Exact and high order discretization schemes for Wishart processes and their affine extensions

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Plan

1. Motivation and notations

2. Exact simulation for Wishart processes
   - Splitting operator property
   - Exact simulation

3. High order discretization for Wishart processes and affine processes
   - Wishart processes
   - Affine processes

4. Numerical results
1 Motivation and notations

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3 High order discretization for Wishart processes and affine processes
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4 Numerical results
Motivation

- Wishart process and affine process defined on the symmetric positive cone $S^+_d(\mathbb{R})$ are a key-tool for:
  - Defining the natural correlation between processes.
  - A generalization of stochastic volatility in multidimension.
  - Pricing complex derivatives taking into account the relationship between spot (Outperformer, Best/Worst of options ...).

- Pricing with Fourier transform methods are less efficient in the multi dimension context.

- To the best of our knowledge, this is the first exact simulation and high order discretization that work without any restriction on parameters.
Definitions

We say that the process \((X^x_t)^{t \geq 0}\) is a \underline{continuous positive affine} process, if it is a solution of the following SDE:

\[
X^x_t = x + \int_0^t (\bar{\alpha} + B(X^x_s)) \, ds + \int_0^t \left( \sqrt{X^x_s} dW_s a + a^T dW_s^T \sqrt{X^x_s} \right),
\]

(1)

where \((W_t, t \geq 0)\) denotes a \(d\)-by-\(d\) square matrix made of independent standard Brownian motions, \(x, \bar{\alpha} \in S^+_d(\mathbb{R}), a \in M_d(\mathbb{R}), B \in \mathcal{L}(S_d(\mathbb{R}))\) (where \(\mathcal{L}(S_d(\mathbb{R}))\) is a linear mapping on \(S_d(\mathbb{R})\)), and \(\forall x \in S^+_d(\mathbb{R})\)

\[
x = o\text{diag}(\lambda_1, \ldots, \lambda_d)o^T \implies \sqrt{x} = o\text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_d})o^T.
\]

\underline{Wishart processes} correspond to the following case:

\[
\exists \alpha \geq 0, \bar{\alpha} = \alpha a^T a \text{ and } \exists b \in M_d(\mathbb{R}), \forall x \in S_d(\mathbb{R}), B(x) = bx + xb^T.
\]

(2)
Application in finance: Gourieroux and Sufana model

- We consider $d$ risky assets $S_1^t, \ldots, S_d^t$. Let $(B_t, t \geq 0)$ denote a standard Brownian motion on $\mathbb{R}^d$ that is independent from $(X_t)_{t \geq 0} \sim WIS_d(x, \alpha, b, a)$. Then, we have

$$t \geq 0, 1 \leq l \leq d, \quad \frac{dS^l_t}{S^l_t} = rdt + (\sqrt{X_t} dB_t)_l.$$ 

- Gourieroux and Sufana model assumes that the Wishart process $(X_t)_{t \geq 0}$ is the Covariance matrix of the spot vector $(S_t)_{t \geq 0}$.

- Da Fonseca and al. have chosen the adequate correlation between spot vector $(S_t)_{t \geq 0}$ and its Covariance matrix $(X_t)_{t \geq 0}$ to observe the smile effect, and to keep the model affine.
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Composition technique for the exact simulation

**Proposition**

If \( (Y^x_t)_{t \geq 0} \) is an affine process starting from \( x \) and associated to the infinitesimal operator \( L_Y \), such that \( L_Y = L_Z + L_X \), and \( L_Z L_X = L_X L_Z \).

Then

\[
Y^x_t \sim X^Z^x_t,
\]

where \( (X^x_t)_{t \geq 0} \) and \( (Z^x_t)_{t \geq 0} \) are two affine independent processes associated respectively to two infinitesimal generators \( L_X \) and \( L_Z \).

**Proof.**

For some class of functions \( f \)

\[
\begin{align*}
\mathbb{E}[f(X^x_t)] &= \sum_{k=0}^{\infty} t^k L_X^k f(x) / k! := e^{tL_X}(f)(x) \\
\mathbb{E}\left[f(X^Z^x_t)\right] &= \mathbb{E}\left[\mathbb{E}\left[f(X^Z^x_t) | Z^x_t\right]\right] \\
&= \sum_{k_1=0}^{+\infty} \frac{t^{k_1}}{k_1!} \mathbb{E}\left[L_X^{k_1} f(Z^x_t)\right] \\
&= \sum_{k_1,k_2=0}^{+\infty} \frac{t^{k_1+k_2}}{k_1! k_2!} L_X^{k_1} L_Z^{k_2} f(x) \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} (L_X + L_Z)^k f(x) \\
&= \mathbb{E}[f(Y^x_t)]
\end{align*}
\]
### Canonical Wishart process transformation

