Exact and high order discretization schemes for Wishart processes and their affine extensions

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Motivation

- Wishart process and affine process defined on the symmetric positive cone $\mathcal{S}_d^+(\mathbb{R})$ are a key-tool for:
  - Defining the natural correlation between processes.
  - A generalization of stochastic volatility in multidimension.
  - Pricing complex derivatives taking into account the relationship between spot (Outperformer, Best/Worst of options ...).

- Pricing with Fourier transform methods are less efficient in the multi dimension context.

- To the best of our knowledge, this is the first exact simulation and high order discretization that work without any restriction on parameters.

Definitions

We say that the process $(X_t^x)_{t\geq 0}$ is a continuous positive affine process, if it is a solution of the following SDE:

$$X_t^x = x + \int_0^t (\overline{\alpha} + B(X_s^x))\, ds + \int_0^t \left(\sqrt{X_s^x} dW_s a + a^T dW_s^T \sqrt{X_s^x}\right), \qquad (1)$$

where $(W_t, t \geq 0)$ denotes a d-by-d square matrix made of independent standard Brownian motions, x, $\bar{\alpha} \in \mathcal{S}_d^+(\mathbb{R})$, $a \in \mathcal{M}_d(\mathbb{R})$, $B \in \mathcal{L}(\mathcal{S}_d(\mathbb{R}))$ (where $\mathcal{L}(\mathcal{S}_d(\mathbb{R}))$ is a linear mapping on $\mathcal{S}_d(\mathbb{R})$), and $\forall x \in \mathcal{S}_d^+(\mathbb{R})$

$$x = o\mathrm{diag}(\lambda_1, \dots, \lambda_d) o^T \implies \sqrt{x} = o\mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d}) o^T.$$

Wishart processes correspond to the following case :

$$\exists \alpha \geq 0, \bar{\alpha} = \alpha a^T a \text{ and } \exists b \in \mathcal{M}_d(\mathbb{R}), \forall x \in \mathcal{S}_d(\mathbb{R}), B(x) = bx + xb^T. \qquad (2)$$

Application in finance : Gourieroux and Sufana model

- We consider d risky assets $S_t^1, \dots, S_t^d$. Let $(B_t, t \geq 0)$ denote a standard Brownian motion on $\mathbb{R}^d$ that is independent from $(X_t)_{t\geq 0} \sim WIS_d(x, \alpha, b, a)$. Then, we have

$$t \geq 0, 1 \leq l \leq d, \; \frac{dS_t^l}{S_t^l} = rdt + (\sqrt{X_t} dB_t)_l.$$

- Gourieroux and Sufana model assumes that the Wishart process $(X_t)_{t\geq 0}$ is the Covariance matrix of the spot vector $(S_t)_{t\geq 0}$.
- Da Fonseca and al. have chosen the adequate correlation between spot vector $(S_t)_{t\geq 0}$ and its Covariance matrix $(X_t)_{t\geq 0}$ to observe the smile effect, and to keep the model affine.

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Composition technique for the exact simulation

Proposition

If $(Y_t^x)_{t\geq 0}$ is an affine process starting from x and associated to the infinitesimal operator L_Y, such that $L_Y = L_Z + L_X$, and $L_Z L_X = L_X L_Z$. Then

$$Y_t^x \sim X_t^{Z_t^x},$$

where $(X_t^x)_{t\geq 0}$ and $(Z_t^x)_{t\geq 0}$ are two affine independent processes associated respectively to two infinitesimal generators L_X and L_Z.

Proof.

