Valuation of credit derivatives in LIBOR models

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Outline

Market size

According to the Bank for International Settlements:

	Jun 2006	Jun 2007	Jun 2008	Jun 2009
Foreign exchange	38,127	48,645	62,983	48,775
Interest rate	262,526	347,312	458,304	437,198
Equity-linked	6,782	8,590	10,177	6,619
Commodity	6,394	7,567	13,229	3,729
Credit default swaps	20,352	42,581	57,403	36,046
Unallocated	35,997	61,713	81,719	72,255
Total	370,178	516,408	683,815	604,622

Table: Notional amounts outstanding for OTC derivatives in billions of US$

Interest rates

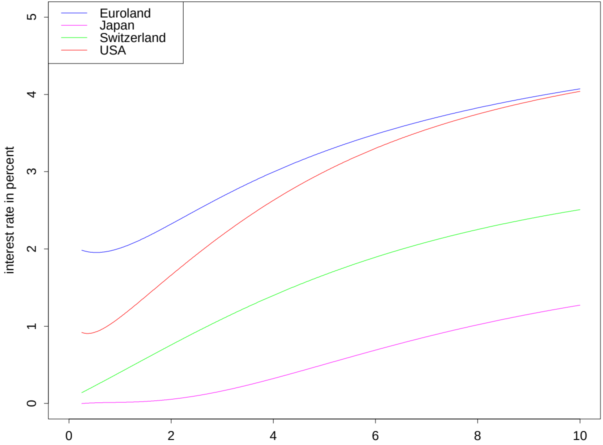

Figure: Term structure of interest rates, Feb 17 2004

Credit spreads

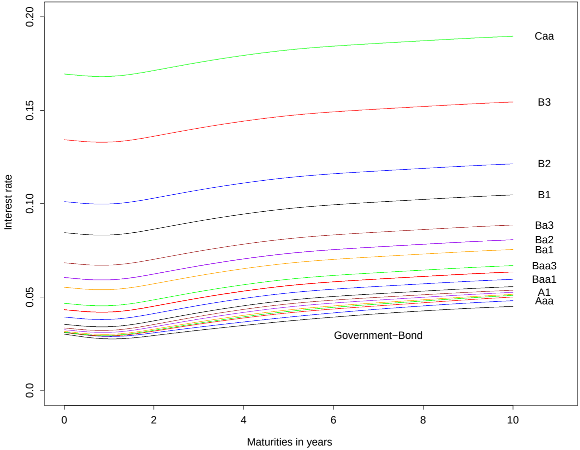

Figure: Term structure of Euro corporate spreads, Dec 20 2002

Evolution of interest rates

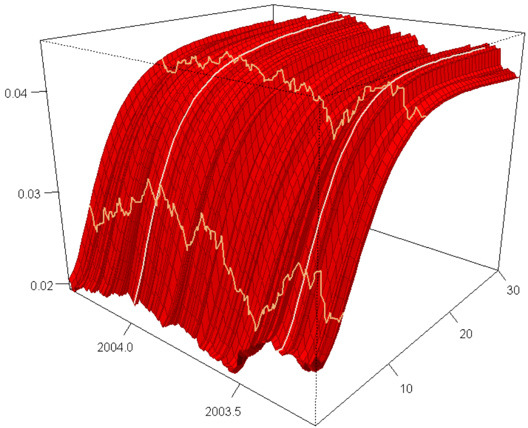

Figure: Evolution of interest rate term structure, 2003–2004

Default-free rates

- Tenor: $0 < T_1 < T_2 < \cdots < T_N < T_{N+1} = T_*$, tenor length δ
- $B(t, T)$: value of a zero coupon bond for T, $B(T, T) = 1$
- $L(t, T)$: forward LIBOR rate for $[T, T + \delta]$

$$L(t, T) = \frac{1}{\delta}\left(\frac{B(t, T)}{B(t, T + \delta)} - 1\right)$$

- $F(t, T, U)$: forward price for T and U; $F(t, T, U) = \frac{B(t,T)}{B(t,U)}$

"Master" relation

$$F(t, T, T + \delta) = \frac{B(t, T)}{B(t, T + \delta)} = 1 + \delta L(t, T) \tag{1}$$