**Proposition**

Let \( t > 0, \, a, \, b \in \mathcal{M}_d(\mathbb{R}) \) and \( \alpha \geq d - 1 \). Let \( m_t = \exp(t b) \), \( q_t = \int_0^t \exp(s b) a^T a \exp(s b^T) ds \) and \( n = \text{Rk}(q_t) \). Then, there is \( \theta_t \in \mathcal{G}_d(\mathbb{R}) \) such that \( q_t = t \theta_t l_n^d \theta_t^T \), and we have:

\[
\text{WIS}_d(x, \alpha, b, a; t) = \text{Law}_{\theta_t} \text{WIS}_d(\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T, \alpha, 0, l_n^d; t) \theta_t^T
\]

**Remark**

- **General /Non central Wishart distribution \(\cong\) Canonical/Central Wishart distribution**
- **In the case of \(d = 1\), we obtain the usual identity of Bessel and CIR processes**

\[
\text{WIS}_1(x, \alpha, b, a; t) = \text{Law}_{\frac{a^2 e^{2bt} - 1}{2bt}} \text{WIS}_1(\frac{2btx}{a^2(1 - e^{-2bt})}, \alpha, 0, 1; t).
\]
A remarkable splitting operator

**Theorem**

Let $L$ be the generator associated to the Wishart process $\text{WIS}_d(x, \alpha, 0, I_n^d)$ and $L_i$ be the generator associated to $\text{WIS}_d(x, \alpha, 0, e_i^d)$ for $i \in \{1, \ldots, d\}$. Then, we have

$$L = \sum_{i=1}^{n} L_i \quad \text{and} \quad \forall i, j \in \{1, \ldots, d\}, \; L_i L_j = L_j L_i,$$

where $\forall 1 \leq i \leq d, \; \forall 1 \leq k, l \leq d, \; (e_i^d)_{k,l} = 1_{\{k=l=i\}}, \; (I_n^d)_{k,l} = 1_{\{k=l, k \leq n\}}$.

**Remark**

- The operators $L_i$ and $L_j$ are the same up to the exchange of coordinates $i$ and $j$.

- The processes $\text{WIS}_d(x, \alpha, 0, e_i^d)$ and $\text{WIS}_d(x, \alpha, 0, I_n^d)$ are well defined on $S_d^+(\mathbb{R})$ under the same hypothesis, namely $\alpha \geq d - 1$ and $x \in S_d^+(\mathbb{R})$. 
Motivation and notations
Exact simulation for Wishart processes
High order discretization for Wishart processes and affine processes
Numerical results

Splitting operator property
Exact simulation

**Exact simulation for the canonical Wishart distribution**

Let us consider $t > 0$ and $x \in S_d^+(\mathbb{R})$. We define iteratively:

$$X_{t}^{1,x} \sim WIS_d(x, \alpha, 0, e_d^1; t),$$
$$X_{t}^{2,x} \sim WIS_d(X_{t}^{1,x}, \alpha, 0, e_d^2; t),$$
$$\ldots$$
$$X_{t}^{n,x} \sim WIS_d(X_{t}^{n-1,x}, \ldots, X_{t}^{1,x}, \alpha, 0, e_d^n; t),$$

where, $X_{t}^{i,x}$ is sampled according to the distribution at time $t$ of an independent Wishart process starting from $X_{t}^{i-1,x}$ and with parameters $(\alpha, 0, e_d^i)$.

We have the following result:

**Proposition**

Let $X_{t}^{n,x}$ be defined as above. Then $X_{t}^{n,x} \sim WIS_d(x, \alpha, 0, I_d^n; t)$. 
Exact simulation of $WIS_d(x, \alpha, 0, e_d^1)$, with $d \in \mathbb{N}^*$

**Theorem**

The solution of $WIS_d(x, \alpha, 0, e_d^1)$ is given explicitly by:

\[
X^x_t = q \begin{pmatrix}
(U^u_t)^{1,1} + \sum_{k=1}^r ((U^u_t)^{1,k+1})^2 & ((U^u_t)^{1,l+1})_{1 \leq l \leq r}^T \\
((U^u_t)^{1,l+1})_{1 \leq l \leq r} & 0 \end{pmatrix}
\begin{pmatrix}
l_r \\
0 \end{pmatrix}
q^T,
\]

where

\[
d(U^u_t)^{1,1} = (\alpha - r) dt + 2 \sqrt{(U^u_t)^{1,1}} dZ_1^1 \geq 0, \\
d((U^u_t)^{1,l+1})_{1 \leq l \leq r} = (dZ_{l+1}^1)_{1 \leq l \leq r}, \\
d((U^u_t)^{k,l})_{2 \leq k,l \leq r} = d((X^x_t)^{k,l})_{2 \leq k,l \leq r} = 0,
\]

and $q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & l_{d-r-1} \end{pmatrix}$. 

