Distorted Optimal Stopping

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Based on joint work with Zuoquan Xu

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Model formulation: optimal stopping with probability distortions
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- Motivation: economical/financial (why?); mathematical (how?)
Outlines

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- Results
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- Results
- Interpretations and implications
First decision: Choose between
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A: Win £5000 with 0.1% chance
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A: Win £5000 with 0.1% chance
B: Win £5 with 100% chance
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A was more popular (lottery)
First decision: Choose between

A: Win £5000 with 0.1% chance
B: Win £5 with 100% chance

A was more popular (lottery)

Second decision: Choose between
First decision: Choose between

- **A**: Win £5000 with 0.1% chance
- **B**: Win £5 with 100% chance

A was more popular (lottery)

Second decision: Choose between

- **C**: Lose £5000 with 0.1% chance
First decision: Choose between
A: Win £5000 with 0.1% chance
B: Win £5 with 100% chance
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Second decision: Choose between
C: Lose £5000 with 0.1% chance
D: Lose £5 with 100% chance
First decision: Choose between
- A: Win £5000 with 0.1% chance
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Second decision: Choose between
- C: Lose £5000 with 0.1% chance
- D: Lose £5 with 100% chance
  - D was more popular (insurance)
First decision: Choose between
  A: Win £5000 with 0.1% chance
  B: Win £5 with 100% chance
    ■ A was more popular (lottery)

Second decision: Choose between
  C: Lose £5000 with 0.1% chance
  D: Lose £5 with 100% chance
    ■ D was more popular (insurance)

People tend to exaggerate, *intentionally or unintentionally*, small probabilities of both winning big *and* losing big
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People tend to exaggerate, intentionally or unintentionally, small probabilities of both winning big and losing big

Exaggerating small probability of huge gains - risk seeking
First decision: Choose between
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People tend to exaggerate, intentionally or unintentionally, small probabilities of both winning big and losing big
- Exaggerating small probability of huge gains - risk seeking
- Exaggerating small probability of huge losses - risk aversion
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People tend to exaggerate, *intentionally or unintentionally*, small probabilities of both winning big and losing big

- Exaggerating small probability of huge gains - risk seeking
- Exaggerating small probability of huge losses - risk aversion
- Present *simultaneously*
Non-linear transformation of the decumulative distribution function

\[
V(X) = \int_{\mathbb{R}^+} w(P(X > x)) \, dx \equiv \int_{\mathbb{R}^+} xd[-w(1 - F_X(x))] \\
= \int_{\mathbb{R}^+} xw'(1 - F_X(x)) \, dF_X(x)
\]
Probability Distortion: Yaari’s Dual Theory

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- \( V(X) \equiv \int_{\mathbb{R}^+} (w \circ P)(X > x)dx: \ \text{Choquet expectation under capacity} \ w \circ P \ - \ \text{non-expected utility} \)
Probability Distortion: Yaari’s Dual Theory

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- Kahneman and Tversky (1979) - prospect theory; Yaari (1987) - dual theory of choice; Lopes (1987) - SP/A theory; insurance literature ...
Distortion (Weighting) Functions

- Kahneman and Tversky (1992) distortion

\[
    w(p) = \frac{p^\gamma}{(p^\gamma + (1 - p)^\gamma)^{1/\gamma}},
\]

- Tversky and Fox (1995) distortion

\[
    w(p) = \frac{\delta p^\gamma}{\delta p^\gamma + (1 - p)^\gamma},
\]

- Prelec (1998) distortion

\[
    w(p) = e^{-\delta(-\ln p)^\gamma}
\]

- Jin and Zhou (2008) distortion

\[
    w(p) = \begin{cases} 
        y_0^{b-a} ke^{a\mu+\frac{(a\sigma)^2}{2}} \Phi \left( \Phi^{-1}(p) - a\sigma \right), & p \leq 1 - z_0, \\
        C + ke^{b\mu+\frac{(b\sigma)^2}{2}} \Phi \left( \Phi^{-1}(p) - b\sigma \right), & z \geq 1 - z_0
    \end{cases}
\]
Distortion Functions (Cont’d): Reverse $S$-Shaped

- Kahneman–Tversky’s distortion
- Tversky–Fox’s distortion
- Prelec’s distortion
- Jin–Zhou’s distortion