Defaultable bonds

- $B^0(t, T)$: value of a defaultable zero coupon bond with zero recovery and maturity T
- τ: time of default
- $\overline{B}(t,T)$: pre-default value of the defaultable bond

$$\Longrightarrow \quad B^0(t,T) = 1_{\{\tau>t\}}\overline{B}(t,T), \quad \overline{B}(T,T) = 1$$

Defaultable rates and spreads

- The defaultable forward LIBOR rate for $[T_k, T_{k+1}]$ is

$$\overline{L}(t, T_k) := \frac{1}{\delta}\left(\frac{\overline{B}(t, T_k)}{\overline{B}(t, T_{k+1})} - 1\right)$$

- The forward LIBOR spread is

$$S(t, T_k) := \overline{L}(t, T_k) - L(t, T_k)$$

- The (discrete-tenor) forward default intensity is

$$H(t, T_k) := \frac{S(t, T_k)}{F(t, T_k, T_{k+1})} = \frac{1}{\delta}\left(\frac{\overline{B}(t, T_k)}{\overline{B}(t, T_{k+1})}\frac{B(t, T_{k+1})}{B(t, T_k)} - 1\right)$$

Aim: consistent modeling of default-free and defaultable rates

Lévy processes

- A time-inhomogeneous Lévy process $X = (X_t)_{0 \leq t \leq T_*}$
- $\mathbb{R}$-valued stochastic process with independent increments
- the law of X_t is

$$\mathbb{E}\left[\mathrm{e}^{iuX_t}\right] = \exp\left(\int_0^t \kappa_s(iu)\mathrm{d}s\right) \tag{2}$$

where

$$\kappa_s(iu) = iub_s - \frac{u^2 c_s}{2} + \int_{\mathbb{R}} (\mathrm{e}^{iux} - 1 - iux) F_s(\mathrm{d}x) \tag{3}$$

with $b_s \in \mathbb{R}$, $c_s \in \mathbb{R}_{\geqslant 0}$ and F_s are Lévy measures, $\forall s \in [0, T_*]$

- Assumptions: exponential moments, abs. continuous characteristics

Lévy processes

- X is a special semimartingale

$$X_t = \int_0^t b_s \mathrm{d}s + \int_0^t \sqrt{c_s}\mathrm{d}W_s + \int_0^t \int_{\mathbb{R}} x(\mu^X - \nu)(\mathrm{d}s, \mathrm{d}x) \tag{4}$$

- W: $\mathbb{P}$-Brownian motion
- μ^X: random measure of jumps of X
- ν: $\mathbb{P}$-compensator of μ^X
- The predictable characteristics (B, C, ν) are deterministic

$$B_t = \int_0^t b_s \mathrm{d}s, \quad C_t = \int_0^t c_s \mathrm{d}s, \quad \nu([0,t] \times A) = \int_0^t \int_A F_s(\mathrm{d}x)\mathrm{d}s$$

The Lévy LIBOR model (BGM, . . . , Eberlein & Özkan)

- Tenor: $0 < T_1 < T_2 < \cdots < T_N < T_{N+1} = T_*$, tenor length δ
- Associate forward measures $\mathbb{P}_{T_k}$ to tenor dates T_k
- Relations:

$$\frac{\mathrm{d}\mathbb{P}_{T_k}}{\mathrm{d}\mathbb{P}_{T_{k+1}}}\Big|_{\mathcal{F}_t} = \frac{F(t, T_k, T_{k+1})}{F(0, T_k, T_{k+1})} \tag{5}$$

- Model: $\mathrm{d}L = L\,\mathrm{d}X$

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- Model: $\mathrm{d}L = L\,\mathrm{d}X$
- Problem: $F = 1 + \delta L$ yields

$$\begin{aligned} \mathrm{d}F = \delta\,\mathrm{d}L = \delta L\,\mathrm{d}X = F\frac{\delta L}{1+\delta L}\,\mathrm{d}X \\ \iff F = F_0\,\mathcal{E}\left(\int \frac{\delta L}{1+\delta L}\,\mathrm{d}X\right) \end{aligned} \tag{6}$$

