Forward equations for option prices in semimartingale models

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Backward Kolmogorov equations for option prices

Consider an asset price/risk factor whose dynamics under a pricing measure is described by a Markov process X with generator L.

- The value $V_t = E^Q[h(X_T)|\mathcal{F}_t]$ at t for a maturity T of European options on X can then be characterized as the solution to the backward Kolmogorov PDE or "generalized Black Scholes" pricing equation
- $V_t = f(t, X_t)$ where

$$\frac{\partial f}{\partial t} + Lf = 0 \qquad f(T, .) = h(.)$$

Dupire equation for call options

In the case where X is a scalar diffusion

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

Bruno Dupire (1994) showed that the prices of *call options*

$$C_t(T, K) = E[(X_T - K)^+ | \mathcal{F}_t]$$

solves another PDE, in the *forward variables* K, T, the **Dupire PDE**:

$$\frac{\partial C_t}{\partial T} = \frac{K^2 \sigma(T, K)^2}{2} \frac{\partial^2 C_t}{\partial K^2} - rK \frac{\partial C_t}{\partial K}$$

on $[t, \infty[\times]0, \infty[$ with the initial condition:
$\forall K > 0 \quad C_t(t, K) = (S_t - K)_+$.

"Unified Theory of Volatility" (Dupire 1993)

Dupire also extended the forward PDE to (non Markovian) models: if X writes

$$dX_t = \delta_t dW_t$$

then, under appropriate conditions on the adapted process $(\delta_t)_{t\geq 0}$ the prices of call options

$$C_t(T,K) = E[(X_T - K)^+|\mathcal{F}_t]$$

solve

$$\frac{\partial C_t}{\partial T} = \frac{K^2\sigma(T,K)^2}{2}\frac{\partial^2 C_t}{\partial K^2} - rK\frac{\partial C_t}{\partial K}$$

where $\sigma(T,K)$ is the *effective volatility* given by

$$\sigma(T,K)^2 = E[\delta_T^2|X_T = K]$$

Forward equations: extensions

- Forward equations are quite useful as a computational/ theoretical tool.
- But...to price n options with payoffs $(h_i, i = 1..n)$ this requires solving n PDEs with different boundary conditions.
- If X is a Markov jump-diffusion process, then the forward PDE becomes an integro-differential equation. The Dupire equation has been extended in various directions:
    1. Jump-diffusion model with compound Poisson jumps (Andersen-Andreasen)
    2. Exponential Lévy processes (Carr & Hirsa, Jourdain)
    3. CDO expected tranche notionals (Cont & Minca)

Outline

- We derive a partial integrodifferential equation for call options in a general semimartingale model, generalizing the result of Dupire (1994), unifying in particular the existing extensions of Dupire equation:
- We allow the case of degenerate (or zero) volatility processes and discontinuities (jumps):
- As an application, we derive a one-dimensional approach to price basket options;

Multi-asset jump-diffusion model

Consider an asset S whose price under the pricing measure $\mathbb{P}$ follows a "stochastic volatility model with random jumps"

$$S_T = S_0 + \int_0^T r(t) S_{t-}\,dt + \int_0^T S_{t-}\delta_t dW_t + \int_0^T \int_{-\infty}^{+\infty} S_{t-}(e^y - 1)\tilde{M}(dtdy)$$

where $r(t)$ is the discount rate, δ_t the spot volatility process and $\tilde{M}$ is a compensated random measure with compensator

$$\mu(\omega; dt\, dy) = m(\omega; t, dy)\, dt;$$

Both the volatility δ_t and $m(t; dy)$ (which represents the intensity of jumps of size y at time t) are allowed to be stochastic. In particular, we do not assume the jumps to be driven by a Lévy process or a process with independent increments.

