Robust Asset Allocation Under Model Uncertainty

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Agenda

- ⋆ Classical approaches to the Asset Allocation Problem.
- ⋆ Factor Model Performances: Empirical Evidence.
- ⋆ Decision Under Ambiguity: Literature Review.
- ⋆ A Robust Alternative Approach to Model Ambiguity.
- ⋆ Evidence from Empirical Study : Outperformance of the ARA Portfolio.
- ⋆ Nonlinear Relative Ambiguity Adjustment.

Classical Approaches to the Asset Allocation Problem

Objective: Presentation of classical models used to represent asset returns.

The Asset Allocation Problem

⋆ An investor wants to allocate her wealth among different assets traded in the capital markets (typically N risky assets and a risk free asset).
Question: Find the proportion of the wealth to invest in each asset.

⋆ *Choice criterion:* choosing the allocation φ as to maximize the expected utility u of the future wealth X^φ at a given time horizon (subject to some potential constraint).

Remark: There is always the constraint $\sum_{i=0}^{N} \varphi^i = 1$.

The Efficient Frontier (Markovitz - 1952)

The optimization problem of the investor is:

$$\boxed{\max_\varphi \mathbb{E}_{\mathbb{P}}[u(X^\varphi, \lambda)]}$$

where $\mathbb{P}$ stands for the prior probability measure of the investor. Depending on her risk aversion λ, the investor chooses a portfolio on the Efficient Frontier.

⇒ Several classical models for asset returns. Among them:

The CAPM (Sharpe 1969)

Any random asset return r^i can be separated into a systematic component and a residual component:

$r^i = r_f + \beta^i(r^M - r_f) + \epsilon^i$

The APT (Ross - 1976)

The APT, unlike the CAPM, is not a consensus model as it depends on the selection of the K factors made by each investor.

$r^i = \mathbb{E}(r^i) + \sum_{k=1}^{K} f^k \beta^{k,i} + \epsilon^i$

-where K is the number of factors selected, f^k stands for the centred return of the factor k and $\beta^{k,i}$ is the loading of the asset i on the factor k.

Factor Model Performances: Empirical Evidence

Objective:

⇒ To compare the performance of classical portfolios on European stock data.

⇒ Empirical evidence that none of the classical models can be fully trusted over an extended period of time.

Data and assumptions

⋆ Daily close prices of Eurostoxx 600 constituents from Jan 2000 to April 2010.

⋆ Cleaning process to get rid of doubtful data.

⋆ Assumptions:

◇ The risk free rate is assumed to be zero (daily data),

◇ No transactions costs, fees and slippage are considered as it is a comparative performance study.

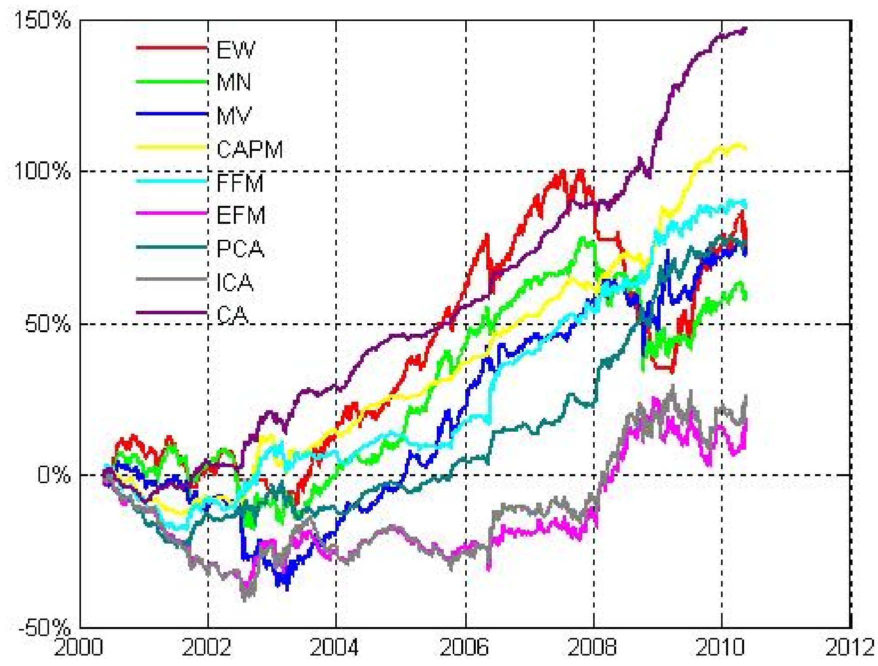
⋆ We run a back test on historical data when at each date the investor re-balances her portfolio by re-estimating the different models over a training window of 120 days.