The Lévy LIBOR model

Backward induction construction

The dynamics of the LIBOR rate $L(\cdot, T_k)$ under $\mathbb{P}_{T_{k+1}}$ is

$$L(t, T_k) = L(0, T_k)\exp\left(\int_0^t b^L(s, T_k)\mathrm{d}s + \int_0^t \lambda(s, T_k)\mathrm{d}X_s^{T_{k+1}}\right) \tag{7}$$

where $X^{T_{k+1}}$ is a $\mathbb{P}_{T_{k+1}}$-semimartingale

$$X^{T_{k+1}} = \int_0^\cdot \sqrt{c_s}\mathrm{d}W_s^{T_{k+1}} + \int_0^\cdot \int_{\mathbb{R}} x(\mu^X - \nu^{T_{k+1}})(\mathrm{d}s, \mathrm{d}x) \tag{8}$$

and

$$b^L(s, T_k) = -\frac{1}{2}\lambda^2(s, T_k)c_s - \int_{\mathbb{R}}\left(\mathrm{e}^{\lambda(s,T_k)x} - 1 - \lambda(s, T_k)x\right)F_s^{T_{k+1}}(\mathrm{d}x).$$

The Lévy LIBOR model

The $\mathbb{P}_{T_{k+1}}$-Brownian motion is

$$W_t^{T_{k+1}} = W_t^{T_*} - \int_0^t \left(\sum_{l=k+1}^{N} \frac{\delta L(t-, T_l)}{1 + \delta L(t-, T_l)} \lambda(t, T_l) \right) \sqrt{c_s} \mathrm{d}s, \tag{9}$$

and the $\mathbb{P}_{T_{k+1}}$-compensator of μ^X is

$$\nu^{T_{k+1}}(\mathrm{d}s, \mathrm{d}x) = \prod_{l=k+1}^{N} \underbrace{\left(1 + \frac{\delta L(s-, T_l)}{1 + \delta L(s-, T_l)} \left(\mathrm{e}^{\lambda(s, T_l)x} - 1 \right) \right)}_{:=\beta(s,x,T_l)} \nu^{T_*}(\mathrm{d}s, \mathrm{d}x).$$

Problem 1: X **has state-dependent characteristics under** $\mathbb{P}_{T_{k+1}}$
Problem 2: **The product term grows exponentially fast**

The defaultable LIBOR model (Eberlein, Kluge & Schönbucher)

Aim: model default intensities (prespecification)

$$H(t, T_k) = H(0, T_k) \exp\left(\int_0^t b^H(s, T_k) \mathrm{d}s + \int_0^t \gamma(s, T_k) \mathrm{d}X_s^{T_{k+1}} \right). \quad (10)$$

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However, we cannot choose H and τ independently

- time-t price of claim paying $B^0(T_k,T_k) = 1_{\{\tau>T_k\}}$ at T_k:

$$B(t,T_k)\mathbb{E}_{\mathbb{P}_{T_k}}[1_{\{\tau>T_k\}}|\widetilde{\mathcal{F}}_t];$$

- on the other hand: $B^0(t,T_k) = 1_{\{\tau>t\}}\overline{B}(t,T_k)$.
- Hence, on $\{\tau>t\}$

$$\overline{B}(t,T_k) = B(t,T_k)\mathbb{E}_{\mathbb{P}_{T_k}}[1_{\{\tau>T_k\}}|\widetilde{\mathcal{F}}_t]$$

- while

$$H(t,T_k) = \frac{1}{\delta}\left(\frac{\overline{B}(t,T_k)}{B(t,T_k)}\frac{B(t,T_{k+1})}{\overline{B}(t,T_{k+1})} - 1\right).$$

The defaultable LIBOR model

- Canonical construction of τ such that $H(\cdot, T_k)$ meets (10)
- X remains a Lévy process on the extended space
- Arbitrage-freeness determines the drift $b^H(\cdot, T_k)$ ...
- ... which leads to an SDE $\quad (G := \log H)$

$$G(t, T_k) = G(0, T_k) + \int_0^t g(\omega; s, G(s-, T_k))\mathrm{d}s + \int_0^t \gamma(s, T_k)\mathrm{d}X_s^{T_{k+1}}$$

- g: locally Lipschitz + growth condition $\Rightarrow$ E & U of solution
- The SDE must be solved numerically!