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Exact and high order discretization schemes
Methodology to sample Exactly Wishart distribution

\[
WIS_d(x, \alpha, b, a; t) \sim \theta_t WIS_d(\theta_t^{-1} m_t m_t^T (\theta_t^{-1})^T, \alpha, 0, I_d; t) \theta_t^T
\]

\[\forall 2 \leq n \leq d, \ WIS_d(x, \alpha, 0, I_d^n), \quad \text{By composition Technique}\]

\[\forall 2 \leq i \leq d, \ WIS_d(x, \alpha, 0, e_i^d), \quad \text{By permutation}\]

\[WIS_d(x, \alpha, 0, e_1^d).\]

Sampling one square Bessel process and \( d - 1 \) Brownian motions.
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A potential $\nu$ order scheme for the operator $L_1$, $d \in \mathbb{N}$

**Theorem**

By replacing the transformation $(U^u_t\{1,1\})_{2 \leq l \leq d}$ (resp. $(U^u_t\{1,1\})$) with $\sqrt{t}(\hat{G}^i)_{1 \leq i \leq r}$ (resp. with $(\hat{U}^u_t\{1,1\})$), then $\hat{X}_t$ is a potential $\nu$-order scheme for the operator $L_1$, where:

- $(\hat{G}^i)_{1 \leq i \leq r}$ is a sequence of independent real variables with finite moments of any order such that:
  $$\forall i \in \{1, \ldots, r\}, \ \forall k \leq 2\nu + 1, \ \mathbb{E}[(\hat{G}^i)^k] = \mathbb{E}[G^k], \text{ where } G \sim \mathcal{N}(0,1).$$

- $(\hat{U}^u_t\{1,1\})$ is sampled independently according to a potential weak $\nu$th-order scheme for the CIR process $d(U^u_t\{1,1\}) = (\alpha - r)dt + 2\sqrt{(U^u_t\{1,1\})}dZ^1_t$ starting from $u\{1,1\}$. 
Methodology to build the scheme of order $\nu$

$$\widehat{WIS}_d(x, \alpha, b, a; t) = \theta_t \widehat{WIS}_d(\theta_t^{-1}m_t x m_t^T (\theta_t^{-1})^T, \alpha, 0, I_d^n; t) \theta_t^T$$

$\forall 2 \leq n \leq d$, $\widehat{WIS}_d(x, \alpha, 0, I_d^n)$, \text{By composition Technique}

$\forall 2 \leq i \leq d$, $\widehat{WIS}_d(x, \alpha, 0, e_i^d)$, \text{By permutation}

$\widehat{WIS}_d(x, \alpha, 0, e_1^d)$.

Schemes of order $\nu$ for: one square Bessel process and $d - 1$ Brownian motions.
Theorem

Let \((X_t^x)_{t \geq 0} \sim \text{WIS}_d(x, \alpha, b, a)\) such that either \(a \in \mathcal{G}_d(\mathbb{R})\) or \(a^T ab = ba^T a\), and \(f \in C^\infty(S_d(\mathbb{R}))\). Let \((\hat{X}_{t_i}^N, 0 \leq i \leq N)\) be sampled with the scheme defined previously with the third order scheme for the CIR given in Alfonsi-2009 and starting from \(x_0 \in S_d^+(\mathbb{R})\). Then,

\[ \exists C, \ N_0 > 0, \ \forall N \geq N_0, \ |\mathbb{E}[f(\hat{X}_{t_i}^N)] - \mathbb{E}[f(X_t^x)]| \leq C/N^3. \]

Remark

New extension of the regularity of the function \(u(t, x) = \mathbb{E}[f(X_t^x)]\), from the CIR process to Wishart process . (Phd thesis A.Alfonsi 2006)
Canonical positive affine process transformation