For some class of functions f

$$\begin{aligned}
\mathbb{E}[f(X_t^x)] &= \textstyle\sum_{k=0}^{\infty} t^k L_X^k f(x)/k! := e^{tL_X}(f)(x) \\
\mathbb{E}\left[f(X_t^{Z_t^x})\right] &= \mathbb{E}\left[\mathbb{E}\left[f(X_t^{Z_t^x})|Z_t^x\right]\right] \\
&= \textstyle\sum_{k_1=0}^{+\infty} \frac{t^{k_1}}{k_1!}\mathbb{E}\left[L_X^{k_1} f(Z_t^x)\right] \\
&= \textstyle\sum_{k_1,k_2=0}^{+\infty} \frac{t^{k_1+k_2}}{k_1!k_2!} L_X^{k_1} L_Z^{k_2} f(x) \\
&= \textstyle\sum_{k=0}^{\infty} \frac{t^k}{k!}(L_X + L_Z)^k f(x) \\
&= \mathbb{E}[f(Y_t^x)]
\end{aligned}$$

Canonical Wishart process transformation

Proposition

Let $t > 0$, $a, b \in \mathcal{M}_d(\mathbb{R})$ and $\alpha \geq d-1$. Let $m_t = \exp(tb)$, $q_t = \int_0^t \exp(sb)a^T a \exp(sb^T)ds$ and $n = \mathrm{Rk}(q_t)$. Then, there is $\theta_t \in \mathcal{G}_d(\mathbb{R})$ such that $q_t = t\theta_t I_d^n \theta_t^T$, and we have:

$$WIS_d(x, \alpha, b, a; t) \underset{Law}{=} \theta_t WIS_d(\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T, \alpha, 0, I_d^n; t)\theta_t^T$$

Remark

- *General /Non central Wishart distribution $\Leftrightarrow$ Canonical/Central Wishart distribution*
- *In the case of $d = 1$, we obtain the usual identity of Bessel and CIR processes*

$$WIS_1(x, \alpha, b, a; t) \underset{Law}{=} a^2 \frac{e^{2bt} - 1}{2bt} WIS_1\left(\frac{2btx}{a^2(1 - e^{-2bt})}, \alpha, 0, 1; t\right).$$

A remarkable splitting operator

Theorem

Let L be the generator associated to the Wishart process $WIS_d(x, \alpha, 0, I_d^n)$ and L_i be the generator associated to $WIS_d(x, \alpha, 0, e_d^i)$ for $i \in \{1, \ldots, d\}$. Then, we have

$$L = \sum_{i=1}^{n} L_i \text{ and } \quad \forall i, j \in \{1, \ldots, d\},\ L_i L_j = L_j L_i, \tag{3}$$

where $\forall 1 \leq i \leq d,\ \forall 1 \leq k, l \leq d,\ (e_d^i)_{k,l} = 1_{\{k=l=i\}},\ (I_d^n)_{k,l} = 1_{\{k=l, k \leq n\}}$

Remark

- *The operators L_i and L_j are the same up to the exchange of coordinates i and j.*
- *The processes $WIS_d(x, \alpha, 0, e_d^i)$ and $WIS_d(x, \alpha, 0, I_d^n)$ are well defined on $\mathcal{S}_d^+(\mathbb{R})$ under the same hypothesis, namely $\alpha \geq d-1$ and $x \in \mathcal{S}_d^+(\mathbb{R})$.*

Exact simulation for the canonical Wishart distribution

Let us consider $t > 0$ and $x \in \mathcal{S}_d^+(\mathbb{R})$. We define iteratively:

$$\begin{aligned} X_t^{1,x} &\sim WIS_d(x, \alpha, 0, e_d^1; t), \\ X_t^{2,X_t^{1,x}} &\sim WIS_d(X_t^{1,x}, \alpha, 0, e_d^2; t), \\ &\dots \\ X_t^{n,\dots^{X_t^{1,x}}} &\sim WIS_d(X_t^{n-1,\dots^{X_t^{1,x}}}, \alpha, 0, e_d^n; t), \end{aligned}$$

where, $X_t^{i,\dots^{X_t^{1,x}}}$ is sampled according to the distribution at time t of an independent Wishart process starting from $X_t^{i-1,\dots^{X_t^{1,x}}}$ and with parameters $(\alpha, 0, e_d^i)$.

We have the following result:

Proposition

Let $X_t^{n,\dots^{X_t^{1,x}}}$ be defined as above. Then $X_t^{n,\dots^{X_t^{1,x}}} \sim WIS_d(x, \alpha, 0, I_d^n; t)$.

