Resilience to contagion in financial networks

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The financial network

A network of financial counterparties can be modeled as a *weighted directed graph* whose

- n vertices (nodes) $i \in V$ represent financial market participants : banks, funds, corporate borrowers and lenders,...
- (directed) links represent counterparty exposures : $e_{i,j}$ is the exposure of i to j.
- In a market-based framework $e_{i,j}$ is understood as the fair market value of the exposure of i to j.
- Each institution i disposes of a *capital buffer* c_i which absorbs market losses : Proxy for c_i : Tier I + Tier II capital.

Balance sheet

Assets	Liabilities
Interbank assets $A_i = \sum_j e_{i,j}$	Interbank liabilities $L_i = \sum_j e_{j,i}$ Deposits D_i
Other assets x_i	Net worth $c_i = \gamma_i A_i$

TABLE: Stylized balance sheet of a bank.

The capital ratio : $\gamma_i = \frac{c_i}{A_i}$

The default dynamics

Cascade definition

The default of a market participant j affects its counterparts in the following way over a short term horizon

- Creditors lose a fraction $(1-R)$ of their exposure. Loss is first absorbed by capital : $c_i \to \min(c_i - (1-R)e_{i,j}, 0)$.
- This leads to a writedown of $(1-R)e_{i,j}$ in the balance sheet of i, which can lead to **default** of i if

$$c_i < (1-R)e_{i,j}.$$

Typically $R \simeq 0$ in the short term (liquidation takes time). Insolvency occurs if Loss(i)$>$ c_i.

Heterogeneity in the structure of interbank networks

Example : Brazil's interbank network (data from Banco Central do Brasil 2008).

- Average number of counterparties (degree) $= 7$
- **Heterogeneity in number of debtors** : In-degree has a heavy-tailed Pareto distribution with exponent $\simeq 2$.
- **Heterogeneity in number of creditors** : Out-degree has a heavy-tailed Pareto distribution with exponent $\simeq 3$.
- Heterogeneous exposures sizes : heavy tailed distribution, a handful of bilateral exposures are > 100 times larger than most of the rest $\rightarrow$ Pareto distribution.

Embedding in a sequence of networks

Financial system : weighted graph $\mathbf{e}$ with the vertex set $[1, \ldots, n]$ and the corresponding sequence of capital ratios $\gamma = (\gamma_i)_{i=1}^n$.
The idea of this paper to embed $(\mathbf{e}, \gamma)$ in a sequence of financial networks, indexed by their size, $(\mathbf{e_n}, \gamma_\mathbf{n})_{n \geq 1}$.

Random financial network

Definition

The random financial network $\mathbf{E_n}$ is a random matrix of size n taken uniformly over all matrices of size n having the following properties :

- For every $1 \leq i \leq n$, line i of network $\mathbf{E_n}$ is a permutation of line i in the network $\mathbf{e_n}$, with the constraint that the main diagonal is zero ;
- On every column, the number of non-zero elements in $\mathbf{E_n}$ is the same as in $\mathbf{e_n}$.

Asymptotic study : idea

Aim : study contagion in the random financial network as its size $n \to \infty$.
We are given : the sequence $\mathbf{e_n}$ of financial networks.
On the probability space $(\Omega, \mathbb{P})$,
we study contagion on the sequence of random networks $\mathbf{E_n}$.
More precisely, we introduce the final fraction of defaults

$$\alpha_n = \frac{N_{def}(\mathbf{E_n}, \gamma_\mathbf{n})}{n}$$

Question 1 : $\alpha_n \xrightarrow{p}$?
and under which assumptions ?
Question 2 : How does the limit depend on the network topology and the individual exposures ?
Question 3 : Is the network resilient to small shocks ?