Proposition

Let \((X_t^x)_{t \geq 0} \sim AFF_d(x, \tilde{\alpha}, B, a)\) and \(n = \text{Rk}(a)\) be the rank of \(a^T a\). Then, there exist a diagonal matrix \(\tilde{\delta}\), and a non singular matrix \(u \in G_d(\mathbb{R})\) such that \(\tilde{\alpha} = u^T \tilde{\delta} u\), and \(a^T a = u^T I_d u\), and we have:

\[
(X_t^x)_{t \geq 0} \overset{\text{Law}}{=} u^T AFF_d \left( (u^{-1})^T x u^{-1}, \tilde{\delta}, B_u, I_d^n \right) u,
\]

where \(\forall y \in S_d(\mathbb{R}), \ B_u(y) = (u^{-1})^T B(u^T y u) u^{-1}\).
The potential second order discretization for a general affine process defined on $\mathcal{S}_d^+(\mathbb{R})$

- It is sufficient to study the affine process $\text{AFF}_d(x, \bar{\delta}, B, I_d^n)$.
- By splitting operator, if $L$ denotes the infinitesimal generator of $\text{AFF}_d(x, \bar{\delta}, B, I_d^n)$, we conclude then that

\[
\begin{align*}
L &= L_{\text{ODE}} + L_{\text{Wishart}}, \\
L_{\text{ODE}} &= \text{Tr}((\bar{\delta} - \delta_{\min} I_d^n + B(x))D^S) \sim X^1_t, \\
L_{\text{Wishart}} &= \text{Tr}((\delta_{\min} I_d^n)D^S) + 2\text{Tr}(xD^S I_d^n D^S) = \sum_{i=1}^n L_i \sim X^2_t.
\end{align*}
\]

Proposition

Both schemes $X^{1, x}_{t/2}$, $X^{2, x}_{t/2}$ and $UX^{1, x}_{t}$, $X^{2, x}_{t} + (1 - U)X^{1, x}_{t}$ are potential second order scheme for $\text{AFF}_d(x, \bar{\delta}, B, I_d^n)$, where $U$ is an independent Bernoulli variable with parameter $\frac{1}{2}$. 
Fast potential second order discretization for a general affine process defined on $S_d^+({\mathbb R})$, $\delta \geq dl_d^n$

- The previous algorithm requires $O(d^4)$, on each step time due to Cholesky decomposition of each transformation $(L_i)_{1 \leq i \leq d}$.

- In the case of $\delta \geq dl_d^n$ we propose an other scheme that costs only $O(d^3)$:

\[
L = L_{ODE} + L_{Wishart},
\]

\[
L_{ODE} = \text{Tr}((\delta - dl_d^n + B(x))D^S),
\]

\[
L_{Wishart} = \text{Tr}((dl_d^n)^S)^S) + 2\text{Tr}(xD^S l_d^n D^S) \sim (c + \sqrt{t\tilde{G}l_d^n})(c + \sqrt{t\tilde{G}l_d^n})^T,
\]

where $\tilde{G}$ is a matrix, in $\mathcal{M}_d({\mathbb R})$, made of independent variables that fit the first five moments of normal random variable.
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4 Numerical results
### Time computation for $\mathbb{E} \left[ \exp (i \text{Tr} (v X_T^X)) \right]$: $Nmc = 10^6$, $a = l_d$, $b = 0$, $x = 10l_d$, $v = 0.09l_d$ and $T = 1$

<table>
<thead>
<tr>
<th>Schemes</th>
<th>R. value</th>
<th>Im. value</th>
<th>Time</th>
<th>R. value</th>
<th>Im. value</th>
<th>Time</th>
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</thead>
<tbody>
<tr>
<td>Exact (1 step)</td>
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<td>-0.527090</td>
<td>-0.228251</td>
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$\alpha = 3.5$, $d = 3$, $\Delta R = 10^{-3}$, $\Delta \text{Im.} = 10^{-3}$, exact value $R. = -0.527090$ and $\text{Im.} = -0.228251$

<table>
<thead>
<tr>
<th>Schemes</th>
<th>R. value</th>
<th>Im. value</th>
<th>Time</th>
<th>R. value</th>
<th>Im. value</th>
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| Exact (N steps)          | -0.589735* | -0.042002* | 223  | -0.591411 and $\text{Im.} = -0.036346$

$\alpha = 2.2$, $d = 3$, $\Delta R = 0.9 \times 10^{-3}$, $\Delta \text{Im.} = 1.3 \times 10^{-3}$, exact value $R. = -0.591411$ and $\text{Im.} = -0.036346$