Exact simulation of $WIS_d(x, \alpha, 0, e_d^1)$, with $d \in \mathbb{N}^*$

Theorem

The solution of $WIS_d(x, \alpha, 0, e_d^1)$ is given explicitly by:

$$X_t^x = q \begin{pmatrix} (U_t^u)_{\{1,1\}} + \sum_{k=1}^{r}((U_t^u)_{\{1,k+1\}})^2 & ((U_t^u)_{\{1,l+1\}})_{1\le l\le r}^T & 0 \\ ((U_t^u)_{\{1,l+1\}})_{1\le l\le r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} q^T,$$

where

$$\begin{aligned} d(U_t^u)_{\{1,1\}} &= (\alpha - r)dt + 2\sqrt{(U_t^u)_{\{1,1\}}}dZ_t^1 \ge 0, \\ d((U_t^u)_{\{1,l+1\}})_{1\le l\le r} &= (dZ_t^{l+1})_{1\le l\le r}. \\ d((U_t^u)_{\{k,l\}})_{2\le k,l\le r} &= d((X_t^x)_{\{k,l\}})_{2\le k,l\le r} = 0, \end{aligned}$$

and $q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix}.$

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Methodology to sample Exactly Wishart distribution

$$WIS_d(x,\alpha,b,a;t) \underset{Law}{\sim} \theta_t WIS_d(\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T, \alpha, 0, I_d^n; t)\theta_t^T$$

$$\Updownarrow$$

$$\forall 2 \leq n \leq d,\ \ WIS_d(x,\alpha,0,I_d^n),\ \ \underline{\text{By composition Technique}}$$

$$\Updownarrow$$

$$\forall 2 \leq i \leq d,\ \ WIS_d(x,\alpha,0,e_d^i),\ \ \underline{\text{By permutation}}$$

$$\Updownarrow$$

$$WIS_d(x,\alpha,0,e_d^1).$$

$$\Updownarrow$$

Sampling one square Bessel process and $d-1$ Brownian motions.

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A potential ν order scheme for the operator L_1, $d \in \mathbb{N}$

Theorem

By replacing in Transformation $((U_t^u)_{\{1,l\}})_{2\leq l\leq d}$ *(resp.* $(U_t^u)_{\{1,1\}}$*) with* $\sqrt{t}(\hat{G}^i)_{1\leq i\leq r}$ *(resp. with* $(\hat{U}_t^u)_{\{1,1\}}$*), then* $\hat{X}_t$ *is a potential* ν*-order scheme for the operator* L_1*, where :*

- $(\hat{G}^i)_{1\leq i\leq r}$ *is a sequence of independent real variables with finite moments of any order such that:*

$$\forall i \in \{1,\ldots,r\},\ \forall k \leq 2\nu+1,\ \mathbb{E}[(\hat{G}^i)^k] = \mathbb{E}[G^k],\ \textit{where } G \sim \mathcal{N}(0,1).$$

- $(\hat{U}_t^u)_{\{1,1\}}$ *is sampled independently according to a potential weak* ν*th-order scheme for the CIR process* $d(U_t^u)_{\{1,1\}} = (\alpha - r)dt + 2\sqrt{(U_t^u)_{\{1,1\}}}dZ_t^1$ *starting from* $u_{\{1,1\}}$.

Methodology to build the scheme of order ν

$$\widehat{WIS}_d(x,\alpha,b,a;t) = \theta_t \widehat{WIS}_d(\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T, \alpha, 0, I_d^n; t)\theta_t^T$$
$$\Updownarrow$$
$$\forall 2 \le n \le d,\ \ \widehat{WIS}_d(x,\alpha,0,I_d^n),\ \ \underline{\text{By composition Technique}}$$
$$\Updownarrow$$
$$\forall 2 \le i \le d,\ \ \widehat{WIS}_d(x,\alpha,0,e_d^i),\ \ \underline{\text{By permutation}}$$
$$\Updownarrow$$
$$\widehat{WIS}_d(x,\alpha,0,e_d^1).$$
$$\Updownarrow$$

Schemes of order ν for: one square Bessel process and $d-1$ Brownian motions.