The value $C_{t_0}(T, K)$ at time t_0 of a call option with expiry $T > t_0$ and strike $K > 0$ is given by

$$C_{t_0}(T, K) = e^{-\int_{t_0}^{T} r(t)\,dt} E^{\mathbb{P}}[\max(S_T - K, 0)|\mathcal{F}_{t_0}];$$

The discounted asset price

$$\hat{S}_T = e^{-\int_0^T r(t)dt}\, S_T,$$

is the stochastic exponential of the martingale U defined by

$$U_T = \int_0^T \delta_t\, dW_t + \int_0^T \int (e^y - 1)\tilde{M}(dt\, dy).$$

Hence, under the assumption

$$\forall T > 0, \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \delta_t^2\, dt + \int_0^T dt \int_{\mathbb{R}} (e^y - 1)^2 m(t, dy)\right)\right] < \infty \tag{H}$$

we have

$$\forall T > 0, \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\langle U^c, U^c\rangle_T + \langle U^d, U^d\rangle_T\right)\right] < \infty$$

which implies that $(\hat{S}_T)$ is a $\mathbb{P}$-martingale.

Let ψ_t be the exponential double tail of the compensator $m(t, dy)$

$$\psi_t(z) = \begin{cases} \int_{-\infty}^{z} dx\ e^x \int_{-\infty}^{x} m(t, du) & z < 0 \\ \int_{z}^{+\infty} dx\ e^x \int_{x}^{\infty} m(t, du) & z > 0 \end{cases}$$

and define

$$\begin{cases} \sigma(t, z) & = \sqrt{\mathbb{E}\left[\delta_t^2 | S_{t^-} = z\right]}; \\ \chi_{t,y}(z) & = \mathbb{E}\left[\psi_t\left(z\right) | S_{t-} = y\right] \end{cases}$$

Theorem (Forward PIDE for call options)

Under assumption (H), the call option price $(T, K) \mapsto C_{t_0}(T, K)$, as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation:

$$
\begin{aligned}
\frac{\partial C_{t_0}}{\partial T} &= -r(T)K\frac{\partial C_{t_0}}{\partial K} + \frac{K^2\sigma(T,K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2} \\
&+ \int_0^{+\infty} y\frac{\partial^2 C_{t_0}}{\partial K^2}(T,dy)\,\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right)
\end{aligned}
$$

on $[t_0,\infty[\times]0,\infty[$ *with the initial condition:*
$\forall K > 0 \quad C_{t_0}(t_0, K) = (S_{t_0} - K)_+.$

Some remarks

The proof of the theorem is essentially based on the application of the Tanak-Meyer formula to $(S_t - K)^+$ between T and $T+h$ in the case of general semimartingales. If $L_t^K = L_t^K(S)$ is the semimartingale local time of S at K under $\mathbb{P}$, then for all $h > 0$

$$(S_{T+h} - K)^+ = (S_T - K)^+ + \int_T^{T+h} 1_{\{S_{t-}>K\}} dS_t + \frac{1}{2}(L_{T+h}^K - L_T^K) + \sum_{T<t\leq T+h} (S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-}>K\}} \Delta S_t.$$

Taking expectations, we get:

$$e^{\int_0^{T+h} r(t)\,dt} C(T+h,K) - e^{\int_0^{T} r(t)\,dt} C(T,K)$$

$$= \mathbb{E}\left[\int_T^{T+h} r(t) S_t\, 1_{\{S_{t-}>K\}} dt + \frac{1}{2}(L^K_{T+h} - L^K_T)\right]$$

$$+ \mathbb{E}\left[\sum_{T<t\leq T+h} (S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-}>K\}} \Delta S_t\right].$$

Let focus on the jump part of the last expression...

$$\mathbb{E}\left[\sum_{T<t\leq T+h}(S_t-K)^+-(S_{t-}-K)^+-1_{\{S_{t-}>K\}}\Delta S_t\right]$$

$$= \int_T^{T+h} dt\mathbb{E}\Big[\int m(t,dx)\Big((S_{t-}e^x-K)^+ \\ -e^x(S_{t-}-K)^+-K1_{\{S_{t-}>K\}}(e^x-1)\Big)\Big]$$

$$= \int_T^{T+h} dt\,\mathbb{E}\left[S_{t-}\,\psi_{t,S_{t-}}\left(\ln\left(\frac{K}{S_{t-}}\right)\right)\right]$$

$$= \int_T^{T+h} dt\,\mathbb{E}\left[S_{t-}\mathbb{E}\left[\psi_{t,S_{t-}}\left(\ln\left(\frac{K}{S_{t-}}\right)\right)|S_{t-}\right]\right]$$

$$= \int_T^{T+h} dt\,\mathbb{E}\left[S_{t-}\chi_{t,S_{t-}}\left(\ln\left(\frac{K}{S_{t-}}\right)\right)\right]$$