Different models

- A market benchmark: the equally weighted portfolio (EW)
- Two Classical Markovitz portfolios:
  - ⋆ The minimum variance portfolio (MN)
  - ⋆ The mean-variance portfolio (MV)
- The CAPM portfolio (CAPM)
- Two exogeneous factors - APT based portfolios:
  - ⋆ A Fundamental Factor Model portfolio (FFM)
  - ⋆ An External Factor Model portfolio (EFM)
- Three endogeneous factors - APT based portfolios:
  - ⋆ The Principal Component Analysis portfolio (PCA)
  - ⋆ The Independant Component Analysis portfolio (ICA)
  - ⋆ The Cluster Analysis portfolio (CA)

Portfolio performances

Figure 1: Factor Models Returns



⇒ Very volatile performances: no model consistantly outperforms others

⇒ Evidence for model ambiguity

Decision Under Ambiguity: Literature Review

Objective:

⇒ Presentation of the different models proposed in the literature to account for ambiguity in a decision problem.

⇒ Often, those models are difficult to implement and restrictive.

Knight Uncertainty (Knight - 1931) Decision makers often consider different models, none of which they can fully trust.

$\Rightarrow$ Necessity to take into account uncertainty in the decision process. We focus on the literature of Asset Allocation problem.

Subjective expected utility (Savage - 1954) The investor considers a set of possible models $\mathcal{Q}$ as well as a distribution μ on these models.

The optimization problem is now:

$$\boxed{\max_{\varphi} \sum_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[u(X^{\varphi})]\mu(\mathbb{Q})}$$

$\Rightarrow$ The investor has several priors as reference, but there exists *no aversion towards this uncertainty*.

$\Rightarrow$ Aversion to model risk (or ambiguity) prevents the investors from using this classical framework of expected utility maximization to compute optimal allocations (Ellsberg paradox - 1961).

Max-min approach (Gilboa & Schmeidler - 1989) Maximizes the minimum expected utility over the set of models $\mathcal{Q}$:

$$\boxed{\max_{\varphi} \min_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[u(X^{\varphi})]}$$

$\Rightarrow$ Too conservative as only considers the worst case scenario.

Penalized max-min approach (Maccheroni, Marinacci & Rustichini - 2006) Penalizes each model differently, using a penalty function α:

$$\boxed{\max_{\varphi} \min_{\mathbb{Q}\in\mathcal{Q}} \{\mathbb{E}_{\mathbb{Q}}[u(X^{\varphi})] - \alpha(\mathbb{Q})\}}$$

Other references: Hansen and Sargent (2001) (entropic penalty function), Garlappi, Uppal and Wang (2007) and Epstein and Schneider (2007) (constrained optimization).

$\Rightarrow$ How to choose the penalty function? Often expressed in terms of a reference model: how to choose the reference?

Integrated approach (Klibanoff, Marinacci and Mukerji (KMM) - 2005)

The idea is to develop a generalized model, by introducing a function Ψ characterizing the ambiguity aversion of the investor through a parameter γ.

The optimization problem is then:

$$\boxed{\max_{\varphi} \sum_{\mathbb{Q}\in\mathcal{Q}} \Psi\left(\mathbb{E}_{\mathbb{Q}}[u(X^{\varphi})], \gamma\right) \mu(\mathbb{Q})}$$

$\Rightarrow$ Almost impossible to implement (calibration, dimension...), especially when some additional constraints are added.

$\Rightarrow$ Our objective: To introduce a *simple* (practical and easily implementable) approach to account for model risk in a *robust* way (i.e. independent of the considered class of models and of the optimization criterion).

A Robust Alternative Approach to Model Ambiguity

Main idea:

⇒ A novel and general method is proposed to account for ambiguity, simple to implement and very flexible.

⇒ The objective is to find a trade-off between optimization and robustness.

$\Rightarrow$ A new approach to model ambiguity that is more flexible, easier to compute, and more tractable than the previous methods.
$\Rightarrow$ Independent of the set of models considered $\mathcal{Q}$, as well as of the choice criterion.