The third order discretization for Wishart process

Theorem

Let $(X_t^x)_{t\geq 0} \sim WIS_d(x,\alpha,b,a)$ such that either $a \in \mathcal{G}_d(\mathbb{R})$ or $a^T ab = ba^T a$, and $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d(\mathbb{R}))$. Let $(\hat{X}^N_{t_i^N}, 0 \leq i \leq N)$ be sampled with the scheme defined previously with the third order scheme for the CIR given in Alfonsi-2009 and starting from $x_0 \in \mathcal{S}_d^+(\mathbb{R})$. Then,

$$\exists C, N_0 > 0, \forall N \geq N_0, |\mathbb{E}[f(\hat{X}^N_{t_N^N})] - \mathbb{E}[f(X_t^{x_0})]| \leq C/N^3.$$

Remark

New extension of the regularity of the function $u(t,x) = \mathbb{E}\left[f(X_t^x)\right]$, from the CIR process to Wishart process . (Phd thesis A.Alfonsi 2006)

Canonical positive affine process transformation

Proposition

Let $(X_t^x)_{t\geq 0} \sim AFF_d(x, \bar{\alpha}, B, a)$ *and* $n = \mathrm{Rk}(a)$ *be the rank of* $a^T a$*. Then, there exist a diagonal matrix* $\bar{\delta}$*, and a non singular matrix* $u \in \mathcal{G}_d(\mathbb{R})$ *such that* $\bar{\alpha} = u^T \bar{\delta} u$*, and* $a^T a = u^T I_d^n u$*, and we have:*

$$(X_t^x)_{t\geq 0} \underset{Law}{=} u^T AFF_d\left((u^{-1})^T x u^{-1}, \bar{\delta}, B_u, I_d^n\right) u,$$

where $\forall y \in \mathcal{S}_d(\mathbb{R}),\ B_u(y) = (u^{-1})^T B(u^T y u) u^{-1}$.

The potential second order discretization for a general affine process defined on $\mathcal{S}_d^+(\mathbb{R})$

- It is sufficient to study the affine process $AFF_d(x, \overline{\delta}, B, I_d^n)$.
- By splitting operator, if L denotes the infinitesimal generator of $AFF_d(x, \overline{\delta}, B, I_d^n)$, we conclude then that

$$\begin{aligned} L &= L_{ODE} + L_{Wishart}, \\ L_{ODE} &= \mathrm{Tr}((\overline{\delta} - \delta_{\min} I_d^n + B(x))D^{\mathcal{S}}) \sim X_t^1, \\ L_{Wishart} &= \mathrm{Tr}((\delta_{\min} I_d^n)D^{\mathcal{S}}) + 2\mathrm{Tr}(xD^{\mathcal{S}} I_d^n D^{\mathcal{S}}) = \textstyle\sum_{i=1}^n L_i \sim X_t^2. \end{aligned}$$

Proposition

Both schemes $X_{t/2}^{1, X_t^{2, X_{t/2}^{1,x}}}$ *and* $UX_t^{1, X_t^{2,x}} + (1-U)X_t^{2, X_t^{1,x}}$ *are potential second order scheme for* $AFF_d(x, \overline{\delta}, B, I_d^n)$*, where* U *is an independent Bernoulli variable with parameter* $\frac{1}{2}$.

Fast potential second order discretization for a general affine process defined on $\mathcal{S}_d^+(\mathbb{R})$, $\overline{\delta} \geq dI_d^n$

- The previous algorithm requires $O(d^4)$, on each step time due to Cholesky decomposition of each transformation $(L_i)_{1\leq i\leq d}$.