Asymptotic study

We have $\mathbf{d_n^+} = (d_n^+(i))_{i=1}^n$ and $\mathbf{d_n^-} = (d_n^-(i))_{i=1}^n$ the sequences of non-negative integers representing the degrees :

$$\sum_{i=1}^n d_n^+(i) = \sum_{i=1}^n d_n^-(i).$$

We introduce the **empirical distribution of the degrees** as

$$\mu_n(j,k) := \frac{1}{n} \# \{ i : d_n^+(i) = j, d_n^-(i) = k \}.$$

Assumptions on the degree sequence

We assume that there exists a probability distribution μ on $\mathbb{N}^2$ such that :

1. The empirical proportion of nodes of degree (j,k) tends to $\mu(j,k)$:
$$\mu_n(j,k) \to \mu(j,k) \text{ as } n \to \infty;$$
2. Finite average degree property :
$$\exists \lambda \in (0,\infty), \qquad \sum_{j,k} j\mu(j,k) = \sum_{j,k} k\mu(j,k) =: \lambda;$$
3. $\sum_{i=1}^n d_n^+(i) = \sum_{i=1}^n d_n^-(i)$;
4. $\sum_{i=1}^n (d_n^+(i))^2 + (d_n^-(i))^2 = O(n)$.

Mapping continuous into discrete variables

For each node i and permutation $\tau \in \Sigma_i^{\mathbf{e_n}}$ of the counterparties of i, we define

$$\Theta(i, \tau, \mathbf{e}) := \min\{k \geq 0, \gamma_i \sum_{j=1}^{d^+(i)} e_{i,j} < \sum_{j=1}^{k} (1-R) e_{i,\tau(j)}\}. \quad (1)$$

$\Theta(i, \tau)$ is the number of counterparty defaults which will generate the default of i if defaults happen in the order prescribed by τ.

$$p_n(j, k, \theta) := \frac{\#\{(i, \tau) \mid \tau \in \Sigma_i^{\mathbf{e_n}}, d_n^+(i) = j, d_n^-(i) = k, \Theta(i, \tau, \mathbf{e}) = \theta\}}{n \mu_n(j, k) j!}.$$

Contagious links

A link is called *contagious* if it generates a default of the end node if the starting node defaults.
$n\mu_n(j,k)jp_n(j,k,1)$ is the total number of contagious links that enter a node with degree (j,k).
The value $p_n(j,k,1)$ gives the proportion of contagious links ending in nodes with degree (j,k).

Assumptions on the exposure sequence

There exists a function $p : \mathbb{N}_+^3 \to [0, 1]$ such that for all $j, k, \theta \in \mathbb{N}$ $(\theta \leq j)$

$$p_n(j, k, \theta) \overset{n\to\infty}{\to} p(j, k, \theta). \tag{2}$$

as $n \to \infty$. This assumption is fulfilled for example in a model where exposures are exchangeable arrays.

The probability limit for the final fraction of defaults

Let us define

$$\beta(n, \pi, \theta) := \mathbb{P}(Bin(n, \pi) \geq \theta) = \sum_{j \geq \theta}^{n} \binom{n}{j} \pi^j (1-\pi)^{n-j}.$$

and

$$I(\pi) := \sum_{j,k} \frac{k\mu(j,k)}{\lambda} \sum_{\theta=0}^{j} p(j,k,\theta)\beta(j,\pi,\theta) \qquad (3)$$

The value $\frac{k\mu(j,k)}{\lambda}$ represents the probability that an edge at random begins in a node with in-degree j and out-degree k.

Define

Theorem

Consider the sequences of exposures and capital ratios $\{(\mathbf{e_n})_{\mathbf{n \geq 1}}, (\gamma_\mathbf{n})_{n \geq 1}\}$ satisfying the Assumptions on the degree and exposure sequence. Let π^* be the smallest fixed point of I. We have

1. *If $\pi^* = 1$, i.e. if $I(\pi) > \pi$ for all $\pi \in [0, 1)$, then asymptotically all nodes default during the cascades*

$$\alpha_n = 1 - o_p(1).$$

2. *If $\pi^* < 1$ and furthermore π^* is a stable fixed point of I, then the asymptotic fraction of defaults*

$$\alpha_n \xrightarrow{p} \sum_{j,k} \mu(j,k) \sum_{\theta=0}^{j} p(j,k,\theta)\beta(j,\pi^*,\theta).$$

The intuition :branching process approximation

We can give the for

$$I(\pi) = \sum_{j,k} \frac{k\mu(j,k)}{\lambda} \sum_{\theta=0}^{j} p(j,k,\theta)\beta(j,\pi,\theta)$$

the following interpretation : if the counterparty of a randomly chosen node defaults with probability π defaults, $I(\pi)$ is the expected fraction of counterparty defaults after one iteration of the cascade.