$w(p)$ vs $p$
A Model of Distorted Optimal Stopping

An asset’s (discounted) price follows GBM on $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t \geq 0})$:

$$dP_t = \mu P_t dt + \sigma P_t dB_t, \quad P_0 = P_0$$
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- $U(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ (payoff/utility) non-decreasing,
- $w(\cdot) : [0, 1] \mapsto [0, 1]$ (probability distortion/weighting) smooth, increasing with $w(0) = 0$ and $w(1) = 1$
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- $\mathcal{T}$: set of $\{\mathcal{F}_t\}_{t \geq 0}$-stopping times $\tau$ with $P(\tau < +\infty) = 1$
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- Problem: To maximise “distorted” mean payoff/utility

\[
J(\tau) := \int_0^\infty w(\mathbb{P}(U(P_\tau) > x)) \, dx \tag{1}
\]

over \( \tau \in \mathcal{T} \)
A Model of Distorted Optimal Stopping

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  $$J(\tau) := \int_{0}^{\infty} w(P(U(P_{\tau}) > x)) \, dx$$

  over $\tau \in \mathcal{T}$

- If $w(x) \equiv x$, then $J(\tau) = E[U(P_{\tau})]$
Application Examples

- Stock selling (liquidation)
- Perpetual American option
- Irreversible investment
- ... all with probability distortions
Optimal Stopping: Literature

  ...
- Shiryaev, Peskir, etc. (2000–): stopping a Brownian motion closest to maximum
- Shiryaev, Xu, Zhou (2008): stopping a GBM to minimise the relative error with respect to the maximum
- Henderson (2009): liquidating a stock with prospect theory payoff/utility (but no distortion); disposition effect
- Nishimura and Ozaki (2007), Riedel (2010): optimal stopping with ambiguity
Conventional approaches in optimal stopping
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- **Martingale**: optional sampling, change of time, change of measure
Conventional approaches in optimal stopping

- *Martingale*: optional sampling, change of time, change of measure
- *Dynamic programming*: variational inequality/HJB equation, free boundary
Mathematical Challenges with Probability Distortions

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  - Nonlinear expectation with Choquet integration: martingale approach fails
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- Distorted optimal stopping
  - Nonlinear expectation with Choquet integration: martingale approach fails
  - Time inconsistency: dynamic programming and HJB fail
Barberis’s Casino Gambling Model

- Barberis (2010): exit strategy in casino gambling with prospect theory preferences
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- Numerical (exhaustive) solutions
Our Goal

- Find *pre-committed* optimal stopping strategies (in continuous time)
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    - Finding the best selling price: quantile/distribution formulation
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  - Mathematical: Take *distribution/quantile* of selling price as decision variable
    - Finding the best selling price: quantile/distribution formulation
    - Recovering optimal selling time: Skorokhod embedding
Skorokhod Embedding

- **Skorokhod embedding problem**: Given a standard Brownian motion $B_t$ and a probability measure $m$ with 0 mean and finite second moment, find an integrable stopping time $\tau$ such that the distribution of $B_{\tau}$ is $m$

- Introduced and solved by Skorokhod (1961)
- Great number of variants, generalisations and applications
- Obłój (2004): a very nice survey!
Asset price:

\[ dP_t = \mu P_t dt + \sigma P_t dB_t \]
Making Martingale

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- Let

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- Define \( u(x) := U(x^{1/\beta}), \quad \forall x \in (0, +\infty) \)
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- Define \( u(x) := U(x^{1/\beta}), \quad \forall x \in (0, +\infty) \)

- Problem (1) is equivalent to
  \[ J(\tau) = \int_0^\infty w(P(U(P_\tau) > x)) \, dx = \int_0^\infty w(P(u(S_\tau) > x)) \, dx, \quad (2) \]
Shape of $u(\cdot)$ and Quality of Asset

- $U(x) = (x - K)^+$ for some $K > 0$: $u(x) = (x^\beta - K)^+$
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  - If $0 < \beta \leq 1$ or $0 \leq \frac{\mu}{\sigma^2} < 0.5$: $u(\cdot)$ is non-decreasing and $S$-shaped.
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  - If $0 < \beta \leq 1$ or $0 \leq \frac{\mu}{\sigma^2} < 0.5$: $u(\cdot)$ is non-decreasing and $S$-shaped.
  - If $\beta > 1$ or $\mu < 0$, asset is “bad”: $u(\cdot)$ is non-decreasing and convex