The defaultable LIBOR model

Forward induction construction

The dynamics of default intensities $H(\cdot, T_k)$ under $\mathbb{P}_{T_{k+1}}$ is

$$H(t, T_k) = H(0, T_k) \exp\Big(\int_0^t b^H(s, T_k)\mathrm{d}s + \int_0^t \gamma(s, T_k)\mathrm{d}X_s^{T_{k+1}} \Big), \quad (11)$$

where $X^{T_{k+1}}$ is a $\mathbb{P}_{T_{k+1}}$-semimartingale

$$X^{T_{k+1}} = \int_0^\cdot \sqrt{c_s}\mathrm{d}W_s^{T_{k+1}} + \int_0^\cdot \int_{\mathbb{R}} x(\mu^X - \nu^{T_{k+1}})(\mathrm{d}s, \mathrm{d}x) \quad (12)$$

and

$$b^H(s, T_k) = \text{function of } \Big(\sum_{l=1}^{k} \frac{\delta H(t-, T_l)}{1+\delta H(t-, T_l)}, \sum_{i=k}^{N} \frac{\delta L(t-, T_i)}{1+\delta L(t-, T_i)} \Big)$$

Complexity of the problem

						$H(t, T_N)$
					$H(t, T_{N-1})$	$H(t, T_{N-1})$
			$H(t, T_k)$	$\dots$	$\dots$	$\dots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$H(t, T_2)$	$\dots$	$H(t, T_2)$	$\dots$	$H(t, T_2)$	$H(t, T_2)$
$H(t, T_1)$	$H(t, T_1)$	$\dots$	$H(t, T_1)$	$\dots$	$H(t, T_1)$	$H(t, T_1)$
$H(t, T_1)$	$H(t, T_2)$	$\dots$	$H(t, T_k)$	$\dots$	$H(t, T_{N-1})$	$H(t, T_N)$
$L(t, T_N)$	$L(t, T_N)$	$\dots$	$L(t, T_N)$	$\dots$	$L(t, T_N)$	$L(t, T_N)$
$L(t, T_{N-1})$	$L(t, T_{N-1})$	$\dots$	$L(t, T_{N-1})$	$\dots$	$L(t, T_{N-1})$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$\dots$	$\dots$	$\dots$	$L(t, T_k)$			
$L(t, T_2)$	$L(t, T_2)$					
$L(t, T_1)$						

Table: Matrix of dependencies for defaultable LIBOR rates

- Forward–Backward dependence in rates and intensities!

What does this mean in practice?

Consequences for LIBOR models

1. caplets can be priced in closed form $\rightsquigarrow$ Black's formula
2. swaptions and multi-LIBOR products cannot be priced in closed form
3. Monte-Carlo pricing is very time consuming $\rightsquigarrow$ coupled high dimensional SDEs!

Consequences for defaultable LIBOR models

1. even "caplets" cannot be priced in closed form!
2. Monte-Carlo pricing is even more time consuming.