General principle

The ambiguity robust adjustment is a two-step procedure, introducing a distinction between two types of ambiguity aversion:

- $\star$ *Absolute ambiguity aversion:* ambiguity aversion for a given model, independently of the other models,
- $\star$ *Relative ambiguity aversion:* ambiguity aversion for a given model, relatively to the other models.

Two-step procedure for the ambiguity robust adjustment:

1. First solve the optimization problem for each model *as if* it was the "true" one. Then adjust the outcome using an *Absolute Ambiguity Robust Adjustment* function ψ.

2. Second, aggregate the adjusted outcomes computed for each model through a *Relative Ambiguity Robust Adjustment* function π.

$$\boxed{\text{simple optimization outcome } x_q \;\Rightarrow\; \psi(x_q) \;\Rightarrow\; \sum_q \psi(x_q)\pi(q)}$$

Comments:

⋆ The decision-maker has to solve a series of simple optimization problems (with constraints) instead of a large complex optimization problem.

⋆ The absolute adjustment is made on the outcome of the optimization problem.

$\Rightarrow$ Flexibility to adjust the same model differently according to the purpose.

⋆ Adding a new model simply modifies the second step.

In the case of the portfolio allocation problem:

1. First, for each model $\mathbb{Q} \in \mathcal{Q}$, get the desired weights $\varphi_i^{\mathbb{Q}}$, for instance by solving the optimization problem

$$\varphi \rightarrow \max_{\varphi} \mathbb{E}_{\mathbb{Q}}[u(X^{\varphi})] \qquad \text{and} \qquad \varphi^{\mathbb{Q}} \equiv \text{argmax} \mathbb{E}_{\mathbb{Q}}[u(X^{\varphi})]$$

Then adjust the weights using the function ψ:

$$\psi\left(\varphi_i^{\mathbb{Q}}\right)$$

2. Finally, the ambiguity adjusted weights are obtained as:

$$\boxed{\varphi_i^{ARA} \equiv \sum_{\mathbb{Q} \in \mathcal{Q}} \psi\left(\varphi_i^{\mathbb{Q}}\right) \pi(\mathbb{Q})}$$

Absolute ambiguity robust adjustment

Idea: Scaling down the optimal weights generated by each model.

Axiomatic characterization of the function ψ:

- ⋆ *Universality:* ψ is identical across all models, but parametrized by an ambiguity aversion parameter γ that may vary and be model specific.
- ⋆ *Monotonicity:* ψ preserves the relative order of the optimal weights obtained for any given model $\mathbb{Q}$.
  The relative preference of the investor towards the different risky assets for a given model $\mathbb{Q}$ is preserved through the transformation ψ.
- ⋆ *Convexity:* The function ψ is concave and then convex, so that ψ reduces more the (absolute) largest weights for each model considered (shrinking effect).

+ Some additional properties (Symmetry, Invariant point, Limit behaviour)

An example for the function ψ is:

$$\psi(x,\gamma) \equiv \begin{cases} \frac{1-\exp^{-\gamma x}}{\gamma}, 0 \leq x \leq 1 \\ \frac{\exp^{\gamma x}-1}{\gamma}, -1 \leq x \leq 0 \end{cases}$$