Where does it come from? One oberves that:

$$\int_{\mathbb{R}} [(ye^x - K)^+ - e^x(y-K)^+ - K(e^x-1)1_{\{y>K\}}]m(t,dx) = y\,\psi_{t,y}\left(\ln\left(\frac{K}{y}\right)\right.$$

- It shows that, **any** arbitrage-free profile of option prices across strike and maturity may be parameterized by a local volatility function $\sigma(t, S_{t-})$ and a local exponential double tail $\chi_{t,S_{t-}}(z)$.
- Intuitively, we would have defined a local Lévy measure $m_{loc}(t, dy, S_{t-})$ by : for $t \geq 0, z \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d) - \{0\}$

$$m_{loc}(t, A, z) = \mathbb{E}\left[m(t, A) | S_{t^-} = z\right].$$

  which would have meant to project all the parameters of (S_t), that is proceed to the *Markovian projection* of (S_t) but it is not obvious that such a projection is well-posed, at least if one doesn't make stronger hypotheses on these parameters.
- We observe that it is sufficient to define the local exponential double tail to price the call option.

Itô diffusions

- Killing the jump part, (H) writes $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \delta_t^2\,dt\right)\right] < \infty$ $a.s$, and one recovers the Dupire PDE.
- This result can be derived from the mimicking theorem of Gyöngy 1986 in the case where the volatility process δ_t verifies a non-degeneracy (i.e. uniform ellipticity) condition.
- In this case our result gives an alternative set of assumptions under which the Dupire equation holds, which do not require this non-degeneracy condition.

Consider the price process S whose dynamics under the pricing measure $\mathbb{P}$ is given by:

$$S_t = S_0 + \int_0^T r(t) S_{t-} dt + \int_0^T S_{t-} \sigma(t, S_{t-}) dB_t + \int_0^T \int_{-\infty}^{+\infty} S_{t-}(e^y - 1) \tilde{N}(dt dy)$$

where B_t is a Brownian motion and N a Poisson random measure on $[0, T] \times \mathbb{R}$ with compensator $\nu(dz)\, dt$, $\tilde{N}$ the associated compensated random measure. Assume:

$$\begin{cases} \sigma(.,.) \quad \text{is bounded} \\ \int_{y>1} e^{2y} \nu(dy) < \infty \end{cases}$$

Proposition

The call option price $C_{t_0}(T,K)$ is a solution (in the sense of distributions) of the partial integro-differential equation:

$$\frac{\partial C_{t_0}}{\partial T} = -r(T)K\frac{\partial C_{t_0}}{\partial K} + \frac{K^2\sigma(T,K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2}$$
$$+ \int_{\mathbb{R}} \nu(dz)\, e^z \left[C_{t_0}(T, Ke^{-z}) - C_{t_0}(T,K) - K(e^{-z}-1)\frac{\partial C_{t_0}}{\partial K} \right]$$

on $[t_0,\infty[\times]0,\infty[$ *with the initial condition:*
$\forall K > 0 \quad C_{t_0}(t_0,K) = (S_{t_0} - K)_+$.

Indeed...in this particular case:

$$\int_0^{+\infty} y \frac{\partial^2 C}{\partial K^2}(T, dy)\, \chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right)$$

$$= \int_{\mathbb{R}} e^z \left[C(T, Ke^{-z}) - C(T, K) - K(e^{-z} - 1)\frac{\partial C}{\partial K} \right] \nu(dz)$$

We now consider price processes with no Brownian component. It is convenient to use the change of variable: $v = \ln y$, $k = \ln K$, $c(k,T) = C(e^k, T)$. Define:

$$\chi_{T,v}(z) = \mathbb{E}\left[\psi_T(z) | S_{T-} = e^v\right]$$

with:

$$\psi_T(z) = \begin{cases} \int_{-\infty}^{z} dx\ e^x \int_{-\infty}^{x} m(T,du) & z < 0 \\ \int_{z}^{+\infty} dx\ e^x \int_{x}^{\infty} m(T,du) & z > 0 \end{cases}$$

