Distorted Optimal Stopping

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Based on joint work with Zuoquan Xu

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Outlines

- Model formulation: optimal stopping with probability distortions

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- Interpretations and implications

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  - Present *simultaneously*

Probability Distortion: Yaari's Dual Theory

- Non-linear transformation of the decumulative distribution function

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- Kahneman and Tversky (1979) - prospect theory; Yaari (1987) - dual theory of choice; Lopes (1987) - SP/A theory; insurance literature ...

Distortion (Weighting) Functions

- Kahneman and Tversky (1992) distortion

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}},$$

- Tversky and Fox (1995) distortion

$$w(p) = \frac{\delta p^\gamma}{\delta p^\gamma + (1-p)^\gamma},$$

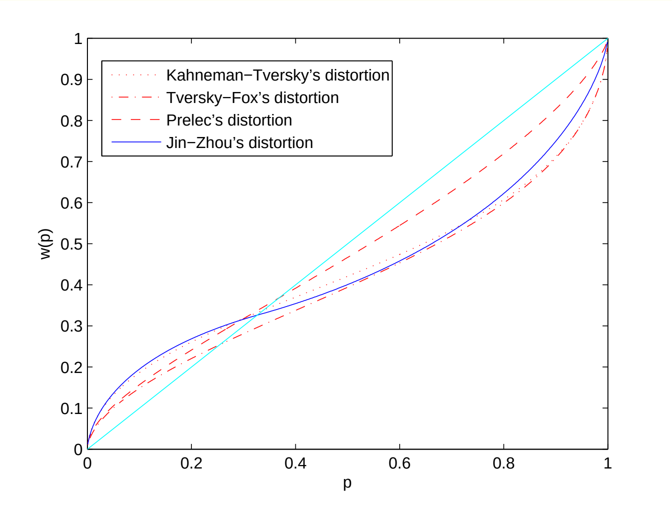
- Prelec (1998) distortion

$$w(p) = e^{-\delta(-\ln p)^\gamma}$$

- Jin and Zhou (2008) distortion

$$w(p) = \begin{cases} y_0^{b-a} k e^{a\mu + \frac{(a\sigma)^2}{2}} \Phi\left(\Phi^{-1}(p) - a\sigma\right), & p \le 1 - z_0, \\ C + k e^{b\mu + \frac{(b\sigma)^2}{2}} \Phi\left(\Phi^{-1}(p) - b\sigma\right), & z \ge 1 - z_0 \end{cases}$$

Distortion Functions (Cont'd): Reverse S-Shaped



A Model of Distorted Optimal Stopping

- An asset's (discounted) price follows GBM on $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t \geqslant 0})$:

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- $U(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ (*payoff/utility*) non-decreasing, $w(\cdot) : [0,1] \mapsto [0,1]$ (*probability distortion/weighting*) smooth, increasing with $w(0) = 0$ and $w(1) = 1$

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- If $w(x) \equiv x$, then $J(\tau) = \mathbf{E}[U(P_\tau)]$

Application Examples

- Stock selling (liquidation)
- Perpetual American option
- Irreversible investment
- ... all with probability distortions

Optimal Stopping: Literature

- Huge literature: Shiryaev (1978), Peskir and Shiryaev (2006) ...
- Shiryaev, Peskir, etc. (2000–): stopping a Brownian motion closest to maximum
- Shiryaev, Xu, Zhou (2008): stopping a GBM to minimise the relative error with respect to the maximum
- Henderson (2009): liquidating a stock with prospect theory payoff/utility (but no distortion); disposition effect
- Nishimura and Ozaki (2007), Riedel (2010): optimal stopping with ambiguity

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  - Nonlinear expectation with Choquet integration: martingale approach fails
  - Time inconsistency: dynamic programming and HJB fail

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- Numerical (exhaustive) solutions

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    - Finding the best selling price: quantile/distribution formulation
    - Recovering optimal selling time: Skorokhod embedding

Skorokhod Embedding

- *Skorokhod embedding problem*: Given a standard Brownian motion B_t and a probability measure m with 0 mean and finite second moment, find an integrable stopping time τ such that the distribution of B_τ is m
- Introduced and solved by Skorokhod (1961)
- Great number of variants, generalisations and applications
- Obłój (2004): a very nice survey!