Log-normal approximation I (Schoenmakers et al)

Terminal measure dynamics

$$H(t,T_k) = H(0,T_k)\exp\left(\int_0^t \overline{b}(s,T_k)\mathrm{d}s + \int_0^t \gamma(s,T_k)\sqrt{c_s}\mathrm{d}W_s\right), \quad (13)$$

where

$$\overline{b}(s,T_k) = -\frac{1}{2}\gamma(s,T_k)^2 c_s - \gamma(s,T_k)c_s \sum_{l=k+1}^{N} V_s^l \lambda(s,T_l) \quad (14)$$

$$+\gamma(s,T_k)c_s\sum_{l=1}^{k} Y_s^l \gamma(s,T_l) + \lambda(s,T_k)c_s V_s^k \frac{1}{Y_s^k}\sum_{l=1}^{k-1} Y_s^l \gamma(s,T_l),$$

with

$$Y_s^l = \frac{\delta H(s,T_l)}{1+\delta H(s,T_l)} \quad \text{and} \quad V_s^l = \frac{\delta L(s,T_l)}{1+\delta L(s,T_l)}.$$

Aim: normal approximation for the drift term $\overline{b}(\cdot,T_k)$.

Log-normal approximation II

We describe how to treat the terms

$$\lambda(s, T_k) c_s V_s^k \frac{1}{Y_s^k} \sum_{l=1}^{k-1} Y_s^l \gamma(s, T_l).$$

Step 1: Introduce the process $M = \left(L^k, H^l, H^k\right) \in (0, \infty)^3$ and the function $C^2 \ni f : (0, \infty)^3 \to \mathbb{R}$ by

$$f(x) = \frac{\delta x_1}{1 + \delta x_1} \frac{\delta x_2}{1 + \delta x_2} \frac{1 + \delta x_3}{\delta x_3}. \tag{15}$$

Thus

$$f(M_s) = V_s^k Y_s^l \frac{1}{Y_s^k}. \tag{16}$$

Log-normal approximation III

Apply Itô's formula to f (and after some calculations):

$$f(M_t) = f(M_0) + \int_0^t \Gamma_s(H(s), L(s))\mathrm{d}s + \int_0^t \Delta_s(H(s), L(s))\mathrm{d}W_s, \quad (17)$$

where

$$\Gamma_s(H(s), L(s)) = \sum_{i\leq 3} \partial_i f(M_s) M_s^i A_s^i + \frac{1}{2} \sum_{i,j\leq 3} \partial_{ij} f(M_s) M_s^i M_s^j B_s^i B_s^j, \quad (18)$$

and

$$\Delta_s(H(s), L(s)) = \sum_{i\leq 3} \partial_i f(M_s) M_s^i B_s^i. \quad (19)$$

Here A^i = function of (H, L) and B^i = deterministic.

Log-normal approximation IV

Step 2: Apply Picard iteration to $f(M)$.

The first Picard iterate is

$$f^{(1)}(M_t) = f(M_0) + \int_0^t \Gamma_s(H(0), L(0))\mathrm{d}s + \int_0^t \Delta_s(H(0), L(0))\mathrm{d}W_s,$$

where

$$\Gamma_s(H(0), L(0)) = \sum_{i\leq 3} \partial_i f(M_0) M_0^i A_0^i + \frac{1}{2} \sum_{i,j\leq 3} \partial_{ij} f(M_0) M_0^i M_0^j B_s^i B_s^j,$$

and

$$\Delta_s(H(0), L(0)) = \sum_{i\leq 3} \partial_i f(M_0) M_0^i B_s^i.$$

Hence $f^{(1)}(M_t)$ is normally distributed.

Log-normal approximation V

Step 3: Approximating the terms

$$
\begin{aligned}
c\lambda_k \sum_{l=1}^{k-1} V_s^k Y_s^l \frac{1}{Y_s^k}\gamma_l &= c\lambda_k \sum_{l=1}^{k-1} f_l(M_s)\gamma_l \\
&\approx c\lambda_k \sum_{l=1}^{k-1} f_l^{(1)}(M_s)\gamma_l =: \sum_{l=1}^{k-1} f_l^{(1)}(M_s)\xi_l, \qquad (20)
\end{aligned}
$$

and applying integration by parts

$$
\begin{aligned}
\int_0^t \xi_l(s) f_l^{(1)}(M_s)\mathrm{d}s &= \int_0^t f_l^{(1)}(M_s)\mathrm{d}\Xi_l(s) \\
&= f_l(M_0)\,\Xi_l(t) + \int_0^t \big(\Xi_l(t) - \Xi_l(s)\big)\Gamma_s(H(0), L(0))\mathrm{d}s \\
&\quad + \int_0^t \big(\Xi_l(t) - \Xi_l(s)\big)\Delta_s(H(0), L(0))\mathrm{d}W_s. \qquad (21)
\end{aligned}
$$