Proposition

If

$$\forall T > 0, \quad \mathbb{E}\left[\exp\left(\int_0^T dt \int (e^y - 1)^2 m(t\, dy)\right)\right] < \infty$$

then the call option price $c(T, k)$ is a solution (in the sense of distributions) of the partial integro-differential equation:

$$\frac{\partial c}{\partial T} + r(T)\frac{\partial c}{\partial k} = \int_{-\infty}^{+\infty} e^{2(v-k)} \left(\frac{\partial^2 c}{\partial k^2} - \frac{\partial c}{\partial k}\right)(T, dv)\, \chi_{T,v}(k - v)$$

In the case considered in Carr, Geman, Madan and Yor 2004, where the Lévy density m_Y has a deterministic separable form:

$$m_Y(t, dz, y)\, dt = \alpha(y, t)\, k(dz)\, dz\, dt$$

The previous PIDE allows us to recover their result becoming:

$$\frac{\partial c}{\partial T} + r(T)\frac{\partial c}{\partial k} = \int_{-\infty}^{+\infty} \kappa(k - v) e^{2(v-k)} \alpha(e^v, T) \left(\frac{\partial^2 c}{\partial k^2} - \frac{\partial c}{\partial k} \right) d(v)$$

where κ is defined as the exponential double tail of $k(u)\, du$, i.e:

$$\kappa(z) = \begin{cases} \int_{-\infty}^{z} dx \; e^x \int_{-\infty}^{x} k(u)\, du & z < 0 \\ \int_{z}^{+\infty} dx \; e^x \int_{x}^{\infty} k(u)\, du & z > 0 \end{cases}$$

Time changed Lévy processes were proposed in Carr, Geman, Madan and Yor 2003, in the context of option pricing. Consider the price process S whose dynamics under the pricing measure $\mathbb{P}$ is given by:

$$\left(S_t \equiv e^{\int_0^t r(s)\,ds} X_t\right) \qquad X_t = \exp\left(L_{\Theta_t}\right) \qquad \Theta_t = \int_0^t \theta_s ds$$

where L_t is a Lévy process with characteristic triplet (b, σ^2, ν), (θ_t) is a locally bounded positive semimartingale. We assume L and θ are $\mathcal{F}_t$-adapted.
$X_t \equiv (e^{-\int_0^t r(s)\,ds} S_t)$ is a martingale under the pricing measure $\mathbb{P}$ if $\exp\left(L_t\right)$ is a martingale which requires the following condition on the characteristic triplet of (L_t):

$$b + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} (e^z - 1 - z\,1_{|z|\leq 1})\nu(dy) = 0$$

Define

$$\alpha(t,x) = E[\theta_t | X_{t-} = x]$$

and χ the exponential double tail of $\nu(du)$

$$\chi(z) = \begin{cases} \int_{-\infty}^{z} dx \; e^x \int_{-\infty}^{x} \nu(du) & z < 0 \\ \int_{z}^{+\infty} dx \; e^x \int_{x}^{\infty} \nu(du) & z > 0 \end{cases}$$

Proposition

If $\int_{y>1} e^{2y}\nu(dy) < \infty$ *then the call option price* $C_{t_0}(T, K)$ *at date* t_0*, as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation:*

$$\frac{\partial C}{\partial T} = -r\alpha(T,K)K\frac{\partial C}{\partial K} + \frac{K^2\alpha(T,K)\sigma^2}{2}\frac{\partial^2 C}{\partial K^2} + \int_0^{+\infty} y\frac{\partial^2 C}{\partial K^2}(T,dy)\,\alpha(T,y)\,\chi\left(\ln\left(\frac{K}{y}\right)\right)$$

on $[t,\infty[\times]0,\infty[$ *with the initial condition:*
$\forall K > 0 \quad C_{t_0}(t_0,K) = (S_{t_0} - K)_+.$

- Note that the same adjustment factor $\alpha(t, X_{t-})$ is applied to the drift, diffusion coefficient and the double exponential tail of the Lévy measure.