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- Problem (1) is equivalent to

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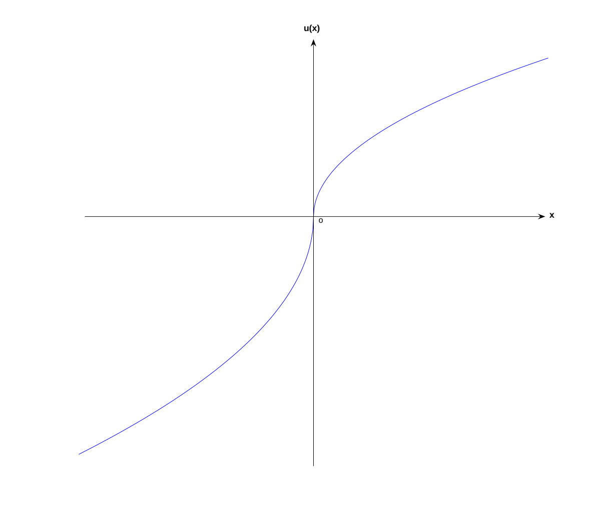
- $U(x) = \frac{1}{1-\gamma}x^\gamma$, $\gamma \in (0,1)$. If $\beta < 0$, $u(x) = \frac{1}{1-\gamma}x^{\gamma/\beta}$ is decreasing convex. If $0 < \beta \leqslant \gamma$, $u(x)$ is increasing convex; If $\beta > 1/\gamma$, then $u(x)$ is increasing concave

S-shaped Function



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- For a general increasing utility function $U(\cdot)$, $u(x) = U(x^{1/\beta})$ is decreasing if $\beta < 0$, and increasing if $\beta > 0$

Thou Shalt Buy and Hold

Theorem

If $u(\cdot)$ *is non-increasing, then* (2) *has the optimal value* $u(0+)$ *and*

$$\lim_{T\to+\infty} J(T) = \sup_{\tau\in\mathcal{T}} J(\tau). \tag{3}$$

Moreover, if $u(\ell) = u(0+)$ *for some* $\ell > 0$*, then*

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- One should hold the asset perpetually
- Consistent with the traditional investment wisdom
- Consistent with Shiryaev, Xu and Zhou (2008)

Distribution/Quantile Sets

- Henceforth we assume $u(\cdot)$ is non-decreasing
- For simplicity, we further assume that $u(\cdot)$ is absolute continuous with $u(0) = 0$

Define *distribution set* $\mathcal{D}$ and *quantile set* $\mathcal{G}$:

$$\mathcal{D} := \left\{ F : \mathbb{R}^+ \mapsto [0,1] \;\middle|\; F \text{ is CDF of } S_\tau, \text{ for some } \tau \in \mathcal{T} \right\}, \quad (5)$$

$$\mathcal{G} := \left\{ G : [0,1] \mapsto \mathbb{R}^+ \;\middle|\; G = F^{-1} \text{ for some } F \in \mathcal{D} \right\}. \quad (6)$$

Distribution/Quantile Formulation

Lemma

For any $\tau \in \mathcal{T}$,

$$J(\tau) = J_D(F) := \int_0^\infty w\,(1 - F(x))\,u'(x)\,\mathrm{d}x, \tag{7}$$

$$J(\tau) = J_Q(G) := \int_0^1 u\,(G(x))\,w'(1 - x)\,\mathrm{d}x, \tag{8}$$

where F and G are the distribution function and quantile function of S_τ, respectively. Moreover,

$$\sup_{\tau \in \mathcal{T}} J(\tau) = \sup_{F \in \mathcal{D}} J_D(F) = \sup_{G \in \mathcal{G}} J_Q(G). \tag{9}$$

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- In the context of utility theory both a probability distortion function and a utility function describe an investor's preference towards risk – they do play some dual roles (Yaari 1987)
- Freedom to choose the convenient formulation

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- This inequality is also *sufficient* for a CDF to be that of S_τ for some $\tau \in \mathcal{T}$!

Characterising Distribution/Quantile Sets

Theorem

We have the following expressions of the distribution set $\mathcal{D}$ and quantile set $\mathcal{G}$:

$$\mathcal{D} = \left\{ F \middle| F \text{ is a CDF, } \int_0^\infty (1 - F(x))\, \mathrm{d}x \leqslant s \right\}, \quad (10)$$

$$\mathcal{G} = \left\{ G \middle| G \text{ is a quantile, } \int_0^1 G(x) dx \leqslant s \right\}. \quad (11)$$

- In particular, both $\mathcal{D}$ and $\mathcal{G}$ are *convex*

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- Dybvig (1988), Schied (2004, 2005), Dana (2005), Carlier and Dana (2005), Jin and Zhou (2008)
- A general framework developed in He and Zhou (2009) for possibly non-concave utility function and non-convex/concave distortions

