Valuation of credit derivatives in LIBOR models

Antonis Papapantoleon

QP Lab & TU Berlin

Joint work with Zorana Grbac and David Skovmand

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Outline

1. Introduction
   - Markets
   - LIBOR rates

2. LIBOR models
   - The driving process
   - Default-free LIBOR model
   - Defaultable LIBOR model

3. Approximations methods
   - Log-normal approximation
   - Numerical example

4. Summary and Outlook
## Market size

According to the Bank for International Settlements:

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<tr>
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<tbody>
<tr>
<td>Foreign exchange</td>
<td>38,127</td>
<td>48,645</td>
<td>62,983</td>
<td>48,775</td>
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<tr>
<td>Interest rate</td>
<td>262,526</td>
<td>347,312</td>
<td>458,304</td>
<td>437,198</td>
</tr>
<tr>
<td>Equity-linked</td>
<td>6,782</td>
<td>8,590</td>
<td>10,177</td>
<td>6,619</td>
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<tr>
<td>Commodity</td>
<td>6,394</td>
<td>7,567</td>
<td>13,229</td>
<td>3,729</td>
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<td>Credit default swaps</td>
<td>20,352</td>
<td>42,581</td>
<td>57,403</td>
<td>36,046</td>
</tr>
<tr>
<td>Unallocated</td>
<td>35,997</td>
<td>61,713</td>
<td>81,719</td>
<td>72,255</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>370,178</strong></td>
<td><strong>516,408</strong></td>
<td><strong>683,815</strong></td>
<td><strong>604,622</strong></td>
</tr>
</tbody>
</table>

**Table:** Notional amounts outstanding for OTC derivatives in billions of US$
Interest rates

Figure: Term structure of interest rates, Feb 17 2004
Credit spreads

Figure: Term structure of Euro corporate spreads, Dec 20 2002
Evolution of interest rates

Figure: Evolution of interest rate term structure, 2003–2004
Default-free rates

- Tenor: $0 < T_1 < T_2 < \cdots < T_N < T_{N+1} = T_*$, tenor length $\delta$
- $B(t, T)$: value of a zero coupon bond for $T$, $B(t, T) = 1$
- $L(t, T)$: forward LIBOR rate for $[T, T + \delta]$

\[
L(t, T) = \frac{1}{\delta} \left( \frac{B(t, T)}{B(t, T + \delta)} - 1 \right)
\]

- $F(t, T, U)$: forward price for $T$ and $U$; $F(t, T, U) = \frac{B(t, T)}{B(t, U)}$

“Master” relation

\[
F(t, T, T + \delta) = \frac{B(t, T)}{B(t, T + \delta)} = 1 + \delta L(t, T)
\]
Defaultable bonds

- $B^0(t, T)$: value of a defaultable zero coupon bond with zero recovery and maturity $T$
- $\tau$: time of default
- $\overline{B}(t, T)$: pre-default value of the defaultable bond

\[
B^0(t, T) = 1_{\{\tau > t\}} \overline{B}(t, T), \quad \overline{B}(T, T) = 1
\]
Defaultable rates and spreads

- The defaultable forward LIBOR rate for $[T_k, T_{k+1}]$ is

  $$
  \overline{L}(t, T_k) := \frac{1}{\delta} \left( \frac{\overline{B}(t, T_k)}{\overline{B}(t, T_{k+1})} - 1 \right)
  $$

- The forward LIBOR spread is

  $$
  S(t, T_k) := \overline{L}(t, T_k) - \overline{L}(t, T_k)
  $$

- The (discrete-tenor) forward default intensity is

  $$
  H(t, T_k) := \frac{S(t, T_k)}{F(t, T_k, T_{k+1})} = \frac{1}{\delta} \left( \frac{\overline{B}(t, T_k)}{\overline{B}(t, T_{k+1})} \frac{B(t, T_{k+1})}{B(t, T_k)} - 1 \right)
  $$

**Aim:** consistent modeling of default-free and defaultable rates
A time-inhomogeneous Lévy process $X = (X_t)_{0 \leq t \leq T_*}$

- $\mathbb{R}$-valued stochastic process with independent increments
- the law of $X_t$ is

$$\mathbb{E}[e^{iuX_t}] = \exp \left( \int_0^t \kappa_s(iu) \, ds \right)$$

(2)

where

$$\kappa_s(iu) = iub_s - \frac{u^2 c_s}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux) F_s(dx)$$

(3)

with $b_s \in \mathbb{R}$, $c_s \in \mathbb{R}_{\geq 0}$ and $F_s$ are Lévy measures, $\forall s \in [0, T_*]$

- Assumptions: exponential moments, abs. continuous characteristics
Lévy processes

- $X$ is a special semimartingale

$$X_t = \int_0^t b_s ds + \int_0^t \sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}} x(\mu^X - \nu)(ds, dx) \quad (4)$$