- In the case of $\overline{\delta} \geq dI_d^n$ we propose an other scheme that costs only $O(d^3)$:

$$\begin{aligned}
L &= L_{ODE} + L_{Wishart},\\
L_{ODE} &= \mathrm{Tr}((\overline{\delta} - dI_d^n + B(x))D^{\mathcal{S}}),\\
L_{Wishart} &= \mathrm{Tr}((dI_d^n)D^{\mathcal{S}}) + 2\mathrm{Tr}(xD^{\mathcal{S}} I_d^n D^{\mathcal{S}}) \sim (c + \sqrt{t}\tilde{G}I_d^n)(c + \sqrt{t}\tilde{G}I_d^n)^T,
\end{aligned}$$

where $\tilde{G}$ is a matrix, in $\mathcal{M}_d(\mathbb{R})$, made of independent variables that fit the first five moments of normal random variable.

Plan

Time computation for $\mathbb{E}\left[\exp(i\mathrm{Tr}(vX_T^x))\right]$: $Nmc = 10^6$, $a = I_d$, $b = 0$, $x = 10I_d$, $v = 0.09I_d$ and $T = 1$

	$N = 10$			$N = 30$		
Schemes	R. value	Im. value	Time	R. value	Im. value	Time
Exact (1 step)	-0.526852	-0.227962	12			
2^{nd} order bis	-0.526229	-0.228663	41	-0.526486	-0.229078	125
2^{nd} order	-0.526577	-0.228923	76	-0.526574	-0.228133	229
3^{rd} order	-0.527021	-0.227286	82	-0.527613	-0.228376	244
Exact (N steps)	-0.526963	-0.228303	123	-0.526891	-0.227729	369
Corrected Euler	-0.525627^*	-0.233863^*	225	-0.525638^*	-0.231449^*	687

$\alpha = 3.5$, $d = 3$, $\Delta_R = 10^{-3}$, $\Delta_{Im} = 10^{-3}$, exact value R. $= -0.527090$ and Im.$= -0.228251$

Exact (1 step)	-0.591579	-0.037651	12			
2^{nd} order	-0.590444	-0.037024	77	-0.590808	-0.036487	229
3^{rd} order	-0.591234	-0.034847	82	-0.590818	-0.036210	246
Exact (N steps)	-0.591169	-0.036618	174	-0.592145	-0.037411	920
Corrected Euler	-0.589735^*	-0.042002^*	223	-0.590079^*	-0.039937^*	680

$\alpha = 2.2$, $d = 3$, $\Delta_R = 0.9 \times 10^{-3}$, $\Delta_{Im} = 1.3 \times 10^{-3}$, exact value R. $= -0.591411$ and Im.$= -0.036346$

Exact (1 step)	0.062712	-0.063757	181			
2^{nd} order bis	0.064237	-0.063825	921	0.064573	-0.062747	2762
2^{nd} order	0.064922	-0.064103	1431	0.063534	-0.063280	4283
3^{rd} order	0.064620	-0.064543	1446	0.064120	-0.063122	4343
Exact (N steps)	0.063418	-0.064636	1806	0.063469	-0.064380	5408
Corrected Euler	0.068298^*	-0.058491^*	2312	0.061732^*	-0.056882^*	7113

$\alpha = 10.5$, $d = 10$, $\Delta_R = 1.4 \times 10^{-3}$, $\Delta_{Im} = 1.3 \times 10^{-3}$, exact value R. $= 0.063960$ and Im.$= -0.063544$

Exact (1 step)	-0.036869	-0.094156	177			
2^{nd} order	-0.036246	-0.094196	1430	-0.035944	-0.092770	4285
3^{rd} order	-0.035408	-0.093479	1441	-0.036277	-0.093178	4327
Exact (N steps)	-0.036478	-0.092860	1866	-0.036145	-0.093003	6385
Corrected Euler	-0.028685^*	-0.094281^*	2321	-0.030118^*	-0.088988^*	7144

$\alpha = 9.2$, $d = 10$, $\Delta_R = 1.4 \times 10^{-3}$, $\Delta_{Im} = 1.4 \times 10^{-3}$, exact value R. $= -0.036064$ and Im.$= -0.093275$

Laplace transform $\mathbb{E}\left[\exp(i\mathrm{Tr}(vX_T^x))\right], \quad d = 3$

