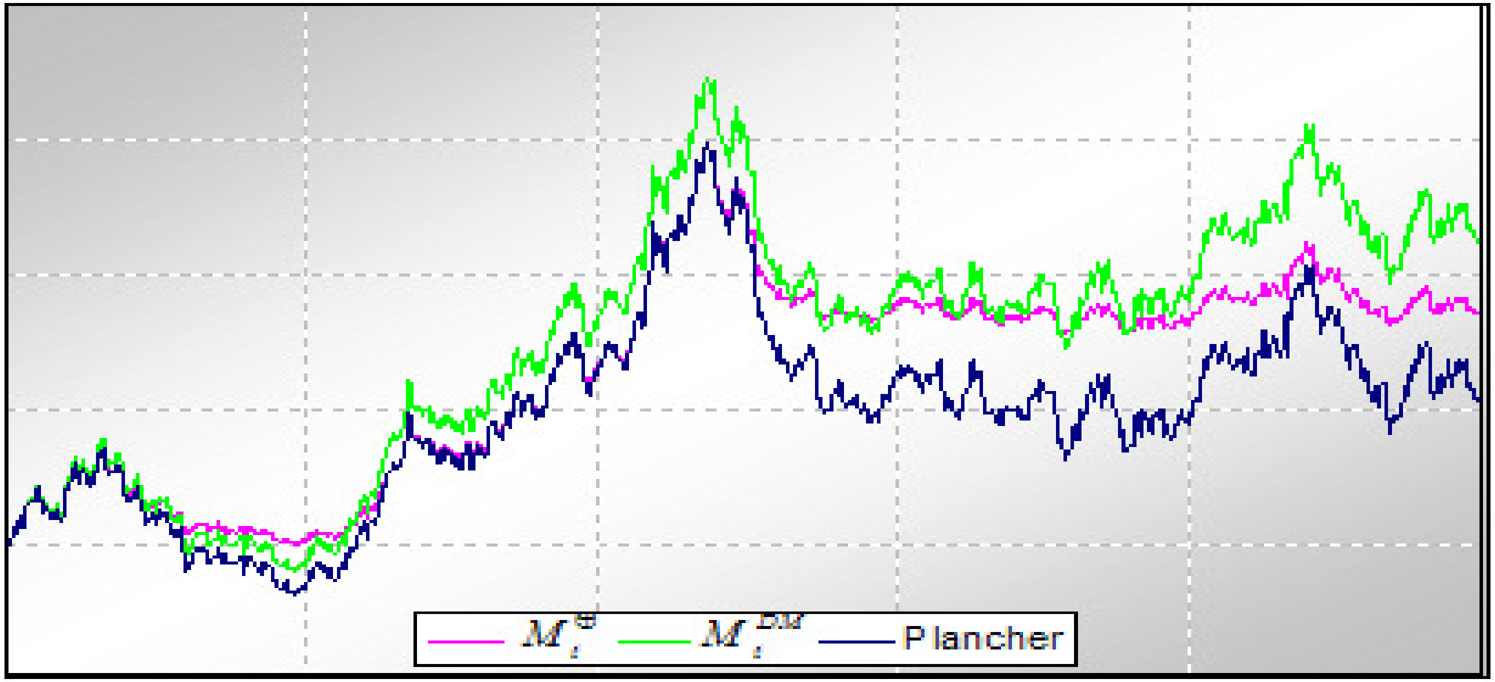
- as g is concave we obtain the inequality

Some pictures



In black a path of the floor, in red the associated path of the AY-martingale

Comparison Azema-Yor and Doob Meyer martingales



In red the associated path of the AY-martingale, in green the Doob Meyer Martingale.

Consumption optimization problem under storage constraints

by **P.Bank**

Ph Thesis Berlin 2000

Durable vs. perishable goods

perishable good	durable good
• chocolate, gas, electricity, ...	• clothes, cars, console, ...
• physically destroyed in process of consumption	• not destroyed, but possibly wears ou when consumed
• affects utility at time of consumption *only*	• provides service flow over extended pe riods of time
• typically bought continually	• typically bought periodically
• Merton, Karatzas et al.	• Hindy, Huang, Kreps et al.

Economic Problem:

Study the **joint** *impact of durable and perishable goods on life time consumption plans!*

Preferences for durable & perishable goods

Consumption plan ...