Log-normal approximation VI

Step 4: Finally, we arrive at the log-normal approximation ($G = \log H$)

$$
\begin{aligned}
G(t, T_k) \approx{} & G(0, T_k) \\
& - \int_0^t \Bigg(\frac{1}{2} c_s \gamma^2(s, T_k) + \sum_{l=k+1}^{N} \Xi_l \cdot \mathrm{E}_s^0(L^l; H, L) \\
& \qquad - \sum_{l=1}^{k} \overline{\Xi}_l \cdot \mathrm{E}_s^0(H^l; H, L) - \sum_{l=1}^{k-1} \Xi_l \cdot \Gamma_s^0(H, L) \Bigg) \, \mathrm{d}s \\
& + \int_0^t \Bigg(\sqrt{c_s} \gamma(s, T_k) + \sum_{l=k+1}^{N} \Xi_l \cdot \mathrm{Z}_s^0(L^l; H, L) \\
& \qquad - \sum_{l=1}^{k} \overline{\Xi}_l \cdot \mathrm{E}_s^0(H^l; H, L) - \sum_{l=1}^{k-1} \Xi_l \cdot \Delta_s^0(H, L) \Bigg) \, \mathrm{d}W_s.
\end{aligned}
\tag{22}
$$

Data for the example

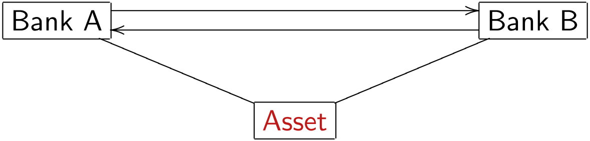
1. Tenor structure: 10 years, semi-annual ($N = 20$)
2. flat volatilities: $\lambda(\cdot, T_i) = 25\%$, $\gamma(\cdot, T_i) = 8\%$
3. flat term structure of interest rates:

$$\overline{B}(0, T_i) = \exp(-0.04\,T_i) \quad \text{and} \quad B(0, T_i) = \exp(-0.02\,T_i)$$

4. Brownian motion
5. Credit derivatives:
   - Reference risk: defaultable FRA, CDS
   - Counterparty risk: vulnerable call

Defaultable FRA

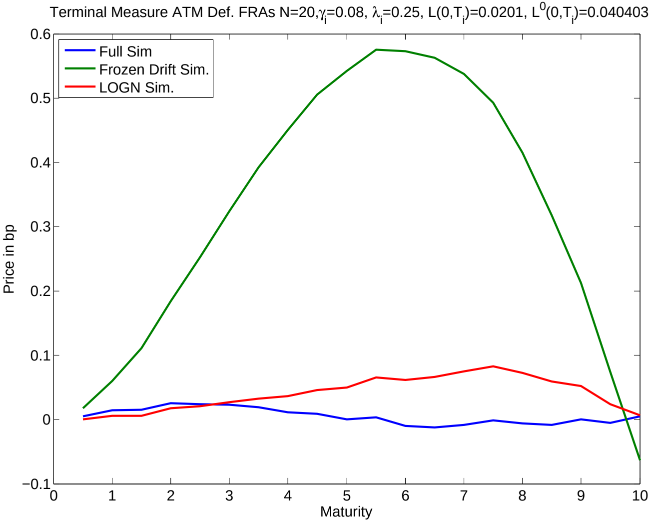
Reference risk



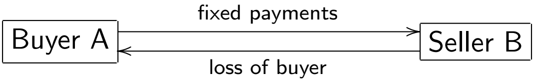
Defaultable FRA

- Payoff: $\overline{L}(T_k, T_k) - K$
- Model-free price: $\overline{L}(\cdot, T_k) \in \mathcal{M}(\overline{\mathbb{P}}_{T_{k+1}})$