- $U(x) = \frac{1}{1-\gamma} x^\gamma$, $\gamma \in (0, 1)$. If $\beta < 0$, $u(x) = \frac{1}{1-\gamma} x^{\gamma/\beta}$ is decreasing convex. If $0 < \beta \leq \gamma$, $u(x)$ is increasing convex; If $\beta > 1/\gamma$, then $u(x)$ is increasing concave
$S$-shaped Function

\[ u(x) \]

\[ x \]

\[ o \]
Shape of $u(\cdot)$ vs Quality of Asset (Cont’d)

- $U(x) = \ln(x + 1)$, $u(x) = \ln(x^{1/\beta} + 1)$ is decreasing if $\beta < 0$, increasing $S$-shaped if $0 < \beta < 1$, and increasing concave if $\beta \geq 1$
Shape of $u(\cdot)$ vs Quality of Asset (Cont’d)

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- $U(x) = 1 - e^{-\alpha x}, \alpha > 0, u(x) = 1 - e^{-\alpha x^{1/\beta}}$ is decreasing if $\beta < 0$, increasing concave if $0 < \beta < 1$, and increasing $S$-shaped if $\beta \geq 1$
Shape of \( u(\cdot) \) vs Quality of Asset (Cont’d)

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- \( U(x) = 1 - e^{-\alpha x}, \ \alpha > 0, \ u(x) = 1 - e^{-\alpha x^{1/\beta}} \) is decreasing if \( \beta < 0 \), increasing concave if \( 0 < \beta < 1 \), and increasing \( S \)-shaped if \( \beta \geq 1 \)

- For a general increasing utility function \( U(\cdot) \), \( u(x) = U(x^{1/\beta}) \) is decreasing if \( \beta < 0 \), and increasing if \( \beta > 0 \)
Theorem

If \( u(\cdot) \) is non-increasing, then (2) has the optimal value \( u(0+) \) and

\[
\lim_{T \to +\infty} J(T) = \sup_{\tau \in T} J(\tau). \tag{3}
\]

Moreover, if \( u(\ell) = u(0+) \) for some \( \ell > 0 \), then

\[
\tau_\ell = \inf\{ t \geq 0 : S_t \leq \ell \} \tag{4}
\]

is an optimal stopping. If \( u(\ell) < u(0+) \) for every \( \ell > 0 \), then (2) has no optimal solution.

- \( u(\cdot) \) being non-increasing corresponds to \( \beta < 0 \) (asset is good)
If $u(\cdot)$ is non-increasing, then (2) has the optimal value $u(0+)$ and

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- One should hold the asset perpetually
### Thou Shalt Buy and Hold

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- Consistent with the traditional investment wisdom
Theorem

If $u(\cdot)$ is non-increasing, then (2) has the optimal value $u(0+)$ and

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\lim_{T \to +\infty} J(T) = \sup_{\tau \in \mathcal{T}} J(\tau). 
$$

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Moreover, if $u(\ell) = u(0+)$ for some $\ell > 0$, then

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- One should hold the asset perpetually
- Consistent with the traditional investment wisdom
- Consistent with Shiryaev, Xu and Zhou (2008)
Henceforth we assume $u(\cdot)$ is non-decreasing.

For simplicity, we further assume that $u(\cdot)$ is absolute continuous with $u(0) = 0$.

Define the distribution set $\mathcal{D}$ and the quantile set $\mathcal{G}$:

$$\mathcal{D} := \left\{ F : \mathbb{R}^+ \mapsto [0, 1] \mid F \text{ is CDF of } S_\tau, \text{ for some } \tau \in \mathcal{T} \right\},$$

(5)

$$\mathcal{G} := \left\{ G : [0, 1] \mapsto \mathbb{R}^+ \mid G = F^{-1} \text{ for some } F \in \mathcal{D} \right\}. \quad (6)$$
Lemma

For any \( \tau \in \mathcal{T} \),

\[
J(\tau) = J_D(F) := \int_0^\infty w(1 - F(x)) u'(x) \, dx, \tag{7}
\]

\[
J(\tau) = J_Q(G) := \int_0^1 u(G(x)) w'(1 - x) \, dx, \tag{8}
\]

where \( F \) and \( G \) are the distribution function and quantile function of \( S_\tau \), respectively. Moreover,

\[
\sup_{\tau \in \mathcal{T}} J(\tau) = \sup_{F \in \mathcal{D}} J_D(F) = \sup_{G \in \mathcal{G}} J_Q(G). \tag{9}
\]
Notice certain symmetry – or duality – between the two formulations.
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$w(\cdot)$ and $u(\cdot)$ play symmetric roles in the two formulations
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In the context of utility theory both a probability distortion function and a utility function describe an investor’s preference towards risk – they do play some dual roles (Yaari 1987)
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Freedom to choose the convenient formulation
Optional Sampling