$$(b\alpha(t, X_{t-}), \sigma^2\alpha(t, X_{t-}), \alpha(t, X_{t-})\chi)$$

Consider a multivariate model with d assets:

$$S_T^i = S_0^i + \int_0^T r(t) S_{t-}^i \, dt + \int_0^T S_{t-} \delta_t^i dW_t^i + \int_0^T \int_{\mathbb{R}^d} S_{t-}^i (e^{y_i} - 1) \tilde{N}(dt\, dy)$$

where δ^i is an adapted process taking values in $\mathbb{R}$ representing the volatility of asset i, W is a d-dimensional Wiener process, N is a Poisson random measure on $[0, T] \times \mathbb{R}^d$ with compensator $\nu(dy)\, dt$, $\tilde{N}$ denotes its compensated random measure.
The Wiener processes W^i are correlated: for all $1 \leq (i, j) \leq d$, $\langle W^i, W^j \rangle_t = \rho_{i,j} t$, with $\rho_{ij} > 0$ and $\rho_{ii} = 1$.
An index is defined as a weighted sum of the asset prices:

$$I_t = \sum_{i=1}^{d} w_i S_t^i$$

The value $C_{t_0}(T,K)$ at time t_0 of an index call option with expiry $T > t_0$ and strike $K > 0$ is given by

$$C_{t_0}(T,K) = e^{-\int_{t_0}^{T} r(t)\,dt} E^{\mathbb{P}}[\max(I_T - K, 0)|\mathcal{F}_{t_0}]$$

Let $k(.,t,dy)$ be the random measure:

$$k(t,dy) = \int \ln\left(\frac{\sum_{1\le i\le d-1} w_i S^i_{t-} e^{y_i} + w_d S^d_{t-} e^{y}}{I_{t-}}\right) \nu(dy_1,..,dy_{d-1},dy)$$

and $\eta_t(z)$ its exponential double tail:

$$\eta_t(z) = \begin{cases} \int_{-\infty}^{z} dx\ e^x \int_{-\infty}^{x} k(t,du) & z < 0 \\ \int_{z}^{+\infty} dx\ e^x \int_{x}^{\infty} k(t,du) & z > 0 \end{cases}$$

Define:

$$\sigma(t,z) = \frac{1}{z}\sqrt{\mathbb{E}\left[\left(\sum_{i,j=1}^{d} w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right) | I_{t-} = z\right]};$$

$$\chi_{t,y}(z) = \mathbb{E}\left[\eta_t\,(z)\,|\,I_{t-} = y\right]$$

Assume

$$\begin{cases} \forall T > 0 \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \|\delta_t\|^2\, dt\right)\right] < \infty \\ \int_{\mathbb{R}^d} (1 \wedge \|y\|)\, \nu(dy) < \infty \\ \int_{\|y\|>1} e^{2\|y\|} \nu(dy) < \infty \end{cases}$$

Theorem

Under these assumptions, the index call price $(T,K) \mapsto C_{t_0}(T,K)$, *as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation:*

$$\frac{\partial C}{\partial T} = -r(T)K\frac{\partial C}{\partial K} + \frac{\sigma(T,K)^2}{2}\frac{\partial^2 C}{\partial K^2} + \int_0^{+\infty} y\frac{\partial^2 C}{\partial K^2}(T,dy)\,\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right)$$

on $[t_0,\infty[\times]0,\infty[$ *with the initial condition:*
$\forall K > 0 \quad C_{t_0}(t_0,K) = (I_{t_0} - K)_+.$

Forward PIDE as dimension reduction

- The following result generalizes the forward PDE studied by Avellaneda et al. 2003 for the diffusion case to a setting with jumps.
- The conditional expectations in the expressions of the effective volatility $\sigma(.,)$ and effective jump intensity $j()$ may be efficiently computed (without simulation) using a *steepest descent* approximation proposed by (Avellaneda Busca Friz Boyer-Olson) in the diffusion case.
- This enables to price index options in a (smile-consistent) multidimensional jump-diffusion model without Monte Carlo simulation, by solving a **one-dimensional** forward PIDE.

Conclusion

- Allows for degenerate/ zero volatility and jumps.
- Extension of the Dupire/forward equation for option prices to a large class of non Markovian models with jumps.
- Allows dimension reduction and use of P(I)DE methods when computing call option prices.

References

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