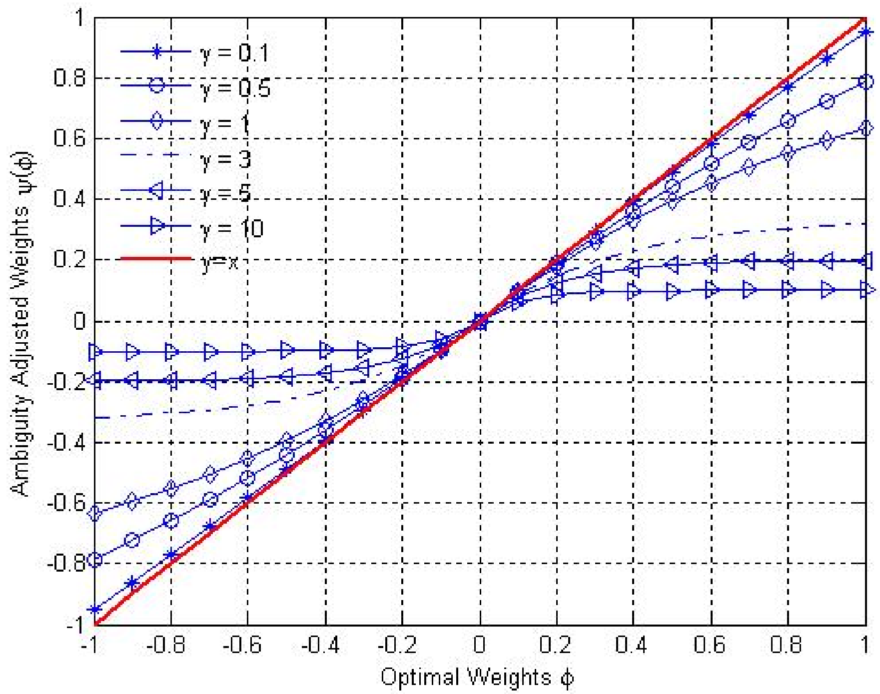


Figure 2: ψ for different values of the ambiguity aversion parameter γ

Relative ambiguity robust adjustment

Characterization of the function π:

- *Second step of the procedure:* after an independent computation and adjustment of the allocation (through ψ), aggregation across all models.
- The relative ambiguity robust adjustment function π represents *how trustworthy each model is* according to the investor's anticipations and *relatively* to the other models of the class $\mathcal{Q}$.
  It measures the *relative ambiguity aversion* the investor displays towards each model $\mathbb{Q}$ among the models in $\mathcal{Q}$.
- The function $\pi : \mathcal{Q} \to [0;1]$ can be seen as a model weighting function.

Comments: π is not necessarily normalized.

$$\forall \mathbb{Q} \in \mathcal{Q}, 0 \leq \pi(\mathbb{Q}) \leq 1 \qquad \text{and} \qquad \sum_{\mathbb{Q} \in \mathcal{Q}} \pi(\mathbb{Q}) \leq 1.$$

If $\sum_{\mathbb{Q} \in \mathcal{Q}} \pi(\mathbb{Q}) < 1$, the investor believes that the set $\mathcal{Q}$ does not give a full understanding of the situation: $1 - \sum_{\mathbb{Q} \in \mathcal{Q}} \pi(\mathbb{Q})$ represents the global aversion for $\mathcal{Q}$.

Role of the risk-free asset

$\star$ The risk-free asset can be seen as a *refuge value*.

The more the investor is averse to ambiguity, the bigger her proportional asset allocation in the risk free asset.

The ambiguity aversion leads the investor to invest less in risky (ambiguous) assets.
$\Rightarrow$ The "desinvested" risky investment value due to the presence of ambiguity is transferred into the risk-free asset.

$\star$ After the ambiguity adjustment, the weight of any risky asset is φ_i^{ARA} and the weight of the risk-free asset is:

$$\varphi_0^{ARA} = 1 - \sum_{i=1}^{N} \varphi_i^{ARA}$$

Evidence from Empirical Study : Outperformance of the ARA Portfolio

Objective: Empirical evidence that the ARA methodology enhances portfolio performances.

Ambiguity robust adjustment - Calibration

$\star$ *Dynamic aversion to ambiguity:* Depending on the considered period, the investor will be more or less confident about the overall set of models she considers (the ambiguity aversion does not necessarily decrease over time).

$\star$ *Simple empirical calibration methodology*, taking into account the relative historical performance of the different models to calibrate the functions ψ and π:

- First, we compute historical time series of some performance measure for the different models and the problem we consider.
- The ambiguity aversion parameter γ can be calibrated as the inverse of the performance measure.
- The measure π can then be computed as a weighted average of the performance measure.

⇒ Same data used as in previous empirical study.

⇒ Several performance measures used to calibrate and test portfolios:

Performance measures used for calibration

- Sharpe ratio: ratio of the mean return of a portfolio over its standard deviation.
- Sortino Price ratio: ratio of the mean return of a portfolio over the standard deviation of its negative returns.
- Gain Loss ratio: ratio of total positive returns over total negative returns.
- Winner Loser ratio: ratio of the number of total positive returns over the number of total negative returns.

Performance measures used to compare models

- Certainty equivalent ratio: equivalent risk-free return (portfolio return adjusted for its risk): the higher, the better.
- Turnover: change in portfolio weights from one re-balancing period to the next: the lower, the better.