- $W$: $\mathbb{P}$-Brownian motion
- $\mu^X$: random measure of jumps of $X$
- $\nu$: $\mathbb{P}$-compensator of $\mu^X$
- The predictable characteristics $(B, C, \nu)$ are deterministic

$$B_t = \int_0^t b_s ds, \quad C_t = \int_0^t c_s ds, \quad \nu([0, t] \times A) = \int_0^t \int_A F_s(dx) ds$$
The Lévy LIBOR model  (BGM, . . . , Eberlein & Özkan)

- Tenor: $0 < T_1 < T_2 < \cdots < T_N < T_{N+1} = T_\ast$, tenor length $\delta$
- Associate forward measures $\mathbb{P}_{T_k}$ to tenor dates $T_k$
- Relations:

\[
\frac{d\mathbb{P}_{T_k}}{d\mathbb{P}_{T_{k+1}}} |_{\mathcal{F}_t} = \frac{F(t, T_k, T_{k+1})}{F(0, T_k, T_{k+1})} \tag{5}
\]

- Model: $dL = L \, dX$
The Lévy LIBOR model (BGM, . . . , Eberlein & Özkan)

- Tenor: \( 0 < T_1 < T_2 < \cdots < T_N < T_{N+1} = T_\ast \), tenor length \( \delta \)
- Associate forward measures \( \mathbb{P}_{T_k} \) to tenor dates \( T_k \)
- Relations:

\[
\frac{d\mathbb{P}_{T_k}}{d\mathbb{P}_{T_{k+1}}} \bigg|_{\mathcal{F}_t} = \frac{F(t, T_k, T_{k+1})}{F(0, T_k, T_{k+1})}
\]  

(5)

- Model: \( dL = L \, dX \)
- Problem: \( F = 1 + \delta L \) yields

\[
dF = \delta \, dL = \delta L \, dX = F \frac{\delta L}{1 + \delta L} \, dX
\]

\[\iff \ \ F = F_0 \, \mathcal{E} \left( \int \frac{\delta L}{1 + \delta L} \, dX \right) \]  

(6)
The Lévy LIBOR model

Backward induction construction

The dynamics of the LIBOR rate $L(\cdot, T_k)$ under $\mathbb{P}_{T_{k+1}}$ is

$$L(t, T_k) = L(0, T_k) \exp \left( \int_0^t b^L(s, T_k) ds + \int_0^t \lambda(s, T_k) dX_s^{T_{k+1}} \right) \quad (7)$$

where $X^{T_{k+1}}$ is a $\mathbb{P}_{T_{k+1}}$-semimartingale

$$X^{T_{k+1}} = \int_0^t \sqrt{c_s} dW_s^{T_{k+1}} + \int_0^t \int_\mathbb{R} x(\mu^X - \nu^{T_{k+1}})(ds, dx) \quad (8)$$

and

$$b^L(s, T_k) = -\frac{1}{2} \lambda^2(s, T_k)c_s - \int_\mathbb{R} \left( e^{\lambda(s, T_k)x} - 1 - \lambda(s, T_k)x \right) F_s^{T_{k+1}}(dx).$$
The Lévy LIBOR model

The $\mathbb{P}_{T_{k+1}}$-Brownian motion is

$$W_{t}^{T_{k+1}} = W_{t}^{T_{*}} - \int_{0}^{t} \left( \sum_{l=k+1}^{N} \frac{\delta L(t-, T_{l})}{1 + \delta L(t-, T_{l})} \lambda(t, T_{l}) \right) \sqrt{c_{s}} ds,$$  \hspace{1cm} (9)

and the $\mathbb{P}_{T_{k+1}}$-compensator of $\mu^{X}$ is

$$\nu^{T_{k+1}}(ds, dx) = \prod_{l=k+1}^{N} \left( 1 + \frac{\delta L(s-, T_{l})}{1 + \delta L(s-, T_{l})} \left( e^{\lambda(s, T_{l})x} - 1 \right) \right) \nu^{T_{*}}(ds, dx).$$

$$:= \beta(s, x, T_{l})$$

**Problem 1:** $X$ has state-dependent characteristics under $\mathbb{P}_{T_{k+1}}$

**Problem 2:** The product term grows exponentially fast
The defaultable LIBOR model  

(Eberlein, Kluge & Schönbucher)

**Aim:** model default intensities *(prespecification)*

\[
H(t, T_k) = H(0, T_k) \exp \left( \int_0^t b^H(s, T_k)ds + \int_0^t \gamma(s, T_k)dX_s^{T_{k+1}} \right). \tag{10}
\]
The defaultable LIBOR model (Eberlein, Kluge & Schönbucher)

**Aim:** model default intensities (prespecification)

\[ H(t, T_k) = H(0, T_k) \exp \left( \int_0^t b^H(s, T_k) ds + \int_0^t \gamma(s, T_k) dX_{s,T_k+1} \right). \] (10)

However, we cannot choose \( H \) and \( \tau \) independently

- time-\( t \) price of claim paying \( B^0(T_k, T_k) = 1_{\{\tau > T_k\}} \) at \( T_k \):
  \[ B(t, T_k) \mathbb{E}_{\mathbb{P}|T_k} [1_{\{\tau > T_k\}} | \tilde{F}_t]; \]