The function

$$\sum_{j,k} \mu(j,k) \sum_{\theta=0}^{j} p(j,k,\theta)\beta(j,\pi,\theta),$$

gives the fraction of defaulted nodes supposing that a counterparty of a randomly chosen node defaults with probability π.

Is the random network robust to small shocks ?

Corollary

If

$$\sum_{j,k} jk \frac{\mu(j,k)}{\lambda} p(j,k,1) < 1 \tag{4}$$

then for every $\epsilon > 0$*, there exists* N_ϵ *and* ρ_ϵ *such that if the initial fraction of defaults is smaller than* ρ_ϵ*, then* $\mathbb{P}(\alpha_n(\mathbf{E_n}, \gamma_\mathbf{n}) \leq \epsilon) > 1 - \epsilon$ *for all* $n \geq N_\epsilon$*.*

The skeleton of contagious links

The converse also holds :

Proposition

If

$$\sum_{j,k} jk \frac{\mu(j,k)}{\lambda} p(j,k,1) > 1,$$

then there exists a connected set C_n of nodes representing a positive fraction of the financial system, i.e. $|C_n|/n \xrightarrow{p} c > 0$ such that, with high probability, any node belonging to this set can trigger the default of all nodes in the set : for any sequence $(\gamma_n)_{n\geq 1}$ such that $\{i, \gamma_n(i) = 0\} \cap C_n \neq \emptyset$,

$$\liminf_n \alpha_n(E_n, \gamma_n) \geq c > 0.$$

Amplification

We suppose that the resilience condition is satisfied.
Let π^*_ϵ be the smallest fixed point of I in $[0,1]$, when a fraction ϵ of all nodes represent fundamental defaults, i.e. $p(j,k,0)=\epsilon$ for all j,k.
First order approximation of the function I :

$$\pi^*_\epsilon = \frac{\epsilon}{1-\sum_{j,k} jk\frac{\mu(j,k)}{\lambda}(j,k)p(j,k,1)} + o(\epsilon).$$

$$\lim_{\epsilon\to 0}\frac{g(\pi^*_\epsilon)}{\epsilon} = 1 + \frac{\sum_{j,k} j\mu(j,k)p(j,k,1)}{1-\sum_{j,k} jk\frac{\mu(j,k)}{\lambda}(j,k)p(j,k,1)}.$$

Amplification - cont.

We denote $\pi_\epsilon^*(d^+, d^-)$ the smallest fixed point of I in $[0, 1]$ in the case where $p(d^+, d^-, 0) = \epsilon$ and $p(j, k, 0) = 0$ for all $(j, k) \neq (d^+, d^-)$.
Then the good measure of how many times is the final fraction of defaults larger than the initial fraction of defaults is

$$\lim_{\epsilon \to 0} \frac{g\left(\pi_\epsilon^*(d^+, d^-)\right)}{\epsilon \mu(d^+, d^-)} = 1 + \frac{d^-}{\lambda} \frac{\sum_{j,k} jk \frac{\mu(j,k)}{\lambda} p(j, k, 1)}{1 - \sum_{j,k} jk \frac{\mu(j,k)}{\lambda} p(j, k, 1)}.$$

The sample network

Size of real financial networks

- US : n = 7969 financial institutions (Source : FDIC)
- Euro area : n = 8350 financial institutions (Source : ECB)

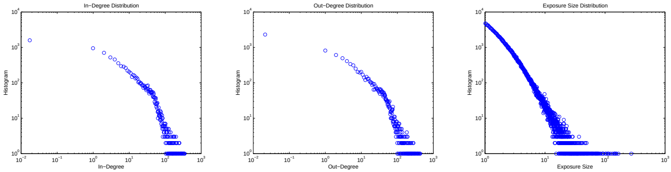

FIGURE: Sample network (a) The distribution of out-degree has a Pareto tail with exponent 2.19, (b) The distribution of the in-degree has a Pareto tail with exponent 1.98, (c) The distribution of the exposures (tail-exponent 2.61).