Single portfolio performances

Figure 3: Strategies Returns

Figure 4: ARA Strategies Returns

	ARA	SEU	Diff(%)
μ(%)	63.34	55.32	14.51
$\overline{\mu}$(Bps)	2.43	2.12	14.51
σ(%)	9.23	9.30	−0.71
max(μ)(Bps)	656.03	656.03	0.00
min(μ)(Bps)	−408.09	−408.09	0.00
Sharpe	0.66	0.57	**15.32**
Sortino	0.81	0.69	**17.51**
GainLoss(%)	53.20	52.73	0.88
WinLose(%)	52.91	53.58	−1.24
CER(Bps)	2.25	1.94	**15.92**
T/O(%)	99.84	113.38	**−11.94**

Table 1: SEU and RA Sharpe Strategies Comparison

$\Rightarrow$ The ARA portfolio beats the SEU portfolio in terms of Sharpe, Sortino and CER by more than 15%.

Nonlinear Relative Ambiguity Adjustment

Objective: Accounting for some none linear effects further improves the ARA portfolio performances empirically.

Motivation

We would like to take into account some additional features such as:

⋆ Non-linear effect when mixing models: e.g, when model A performs well, model B tends to perform even better.

⋆ If the various models disagree particularly on a specific asset, this asset should be even more penalized and vice versa.

⇒ Model the Relative Ambiguity Robust Adjustment as a non-linear function.

The Support Vector Machines

⋆ A nonlinear, non-parametric method to estimate π, the SVM separate non linearly transformed data into hyperplanes:

$$f(x) = < \omega, \pi(x) > + \omega_0$$

where π is defined by a non linear kernel k: $k(x, y) \equiv < \pi(x), \pi(y) >$, and ω is the support vector.

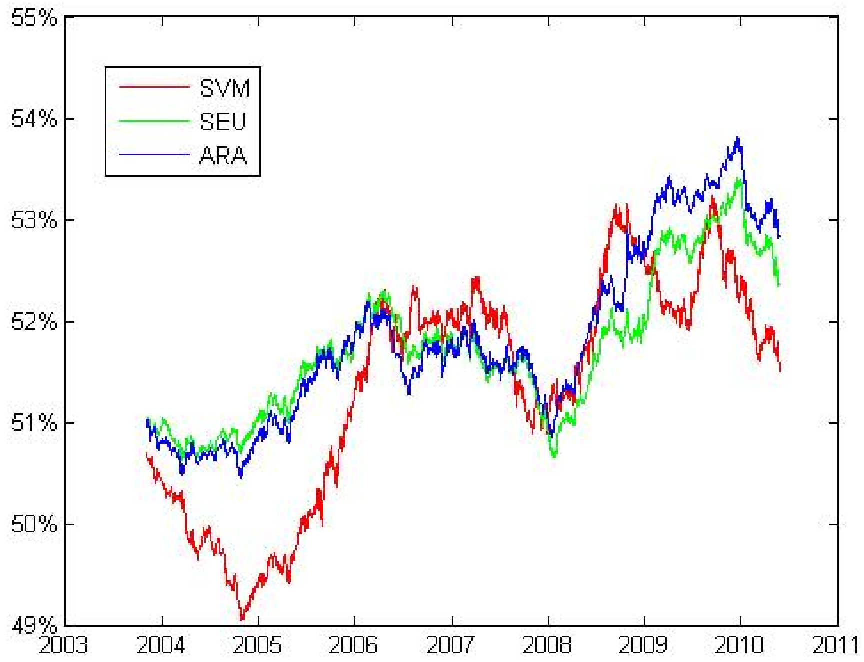
Backtest

⋆ We run a back test on historical data, where SVM predicts the sign of the next day return:

⋆ Some issues:

- Better in some periods than SEU or ARA with linear π, but much worse other times. Also, more volatile.
- Heavy in terms of computation.