- For perishable good C: nonnegative, absolutely continuous process with optional density c_t
- For durable good D: nonnegative, right continuous, increasing, optional process

Utility functional:

$$U(C, D) = \mathbb{E} \int_0^{\hat{T}} u(t, c_t, D_t)\, dt$$

- $\hat{T}$ denotes agent's time horizon
- $\mathbf{u(t, ., .)}$ is his time t period utility function: strictly concave, increasing, satisfying Inada conditions

- **Example** : Cobb-Douglas Utility

$$u(t,c,d) = e^{-\rho t}\Big(\frac{1}{\gamma}c^{\gamma}\Big)\Big(\frac{1}{\delta}d^{\delta}\Big) \text{ with } \gamma,\delta > 0,\ \gamma+\delta < 1\,.$$

The agent's optimization problem

Price of consumption plan (C, D):

$$\pi(C,D) = \mathbb{E}\int_0^{\hat{T}} H_t\, c_t\, dt + \mathbb{E}\int_0^{\hat{T}} \widehat{H}_t\, dD_t\,.$$

where H_t, $\widehat{H}_t > 0$ are **state price density processes** for durable & perishable goods.

agent's budget: $w > 0$

Utility maximization problem:

Maximize $U(C,D)$ *over all consumption plans* (C,D) *satisfying the budget constraint* $\pi(C,D) \leq w$*.*

First order conditions for optimality

A consumption plan (C^*, D^*) is cost efficient iff there exists a Lagrange parameter $M > 0$ such that

$\Rightarrow$ $\nabla_C U(C^*, D^*)_t \leq M H_t$ for all $t \in [0, \hat{T}]$ with '=' whenever $c_t^* > 0$,

$\Rightarrow$ $\nabla_D U(C^*, D^*)_t \leq M \widehat{H}_t$ with '=' whenever $dD_t^* > 0$

where the gradients are given by

$$\nabla_{\mathbf{C}} \mathbf{U}(\mathbf{C}, \mathbf{D})_{\mathbf{t}} = \partial_{\mathbf{c}} \mathbf{u}(\mathbf{t}, \mathbf{c}_{\mathbf{t}}, \mathbf{D}_{\mathbf{t}}) \quad (\mathbf{0} \leq \mathbf{t} \leq \hat{\mathbf{T}})$$

and

$$\nabla_{\mathbf{D}} \mathbf{U}(\mathbf{C}, \mathbf{D})_{\mathbf{t}} = \mathbb{E}\Big(\int_{\mathbf{t}}^{\hat{\mathbf{T}}} \partial_{\mathbf{d}} \mathbf{u}(\mathbf{s}, \mathbf{c}_{\mathbf{s}}, \mathbf{D}_{\mathbf{s}})\, \mathbf{ds} | \mathcal{F}_{\mathbf{t}} \Big)$$

Solution of first order conditions

Step 1 Solve in (i) for $\dot{C}^*$:

$$\mathbf{c_t^*} = i_c(t, MH_t, D_t^*) \quad \text{where } i_c(t,.,d) = (\partial_c u(t,.,d))^{-1}$$

Step 2 Employ this in (ii) to obtain a condition involving D^* only:

$$\begin{cases} Y_t^* := \mathbb{E}\Big(\int_t^{\hat{T}} f(s, D_s^*)\, ds | \mathcal{F}_t \Big) \leq M\widehat{H}_t \\ \int_t^{\hat{T}} (M\widehat{H}_t - Y_t^*) dD_t^* = 0 \end{cases}$$

where $f(s,l) = \partial_d u\,(s, i_c(s, MH_s, l), l)$.

Step 3 Find the solution by using Skorohod–type representation theorem

Representation theorem

Theorem:

Let f be a continuous, strictly decreasing function.

For a given optional process X, there exists an adapted process L^f with upper-right continuous paths such that

$$X_T = \mathbb{E}\Big[\int_{(T,+\infty]} f\big(t, \sup_{v\in[T,t)} L^f_v\big)|\mathcal{F}_T\Big]$$

for any stopping time $T \in \mathcal{T}$. Then

$\Rightarrow$ $D^*_t = \sup_{0\le s\le t} L^f_s$ where $L = (L_s)_{0\le s<\hat{T}}$ is a **storage index** determined by

$$\mathbb{E}\Big(\int_t^{\hat{T}} f(s, \sup_{v\in[t,s]} L_v)\, ds|\mathcal{F}_t\Big) = M\widehat{H}_t \quad (0 \le t < \hat{T})\,.$$