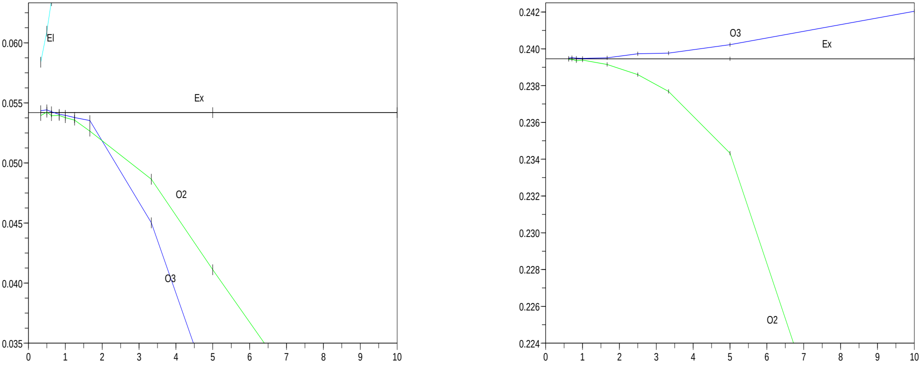
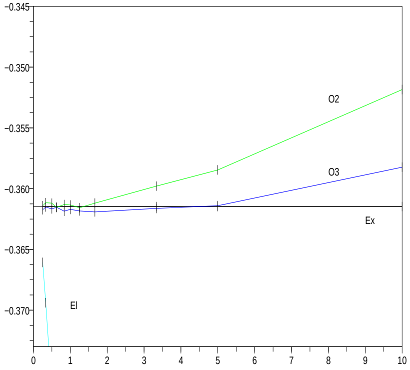


Figure: $d = 3$, 10^7 MC, $T = 10$. The RV of $\mathbb{E}[\exp(-\mathrm{Tr}(iv\hat{X}^N_{t^N_N}))]$ in function of T/N. Left: $v = 0.05I_d$ and $x = 0.4I_d$, $\alpha = 4.5$, $a = I_d$ and $b = 0$. Ex.Val.: 0.054277. Right: $v = 0.2I_d + 0.04q$ and $x = 0.4I_d + 0.2q$, $\alpha = 2.22$, $a = I_d$ and $b = -0.5I_d$. Ex.Val: 0.239836. Here, q is the matrix defined by: $q_{i,j} = \mathbf{1}_{i \neq j}$. The width of each point represents the 95% confidence interval.

Laplace transform $\mathbb{E}\left[\exp(i\mathrm{Tr}(vX_T^x))\right],\quad d=10$



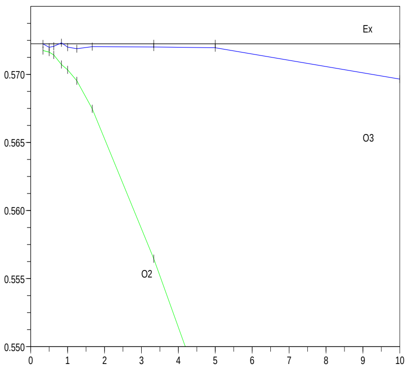


Figure: $d=10$, 10^7 MC, $T=10$. Left: IM of $\mathbb{E}[\exp(-\mathrm{Tr}(iv\hat{X}^N_{t^N_N}))]$ with $v=0.009I_d$ in function of T/N, $x=0.4I_d$, $\alpha=12.5$, $b=0$ and $a=I_d$. Ex.Val: -0.361586. Right: RV of $\mathbb{E}[\exp(-\mathrm{Tr}(iv\hat{X}^N_{t^N_N}))]$ with $v=0.009I_d$ in function of T/N, $x=0.4I_d$, $\alpha=9.2$, $b=-0.5I_d$ and $a=I_d$. Ex.Val 0.572241. The width of each point represents the 95% confidence interval.