D-FRA: numerical results



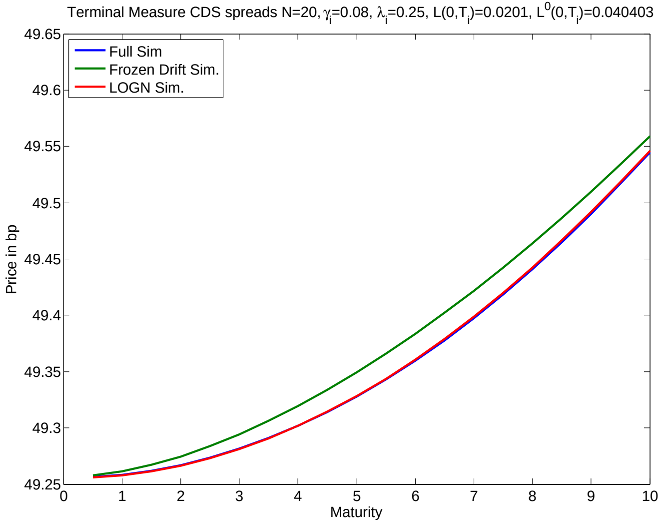
Credit default swap (CDS)



- Credit event: default of fixed coupon bond C
- In case of default: A receives $1 - \pi(1 + c)$ from B
- Time-0 value of payments: $\mathcal{S} \sum_{k=1}^{n} \overline{B}(0, T_{k-1})$
- CDS rate

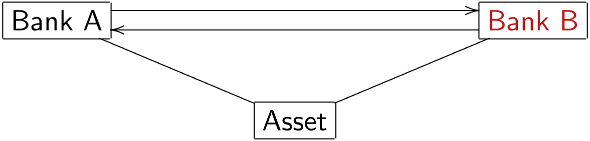
$$\mathcal{S} = \frac{1 - \pi(1 + c)}{\sum_{k=1}^{n} \overline{B}(0, T_{k-1})} \sum_{k=1}^{n} \left(\overline{B}(0, T_k) \delta \mathbb{E}_{\overline{\mathbb{P}}_{T_k}} [H(T_{k-1}, T_{k-1})] \right).$$

CDS: numerical results



Vulnerable call option

Counterparty risk



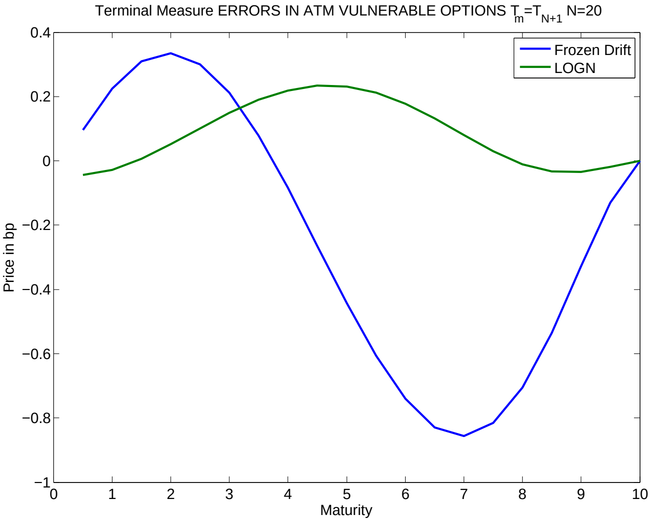
Vulnerable call

Call option written by default-prone issuer:

$$C_{T_k} 1_{\{\tau > T_k\}} + q C_{T_k} 1_{\{\tau \leq T_k\}}, \tag{23}$$

where $C_{T_k} = (B(T_k, T_m) - K)^+$, $q =$ recovery rate

V-Call: numerical results



Summary and Outlook

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   - closed, or semi-closed, form solutions
   - applicable for many driving processes
   - empirical evidence supportive – more work needed

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2. - Z. Grbac, A. Papapantoleon, D. Skovmand
     Valuation of derivatives with reference and counterparty default risk in LIBOR models.
     In preparation, 2011

Thank you for your attention!