<table>
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<tr>
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<th>Time</th>
<th>R. value</th>
<th>Im. value</th>
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</table>
| Corrected Euler          | 0.068298* | -0.058491* | 2312 | 0.063960 and $\text{Im.} = -0.063544$

$\alpha = 10.5$, $d = 10$, $\Delta R = 1.4 \times 10^{-3}$, $\Delta \text{Im.} = 1.3 \times 10^{-3}$, exact value $R. = 0.063960$ and $\text{Im.} = -0.063544$

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<th>Im. value</th>
<th>Time</th>
<th>R. value</th>
<th>Im. value</th>
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| Corrected Euler          | -0.028685* | -0.094281* | 2321 | -0.036064 and $\text{Im.} = -0.093275$

$\alpha = 9.2$, $d = 10$, $\Delta R = 1.4 \times 10^{-3}$, $\Delta \text{Im.} = 1.4 \times 10^{-3}$, exact value $R. = -0.036064$ and $\text{Im.} = -0.093275$
Motivation and notations
Exact simulation for Wishart processes
High order discretization for Wishart processes and affine processes
Numerical results

Laplace transform $\mathbb{E} [\exp(i \text{Tr}(\nu X_t^X))]$, $d = 3$

Figure: $d = 3$, $10^7$ MC, $T = 10$. The RV of $\mathbb{E}[\exp(-\text{Tr}(\nu X_t^N))]$ in function of $T/N$. Left: $\nu = 0.05I_d$ and $x = 0.4I_d$, $\alpha = 4.5$, $a = I_d$ and $b = 0$. Ex.Val.: 0.054277. Right: $\nu = 0.2I_d + 0.04q$ and $x = 0.4I_d + 0.2q$, $\alpha = 2.22$, $a = I_d$ and $b = -0.5I_d$. Ex.Val: 0.239836. Here, $q$ is the matrix defined by: $q_{i,j} = 1_{i \neq j}$. The width of each point represents the 95% confidence interval.
Laplace transform $\mathbb{E}[\exp(i\text{Tr}(vX^\mathcal{X}_{t_f}))], \ d = 10$

Figure: $d = 10$, $10^7$ MC, $T = 10$. Left: IM of $\mathbb{E}[\exp(-\text{Tr}(iv\hat{X}_N^t))]$ with $v = 0.009l_d$ in function of $T/N$, $x = 0.4l_d$, $\alpha = 12.5$, $b = 0$ and $a = l_d$. Ex.Val: $-0.361586$. Right: RV of $\mathbb{E}[\exp(-\text{Tr}(iv\hat{X}_N^t))]$ with $v = 0.009l_d$ in function of $T/N$, $x = 0.4l_d$, $\alpha = 9.2$, $b = -0.5l_d$ and $a = l_d$. Ex.Val 0.572241. The width of each point represents the 95% confidence interval.
Trajectory error $\mathbb{E} \left[ \max_{0 \leq s \leq T} \text{Tr}(X_s^x) \right]$  

Figure: $d = 3$, $10^7$ MC, $T = 1$, $x = 0.4l_d + 0.2q$ with $q_{i,j} = 1_{i \neq j}$, $\alpha = 2.2$, $b = 0$ and $a = l_d$. Left, $\mathbb{E} [\max_{0 \leq k \leq N} \text{Tr}(\hat{X}_k^{N})]$, right: $\mathbb{E} [\max_{0 \leq k \leq N} \text{Tr}(X_k^{x_N})]$ in function of $T/N$. The width of each point gives the precision up to two standard deviations.
**Gourieroux Sufana Model - Put Best of Option**

**Figure:** \( E[e^{-rT}(K - \max(\hat{S}^{1,T}_N, \hat{S}^{2,T}_N))^+] \) in function of \( T/N \). \( d = 2, T = 1, K = 120, S^1_0 = S^2_0 = 100, \) and \( r = 0.02 \), \( x = 0.04I_d + 0.02q \) with \( q_{i,j} = 1_{i \neq j} \), \( a = 0.2I_d \), \( b = 0.5I_d \) and \( \alpha = 4.5 \) (left), \( \alpha = 1.05 \) (right). The width of each point gives the precision up to two standard deviations (10^6 MC).
In this work, we have presented:

- Exact scheme for Wishart process.
- Second and third order scheme for Wishart process.
- Potential second order scheme for a general affine process defined on $S_d^+(\mathbb{R})$. 
Thank you !!