Let $F \in \mathcal{D}$: $F$ is CDF of $S_\tau$ for some $\tau \in \mathcal{T}$
Optional Sampling

- Let $F \in \mathcal{D}$: $F$ is CDF of $S_\tau$ for some $\tau \in \mathcal{T}$
- Since $S_t$ is a nonnegative martingale, optional sampling theorem and Fatou’s lemma yield, necessarily,
  $\int_0^\infty (1 - F(x)) \, dx \equiv \mathbb{E}[S_\tau] \leq s$
Let $F \in \mathcal{D}$: $F$ is CDF of $S_\tau$ for some $\tau \in \mathcal{T}$

Since $S_t$ is a nonnegative martingale, optional sampling theorem and Fatou’s lemma yield, necessarily,
\[ \int_0^\infty (1 - F(x)) \, dx \equiv \mathbb{E}[S_\tau] \leq s \]

This inequality is also sufficient for a CDF to be that of $S_\tau$ for some $\tau \in \mathcal{T}$!
Characterising Distribution/Quantile Sets

Theorem

We have the following expressions of the distribution set $D$ and quantile set $G$:

$$D = \left\{ F \left| F \text{ is a CDF, } \int_0^\infty (1 - F(x)) \, dx \leq s \right\} \right., \quad (10)$$

$$G = \left\{ G \left| G \text{ is a quantile, } \int_0^1 G(x) \, dx \leq s \right\} \right.. \quad (11)$$

- In particular, both $D$ and $G$ are convex
Step 1: Choose either distribution or quantile formulation
Solution Scheme

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- Step 2: Obtain the optimal distribution or quantile function
Solution Scheme

- Step 1: Choose either distribution or quantile formulation
- Step 2: Obtain the optimal distribution or quantile function
- Step 3: Recover optimal stopping via Skorokhod embedding
In the realm of portfolio selection
Quantile Formulation: History

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- In the realm of portfolio selection
- A general framework developed in He and Zhou (2009) for possibly non-concave utility function and non-convex/concave distortions
Solutions Based on Shapes of $u(\cdot)$ and $w(\cdot)$

- $w(\cdot)$:
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- $w(\cdot)$:
  - convex (risk averse)
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Solutions Based on Shapes of $u(\cdot)$ and $w(\cdot)$

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Convex $w(\cdot)$: Distribution Formulation

Let $w(\cdot)$ be convex (whereas $u(\cdot)$ may have any shape): i.e. the agent is risk _averse_
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Recall

$$J_D(F) := \int_0^\infty w(1 - F(x)) u'(x) \, dx,$$

$$J_Q(G) := \int_0^1 u(G(x)) w'(1 - x) \, dx,$$
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  \[
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  \]

- Distribution formulation is easier to study than quantile formulation
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Distribution formulation is easier to study than quantile formulation

Maximising a convex functional over a convex set $\mathcal{D}$

\[
(\int_0^1 (1 - F(x)) \, dx \leq s)
\]

A maximum should be at “corners” of $\mathcal{D}$
Theorem

If $w(\cdot)$ is convex and $u(\cdot)$ is non-decreasing, then

$$\sup_{\tau \in \mathcal{T}} J(\tau) = \sup_{0 < a \leq s \leq b} \left[ \left( 1 - w \left( \frac{s - a}{b - a} \right) \right) u(a) + w \left( \frac{s - a}{b - a} \right) u(b) \right]. \quad (14)$$

Moreover, if (14) has an optimal pair $(a^*, b^*)$ with $a^* > 0$, then

$$\tau_{(a^*, b^*)} := \inf \{ t \geq 0 : S_t \notin (a^*, b^*) \} \quad (15)$$

is an optimal stopping to problem (2).
By symmetry, if $u(\cdot)$ is convex, one uses quantile formulation

**Theorem**

*If $u(\cdot)$ is convex, then*

$$
\sup_{\tau \in \mathcal{T}} J(\tau) = \sup_{0 < a \leq s \leq b} \left[ \left( 1 - w \left( \frac{s - a}{b - a} \right) \right) u(a) + w \left( \frac{s - a}{b - a} \right) u(b) \right]. \quad (16)
$$