Relevance of asymptotics - cont.

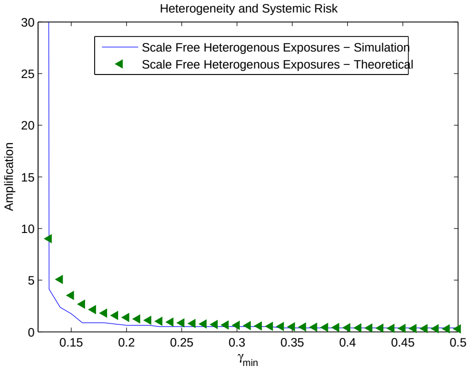

FIGURE: Amplification of the default number in a Scale-Free Network. The in and out-degree of the scale-free network are Pareto distributed with tail coefficients 2.19 and 1.98 respectively, the exposures are Pareto distributed with tail coefficient 2.61, $n = 10000$.

Too interconnected to fail ?

We plot the simulated final fraction of defaults starting from one fundamental default in a simulated, scale free network as a function of the out-degree, versus the theoretical slope given above.

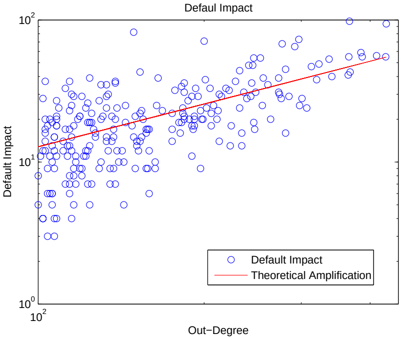

FIGURE: Number of defaulted nodes

The Impact of heterogeneity

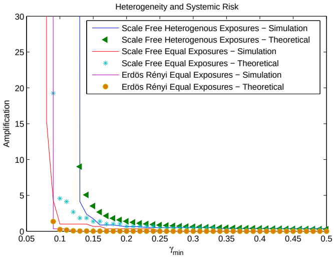

FIGURE: Amplification of the number of defaults in a Scale-Free Network (in and out-degree of the scale-free network are Pareto distributed with tail coefficients 2.19 and 1.98 respectively, the exposures are Pareto distributed with tail coefficient 2.61), the same network with equal weights and an Erdös Rényi Network with equal exposures $n = 10000$.

Immediate conclusion : average connectivity alone cannot measure systemic risk.

Stress tests

In the Supervisory Capital Assessment Program, implemented by the Board of Governors of the Federal Reserve System in 2009, the top 19 banks in the US were asked to project their losses and resources under a macroeconomic shock scenario. The program determined which of the large banks needed to augment its capital base in order to withstand the projected losses.

A simple external shock model

LR : ratio between the total interbank assets and total assets

$$c_i = \gamma_{min} A_i \frac{1}{LR}.$$

Under a stress test scenario, a macroeconomic shock Z, constant over all banks affects the banks external assets (defined as the difference between total and interbank assets). After the shock, the capital ratio becomes

$$c_i(Z) = \gamma_{min} A_i \left(1 + \left(\frac{1}{LR} - 1\right)(1 - Z)\right)\epsilon(i), \tag{5}$$

where $\epsilon(i)$ are independent variables with

$$\mathbb{P}(\epsilon(i) = 1) = \epsilon = 1 - \mathbb{P}(\epsilon(i) = 0),$$

$\epsilon(i) = 1$ indicating whether i is in default in the stress scenario under consideration.