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- A maximum should be at "corners" of $\mathcal{D}$

Convex $w(\cdot)$: Results

Theorem

If $w(\cdot)$ is convex and $u(\cdot)$ is non-decreasing, then

$$\sup_{\tau \in \mathcal{T}} J(\tau) = \sup_{0<a\leqslant s\leqslant b} \left[\left(1 - w\left(\frac{s-a}{b-a}\right)\right) u(a) + w\left(\frac{s-a}{b-a}\right) u(b) \right]. \quad (14)$$

Moreover, if (14) has an optimal pair (a^*, b^*) with $a^* > 0$, then

$$\tau_{(a^*,b^*)} := \inf\{t \geqslant 0 : S_t \notin (a^*, b^*)\} \quad (15)$$

is an optimal stopping to problem (2).

Convex $u(\cdot)$: Quantile Formulation

By symmetry, if $u(\cdot)$ is convex, one uses quantile formulation

Theorem

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- Zhang (2001): cut-loss and take-profit thresholds are exogenously set
- Here we show that this strategy is endogenous for broad classes of problems
- It also recovers Henderson (2010) where probability distortion is absent

Concave $u(\cdot)$ *and* Convex $w(\cdot)$: *Dump That Bloody Stock!*

Corollary

If $u(\cdot)$ *is concave and* $w(\cdot)$ *is convex, then*

$$\sup_{\tau \in \mathcal{T}} J(\tau) \quad = \quad u(s). \tag{18}$$

Moreover, $\tau \equiv 0$ *is an optimal stopping.*

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- $w(\cdot)$ being convex corresponds to risk aversion
- A risk averse agent will sell a bad stock immediately

Concave $u(\cdot)$

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- Lagrange!

Concave $u(\cdot)$: Lagrange

- Consider a family of relaxed problems

$$\begin{aligned} J_Q^{\lambda}(G) &= \int_0^1 u(G(x))w'(1-x)\,\mathrm{d}x - \lambda\left(\int_0^1 G(x)\,\mathrm{d}x - s\right) \\ &= \int_0^1 f^{\lambda}(x, G(x))\,\mathrm{d}x + \lambda s, \end{aligned}$$

where $\lambda \geqslant 0$ and $f^{\lambda}(x,y) := u(y)w'(1-x) - \lambda y$

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  $G_\lambda(x) := (u')^{-1}\left(\frac{\lambda}{w'(1-x)}\right),\ x \in (0,1)$

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- Maximising $f^\lambda(x,\cdot)$ for each x we get $G_\lambda(x) := (u')^{-1}\left(\frac{\lambda}{w'(1-x)}\right),\ x \in (0,1)$
- Is G_λ qualified as a quantile function (primarily, non-decreasing)?

Concave $u(\cdot)$ and $w(\cdot)$

- When $w(\cdot)$ is also concave, then $G_\lambda(x) := (u')^{-1}\left(\frac{\lambda}{w'(1-x)}\right)$ is indeed non-decreasing, hence a quantile

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Concave $u(\cdot)$ and $w(\cdot)$: An Example

- Consider $u(x) = \frac{1}{\gamma}x^{\gamma}$, $0 < \gamma < 1$, and $w(x) = x^{\alpha}$, $0 < \gamma < \alpha < 1$: both concave

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$$F^*(x) = \begin{cases} 1 - \left(s\frac{\alpha-\gamma}{1-\gamma}\right)^{\frac{1-\gamma}{1-\alpha}} x^{-\frac{1-\gamma}{1-\alpha}}, & x \geqslant s\frac{\alpha-\gamma}{1-\gamma}; \\ 0, & x < s\frac{\alpha-\gamma}{1-\gamma}, \end{cases} \tag{20}$$

a *Pareto distribution* with Pareto index $\frac{1-\gamma}{1-\alpha} > 1$

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$$\tau_{AY} = \inf\left\{ t \geqslant 0 \; : \; S_t \leq \frac{\alpha-\gamma}{1-\gamma} \max_{0 \leqslant s \leqslant t} S_s \right\} \tag{21}$$

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Concave $u(\cdot)$ and Reverse S-shaped $w(\cdot)$

- Reverse S-shaped $w(\cdot)$: fear and hope are present simultaneously (He and Zhou 2009)

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$$G^*(x) = a\mathbf{1}_{(0,c]}(x) + \left(a \vee (u')^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\right)\mathbf{1}_{(c,1]}(x),$$

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  - Finite time horizon
  - Discounting factor explicit in objective functional
  - Time consistent strategies (formulation?)