Figure 5: SEU, ARA and SVM strategies Hit ratios



A more ad hoc method to calibrate π

⋆ Start from a list of desired nonlinear properties to construct an ad hoc function π:

- The weight dispersion: If $v^i < v^j$ (meaning that the models disagree more on asset j than on asset i), then the final ARA weight $\phi^{i,ARA}$ should be closer relatively to the mean μ^i than the weight $\phi^{j,ARA}$ to the mean μ^j.
- The precautionary principle: The investor is able to set a threshold, v^{max}, such that if the dispersion of the different models considered is above this threshold for a given asset, the final robust ARA weight should be set to zero.
- The global ambiguity aversion: Represented by a cash buffer or an overall cap on the sum of total asset allocations for the ARA portfolio.

More specifically, considering a cap value of 1 and the variance as dispersion measure, the final nonlinear ARA allocation can be defined as:

$\forall i \in [1, N], \phi^{ARA,i} \equiv \max\{1, \frac{\mu_{\mathbb{Q}\in\mathcal{Q}}[\psi(\phi^{\mathbb{Q},i}, \gamma^{\mathbb{Q}})]}{\sigma^2_{\mathbb{Q}\in\mathcal{Q}}[\psi(\phi^{\mathbb{Q},i}, \gamma^{\mathbb{Q}})]}\}.$

Figure 6: Ad hoc nonlinear Sharpe ARA strategy versus linear Sharpe ARA strategy

	non linear	sharpe
$\mu(\%)$	85.88	63.34
$\overline{\mu}$(Bps)	3.29	2.43
$\sigma(\%)$	12.18	9.23
max(μ)(Bps)	820.39	656.03
min(μ)(Bps)	−504.85	−408.09
Sharpe	0.67	0.66
Sortino	0.86	0.81
GainLoss(%)	53.43	53.20
WinLose(%)	52.84	52.91
CER(Bps)	3.23	2.39
T/O(%)	104.32	99.84

Table 2: Non Linear Strategy Performance

⇒ The ad hoc nonlinear form for π greatly enhance the ARA portfolio.

Conclusion and further research

⋆ ARA is a new method to account for ambiguity in a decision problem.

⋆ Applied to the specific asset allocation problem, empirical evidence that the ARA methodology enhance portfolio performances.

⋆ Further research areas include:

- Calibration of the absolute and relative ambiguity aversion parameters.
- Other forms for the RARA function.
- Application of the ARA methodology on other fields.

Appendix

	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
EW	**1.54**	−0.58	−1.13	**1.76**	2.45	1.57	0.42	−3.69	**1.98**	−0.44
MN	**0.75**	**0.00**	−0.77	0.75	**2.82**	2.49	1.52	−1.51	0.98	−0.04
MV	**0.01**	−1.28	−1.30	**0.87**	2.46	2.10	1.57	−0.42	1.28	**0.23**
CAPM	−1.88	−0.01	**2.29**	0.18	**3.89**	**4.70**	**2.35**	1.45	**3.51**	−0.57
FFM	−2.07	**0.17**	**2.03**	−0.13	1.21	**2.63**	**2.08**	**1.82**	0.82	−0.28
EFM	−2.15	−1.88	0.84	−0.32	0.18	0.36	0.55	**1.75**	−0.45	0.10
PCA	−3.55	−0.09	1.06	−0.84	2.36	2.26	1.52	**3.93**	1.40	−0.26
ICA	−2.11	−1.97	0.63	−0.05	−0.12	1.15	0.28	1.32	0.06	**0.52**
CA	−2.47	**2.18**	**1.90**	**1.07**	**4.36**	**4.42**	**3.85**	2.02	**4.10**	**1.45**

Table 3: Sharpe per Strategy per Period no Transaction Costs

	EW	MN	MV	CAPM	FFM	EFM	PCA	ICA	CA
μ	60.83	−40.36	−23.04	−3.06	−29.52	−97.29	−39.33	−90.69	32.07
$\overline{\mu}$	2.33	−1.55	−0.88	−0.12	−1.13	−3.72	−1.51	−3.47	1.23
σ	13.69	12.91	12.50	6.30	8.14	12.06	7.14	12.57	6.29
max(μ)	656.03	805.77	766.45	373.50	539.07	643.67	503.37	640.99	361.20
min(μ)	−505.07	−710.86	−948.33	−380.61	−367.13	−560.71	−355.11	−510.23	−295.45
Sharpe	0.43	−0.30	−0.18	−0.05	−0.35	**−0.77**	−0.53	−0.69	**0.49**
Sortino	0.48	−0.37	−0.22	−0.06	−0.48	**−1.07**	−0.71	−0.95	**0.68**
GainLoss	52.13	48.51	49.09	49.77	48.31	**46.23**	47.42	46.59	**52.43**
WinLose	**55.35**	50.98	48.66	47.90	46.69	46.11	47.40	**45.75**	50.47
CER	**1.95**	−1.88	−1.19	−0.20	−1.26	**−4.02**	−1.61	−3.79	1.15
T/O	**19.33**	114.29	108.03	143.49	150.36	148.89	148.60	148.88	**150.53**