*Moreover, if (16) has an optimal pair $(a^*, b^*)$ with $a^* > 0$, then*

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\tau(a^*, b^*) := \inf \{ t \geq 0 : S_t \notin (a^*, b^*) \} \quad (17)
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*is an optimal stopping to problem (2).*
Traditional investment wisdom/advice: one should set a target price and a cut-loss price
Convex $w(\cdot)$ or $u(\cdot)$: Cut-Loss and Take-Profit Strategy

- Traditional investment wisdom/advice: one should set a target price and a cut-loss price
- $u(\cdot)$ being convex corresponds to “intermediate” and “bad” asset
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- Zhang (2001): cut-loss and take-profit thresholds are exogenously set
- Here we show that this strategy is endogenous for broad classes of problems
- It also recovers Henderson (2010) where probability distortion is absent
Corollary

If \( u(\cdot) \) is concave and \( w(\cdot) \) is convex, then

\[
\sup_{\tau \in T} J(\tau) = u(s). \tag{18}
\]

Moreover, \( \tau \equiv 0 \) is an optimal stopping.

- For power and log utility \( U(\cdot), u(\cdot) \) being concave corresponds to bad asset.
Corollary

If $u(\cdot)$ is concave and $w(\cdot)$ is convex, then

$$\sup_{\tau \in \mathcal{T}} J(\tau) = u(s).$$  \hspace{1cm} (18)

Moreover, $\tau \equiv 0$ is an optimal stopping.

- For power and log utility $U(\cdot)$, $u(\cdot)$ being concave corresponds to bad asset
- $w(\cdot)$ being convex corresponds to risk aversion
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If \( u(\cdot) \) is concave and \( w(\cdot) \) is convex, then

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\sup_{\tau \in \mathcal{T}} J(\tau) = u(s).
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Moreover, \( \tau \equiv 0 \) is an optimal stopping.

- For power and log utility \( U(\cdot) \), \( u(\cdot) \) being concave corresponds to bad asset
- \( w(\cdot) \) being convex corresponds to risk aversion
- A risk averse agent will sell a bad stock immediately
Let \( u(\cdot) \) be concave
Let $u(\cdot)$ be concave

Quantile formulation

$$J_Q(G) = \int_0^1 u(G(x)) w'(1 - x) \, dx$$ (19)
Let $u(\cdot)$ be concave

Quantile formulation

$$J_Q(G) = \int_0^1 u(G(x)) w'(1 - x) \, dx$$  \hspace{1cm} (19)

Maximising a concave functional over a convex set
Let $u(\cdot)$ be concave.

Quantile formulation

\[ J_Q(G) = \int_0^1 u(G(x)) w'(1 - x) \, dx \quad (19) \]

Maximising a *concave* functional over a convex set.

Lagrange!
Consider a family of relaxed problems

\[
J^\lambda_Q(G) = \int_0^1 u(G(x))w'(1 - x) \, dx - \lambda \left( \int_0^1 G(x) \, dx - s \right) \\
= \int_0^1 f^\lambda(x, G(x)) \, dx + \lambda s,
\]

where \( \lambda \geq 0 \) and \( f^\lambda(x, y) := u(y)w'(1 - x) - \lambda y \).
Consider a family of relaxed problems

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where \( \lambda \geq 0 \) and \( f^\lambda(x, y) := u(y)w'(1 - x) - \lambda y \)

Maximising \( f^\lambda(x, \cdot) \) for each \( x \) we get

\[ G_\lambda(x) := (u')^{-1} \left( \frac{\lambda}{w'(1-x)} \right), \quad x \in (0, 1) \]
Consider a family of relaxed problems

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G^\lambda(x) := (u')^{-1} \left( \frac{\lambda}{w'(1-x)} \right), \quad x \in (0, 1)
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Is \( G^\lambda \) qualified as a quantile function (primarily, non-decreasing)?
When $w(\cdot)$ is also concave, then $G_\lambda(x) := (u')^{-1} \left( \frac{\lambda}{w'(1-x)} \right)$ is indeed non-decreasing, hence a quantile.
Concave $u(\cdot)$ and $w(\cdot)$

- When $w(\cdot)$ is also concave, then $G_\lambda(x) := (u')^{-1} \left( \frac{\lambda}{w'(1-x)} \right)$ is indeed non-decreasing, hence a quantile.
- Finding $0 \leq \lambda^* < \infty$ so that
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- Then $G^* = G_{\lambda^*}$ is optimal to problem (8).
When \( w(\cdot) \) is also concave, then \( G_\lambda(x) := (u')^{-1} \left( \lambda \frac{w'}{w'(1-x)} \right) \) is indeed non-decreasing, hence a quantile.