Trajectory error $\mathbb{E}\left[\max_{0\leq s\leq T} \mathrm{Tr}(X_s^x)\right]$

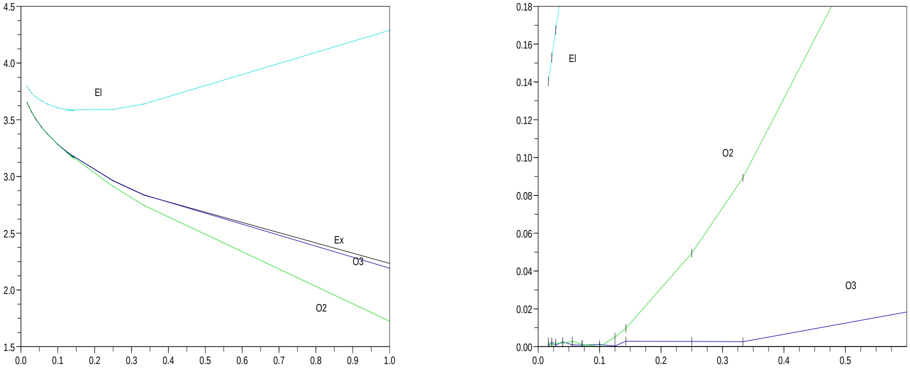


Figure: $d = 3$, 10^7 MC, $T = 1$, $x = 0.4I_d + 0.2q$ with $q_{i,j} = \mathbf{1}_{i\neq j}$, $\alpha = 2.2$, $b = 0$ and $a = I_d$. Left, $\mathbb{E}[\max_{0\leq k\leq N} \mathrm{Tr}(\hat{X}_{t_k^N}^N)]$, right: $\mathbb{E}[\max_{0\leq k\leq N} \mathrm{Tr}(\hat{X}_{t_k^N}^N)] - \mathbb{E}[\max_{0\leq k\leq N} \mathrm{Tr}(X_{t_k^N}^x)]$ in function of T/N. The width of each point gives the precision up to two standard deviations.

Gourieroux Sufana Model - Put Best of Option

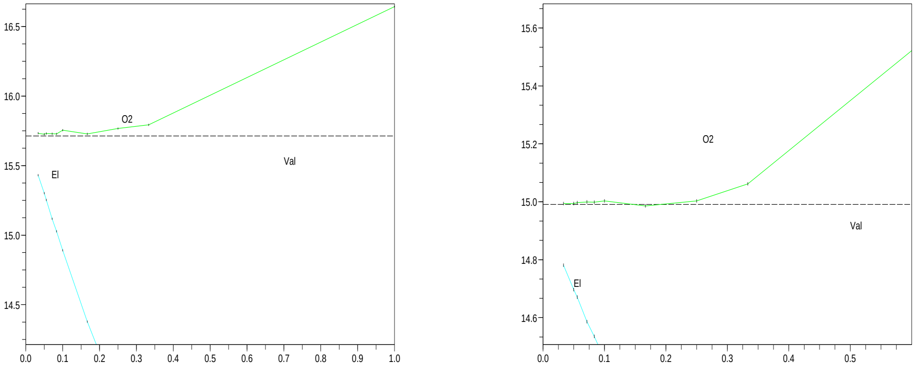


Figure: $\mathbb{E}[e^{-rT}(K - \max(\hat{S}^{1,N}_{t_N^N}, \hat{S}^{2,N}_{t_N^N}))^+]$ in function of T/N. $d = 2$, $T = 1$, $K = 120$, $S_0^1 = S_0^2 = 100$, and $r = 0.02$, $x = 0.04I_d + 0.02q$ with $q_{i,j} = \mathbf{1}_{i \neq j}$, $a = 0.2I_d$, $b = 0.5I_d$ and $\alpha = 4.5$ (left), $\alpha = 1.05$ (right). The width of each point gives the precision up to two standard deviations (10^6 MC).

Summary

In this work, we have presented :

- Exact scheme for Wishart process.
- Second and third order scheme for Wishart process.
- Potential second order scheme for a general affine process defined on $\mathcal{S}_d^+(\mathbb{R})$.

Thank you !!