Table 4: Strategies Performances 3bps Transaction Costs

	sharpe	sortino	gainloss	winlose
μ(%)	55.32	51.28	47.80	49.10
$\overline{\mu}$(Bps)	2.12	1.96	1.83	1.88
σ(%)	9.30	9.04	9.26	9.31
max(μ)(Bps)	656.03	656.03	656.03	656.03
min(μ)(Bps)	−408.09	−408.09	−408.09	−408.09
Sharpe	0.57	0.54	0.49	0.50
Sortino	0.69	0.66	0.60	0.61
GainLoss(%)	52.73	52.62	52.39	52.44
WinLose(%)	53.58	53.48	52.96	53.12
CER(Bps)	1.94	1.80	1.66	1.71
T/O(%)	113.38	115.09	117.08	116.84

Table 5: SEU Strategies Performances 3 bps Transaction Costs

	sharpe	sortino	gainloss	winlose
μ(%)	63.34	57.65	55.12	58.30
$\overline{\mu}$(Bps)	2.43	2.21	2.11	2.23
σ(%)	9.23	8.75	9.69	9.91
max(μ)(Bps)	656.03	656.03	656.03	656.03
min(μ)(Bps)	−408.09	−408.09	−408.09	−430.06
Sharpe	0.66	0.63	0.54	0.56
Sortino	0.81	0.79	0.66	0.69
GainLoss(%)	53.20	53.10	52.69	52.77
WinLose(%)	52.91	52.77	52.91	52.83
CER(Bps)	2.25	2.05	1.92	2.04
T/O(%)	99.84	104.60	108.21	107.42
AAA	0.70	0.62	0.91	0.92
RAA	0.59	0.53	0.73	0.73

Table 6: ARA Strategies Performances 3 bps Transaction Costs

	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
EW	**1.41**	**−0.71**	−1.25	**1.68**	**2.34**	**1.50**	**0.35**	−3.75	**1.94**	**−0.50**
MN	**−0.28**	**−0.62**	−1.25	−0.75	**1.76**	1.36	0.34	−1.88	0.38	−0.98
MV	**−1.02**	−2.32	−1.90	**0.20**	**1.62**	0.82	**0.37**	−1.06	0.74	−0.79
CAPM	−3.65	−1.88	**1.04**	−1.28	0.08	**1.46**	−0.39	0.41	**2.19**	−3.41
FFM	−3.70	−1.38	**0.60**	−1.42	−1.68	1.30	0.28	**0.92**	−0.22	−1.89
EFM	−3.45	−3.01	−0.20	−1.33	−1.65	−0.65	−0.37	**1.17**	−1.12	**−0.52**
PCA	−5.49	−1.84	−0.13	−2.13	−1.08	−0.22	−1.16	**2.72**	0.39	−2.43
ICA	−3.39	−3.18	−0.37	−1.08	−1.90	0.19	−0.64	0.75	−0.54	**−0.12**
CA	−4.50	**0.25**	**0.55**	**−0.43**	0.49	**1.53**	**1.21**	0.91	**2.69**	−1.03

Table 7: Sharpe per strategy per periods 3 bps Transaction Costs

	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
sharpe	**0.53**	−1.52	**0.03**	−0.97	**1.94**	**1.59**	**0.21**	**0.68**	**1.48**	−1.19
sortino	**0.51**	−1.54	−0.03	−0.99	**1.98**	**1.73**	**0.26**	**0.53**	**1.60**	−1.32
gainloss	**0.29**	−1.30	**1.07**	−0.69	**1.73**	**1.11**	−0.12	**0.63**	**1.36**	−1.25
winlose	**0.36**	−1.26	**1.15**	−0.80	**1.75**	**0.98**	−0.03	**0.68**	**1.34**	−1.11

Table 8: Sharpe per ARA strategy per periods