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Then \( G^* = G_{\lambda^*} \) is optimal to problem (8).

Bad asset but risk-seeking agent
Consider $u(x) = \frac{1}{\gamma} x^\gamma$, $0 < \gamma < 1$, and $w(x) = x^\alpha$, $0 < \gamma < \alpha < 1$: both concave.
Concave $u(\cdot)$ and $w(\cdot)$: An Example

- Consider $u(x) = \frac{1}{\gamma}x^\gamma$, $0 < \gamma < 1$, and $w(x) = x^\alpha$, $0 < \gamma < \alpha < 1$: both concave
- $G^*(x) = s^{\frac{\alpha-\gamma}{1-\gamma}} \left( \frac{1}{1-x} \right)^{\frac{1-\alpha}{1-\gamma}}$ is optimal quantile
Concave \( u(\cdot) \) and \( w(\cdot) \): An Example

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- Corresponding CDF of optimally stopped price is

\[
F^*(x) = \begin{cases} 
1 - \left( s \frac{\alpha-\gamma}{1-\gamma} \right)^{\frac{1-\gamma}{1-\alpha}} x^{-\frac{1-\gamma}{1-\alpha}}, & x \geq s \frac{\alpha-\gamma}{1-\gamma}; \\
0, & x < s \frac{\alpha-\gamma}{1-\gamma},
\end{cases}
\]

(20)

a *Pareto distribution* with Pareto index \( \frac{1-\gamma}{1-\alpha} > 1 \)
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a *Pareto distribution* with Pareto index $\frac{1-\gamma}{1-\alpha} > 1$

- One never stops when the asset price is below $s \frac{\alpha-\gamma}{1-\gamma}$

- Azéma–Yor stopping time

$$\tau_{AY} = \inf \left\{ t \geq 0 : S_t \leq \frac{\alpha - \gamma}{1 - \gamma} \max_{0 \leq s \leq t} S_s \right\}$$

(21)

is an optimal solution to problem (2)
Concave $u(\cdot)$ and Reverse $S$-shaped $w(\cdot)$

- Reverse $S$-shaped $w(\cdot)$: fear and hope are present simultaneously (He and Zhou 2009)
Concave $u(\cdot)$ and Reverse $S$-shaped $w(\cdot)$

- Reverse $S$-shaped $w(\cdot)$: fear and hope are present simultaneously (He and Zhou 2009)
- Optimal quantile is of the form

$$G^*(x) = a \mathbf{1}_{(0,c]}(x) + \left( a \vee (u')^{-1} \left( \frac{\lambda}{w'(1-x)} \right) \right) \mathbf{1}_{(c,1]}(x),$$

where parameters $a$, $c$ and $\lambda$ are subject to

$$\lambda \geq 0, \quad q \leq c \leq 1, \quad a \geq 0,$$

$$ac + \int_c^1 \left( a \vee (u')^{-1} \left( \frac{\lambda}{w'(1-x)} \right) \right) dx \leq s.$$
Concave $u(\cdot)$ and Reverse $S$-shaped $w(\cdot)$

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$$G^*(x) = a \mathbf{1}_{(0,c]}(x) + \left( a \lor (u')^{-1} \left( \frac{\lambda}{w'(1-x)} \right) \right) \mathbf{1}_{(c,1]}(x),$$

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- Payoff under $G^*$ is

$$(1-w(1-c))u(a) + \int_c^1 u \left( a \lor (u')^{-1} \left( \frac{\lambda}{w'(1-x)} \right) \right) w'(1-x) dx.$$
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- **Reverse $S$-shaped $w(\cdot)$**: fear and hope are present simultaneously (He and Zhou 2009)
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- A mathematical programme!
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Optimal stopping with probability distortion/weighting is formulated and pre-committed strategies obtained.
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A general machinery introduced and developed – distribution/quantile formulation + Skorokhod embedding.
Conclusions

- Optimal stopping with probability distortion/weighting is formulated and pre-committed strategies obtained
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  - Time consistent strategies (formulation?)