An infinite random scale free network

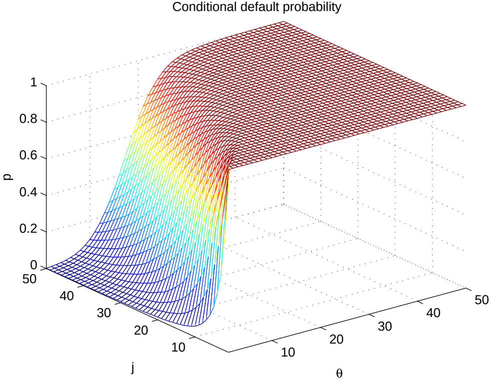

FIGURE: The conditional probability of default, Minimal capital ratio = 8%, Macroeconomic shock = 20%

Phase transitions : Armageddon ?

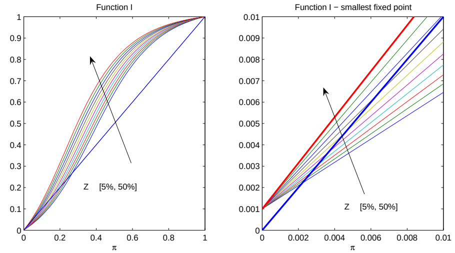

FIGURE: Function I for increasing magnitude of the macroeconomic shock. As the common factor increases, the smallest fixed point becomes 1 and the phase transition occurs.

Phase transitions : Armageddon ?

Resilience function : $1 - \sum_{j,k} jk \frac{\mu(j,k)}{\lambda} p(j,k,1)$

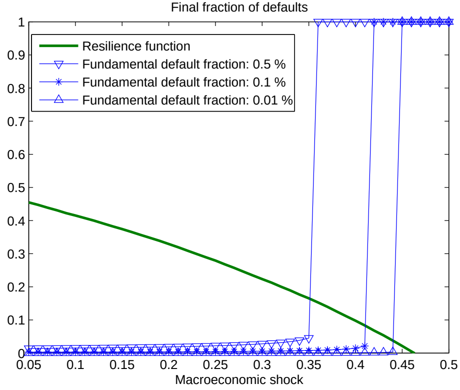

FIGURE: Final fraction of defaults : infinite network

The finite sample

In a finite network the resilience condition becomes

$$\frac{1}{m_n}\sum_i d_n^-(i) q_i < 1, \tag{6}$$

with q_i : the number of 'contagious' links.

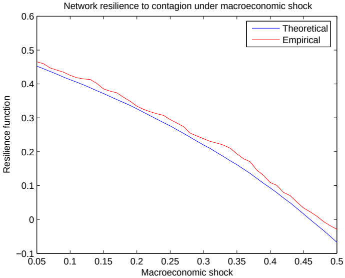

FIGURE: (a)Proportion of contagious links. (b)Resilience function for varying size of macroeconomic shock in the sample and limit random network.

Resilience and phase transitions

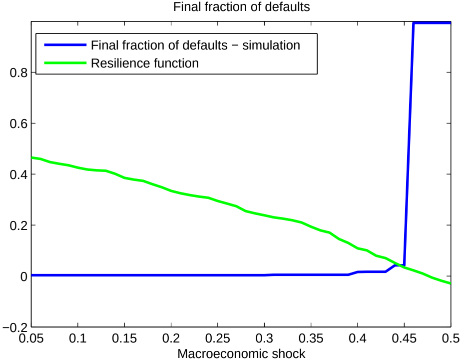

FIGURE: Final fraction of defaults.

Conclusions

- Our approach complements existing stress tests used by regulators and suggests to monitor the capital adequacy of each institution with regard to its *largest exposures*.
- As the banks are asked to project the effect of the macroeconomic shock on their balance sheets, specific values for the quantities of interest (number of contagious links and connectivity) can be reported to the regulator and the resilience can be then assessed by our criterion.
- Our results also suggest that one need not monitor/know the *entire* network of counterparty exposures but simply the *skeleton*/ subgraph of contagious links.

Conclusions - cont.

- The regulator can efficiently contain contagion by focusing on fragile nodes, especially those with high connectivity, and their counterparties.
- Higher capital requirements could be imposed on them to reduce their number of contagious links and insure that the danger of phase transitions as described above is avoided.

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