- on the other hand: \( B^0(t, T_k) = 1_{\{\tau > t\}} \overline{B}(t, T_k). \)

- Hence, on \( \{\tau > t\} \)
  \[ \overline{B}(t, T_k) = B(t, T_k) \mathbb{E}_{\mathbb{P}|T_k} [1_{\{\tau > T_k\}} | \tilde{F}_t] \]

- while
  \[ H(t, T_k) = \frac{1}{\delta} \left( \frac{\overline{B}(t, T_k) B(t, T_{k+1})}{B(t, T_k) \overline{B}(t, T_{k+1})} - 1 \right). \]
The defaultable LIBOR model

- Canonical construction of $\tau$ such that $H(\cdot, T_k)$ meets (10)
- $X$ remains a Lévy process on the extended space
- Arbitrage-freeness determines the drift $b^H(\cdot, T_k)$ . . .
- . . . which leads to an SDE $(G := \log H)$

$$G(t, T_k) = G(0, T_k) + \int_0^t g(\omega; s, G(s, T_k))ds + \int_0^t \gamma(s, T_k)dX_s^{T_{k+1}}$$

- $g$: locally Lipschitz + growth condition $\Rightarrow$ E & U of solution
- The SDE must be solved numerically!
The defaultable LIBOR model

Forward induction construction

The dynamics of default intensities \( H(\cdot, T_k) \) under \( \mathbb{P}_{T_{k+1}} \) is

\[
H(t, T_k) = H(0, T_k) \exp \left( \int_0^t b^H(s, T_k) ds + \int_0^t \gamma(s, T_k) dX^{T_{k+1}}_s \right), \tag{11}
\]

where \( X^{T_{k+1}} \) is a \( \mathbb{P}_{T_{k+1}} \)-semimartingale

\[
X^{T_{k+1}} = \int_0^t \sqrt{c_s} dW^{T_{k+1}}_s + \int_0^t \int_{\mathbb{R}} x(\mu^X - \nu^{T_{k+1}})(ds, dx) \tag{12}
\]

and

\[
b^H(s, T_k) = \text{function of} \left( \sum_{l=1}^k \frac{\delta H(t-, T_l)}{1 + \delta H(t-, T_l)}, \sum_{i=k}^N \frac{\delta L(t-, T_i)}{1 + \delta L(t-, T_i)} \right)
\]
Complexity of the problem

<table>
<thead>
<tr>
<th></th>
<th>$H(t,T_1)$</th>
<th>$H(t,T_2)$</th>
<th>$H(t,T_k)$</th>
<th>$H(t,T_{N-1})$</th>
<th>$H(t,T_N)$</th>
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<td>$H(t,T_2)$</td>
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<td>$L(t,T_N)$</td>
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<td>$L(t,T_{N-1})$</td>
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<td>$L(t,T_{N-1})$</td>
<td>$L(t,T_{N-1})$</td>
</tr>
<tr>
<td>$L(t,T_2)$</td>
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<td>...</td>
<td>$L(t,T_k)$</td>
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</tr>
<tr>
<td>$L(t,T_1)$</td>
<td>$L(t,T_2)$</td>
<td>...</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

**Table:** Matrix of dependencies for defaultable LIBOR rates

- **Forward–Backward dependence** in rates and intensities!
What does this mean in practice?

Consequences for LIBOR models

1. Caplets can be priced in closed form \(\leadsto\) Black’s formula
2. Swaptions and multi-LIBOR products cannot be priced in closed form
3. Monte-Carlo pricing is very time consuming \(\leadsto\) coupled high dimensional SDEs!

Consequences for defaultable LIBOR models

1. Even “caplets” cannot be priced in closed form!
2. Monte-Carlo pricing is even more time consuming.
Log-normal approximation I (Schoenmakers et al)

Terminal measure dynamics

\[ H(t, T_k) = H(0, T_k) \exp \left( \int_0^t \bar{b}(s, T_k) \, ds + \int_0^t \gamma(s, T_k) \sqrt{c_s} \, dW_s \right), \quad (13) \]

where

\[ \bar{b}(s, T_k) = -\frac{1}{2} \gamma(s, T_k)^2 c_s - \gamma(s, T_k) c_s \sum_{l=k+1}^{N} V_s^l \lambda(s, T_l) \quad (14) \]

\[ + \gamma(s, T_k) c_s \sum_{l=1}^{k} Y_s^l \gamma(s, T_l) + \lambda(s, T_k) c_s \sum_{l=1}^{k-1} \frac{1}{Y_s^k} \sum_{l=1}^{k} Y_s^l \gamma(s, T_l), \]

with

\[ Y_s^l = \frac{\delta H(s, T_l)}{1 + \delta H(s, T_l)} \quad \text{and} \quad V_s^l = \frac{\delta L(s, T_l)}{1 + \delta L(s, T_l)}. \]

**Aim:** normal approximation for the drift term \( \bar{b}(\cdot, T_k) \).
Log-normal approximation II

We describe how to treat the terms

$$\lambda(s, T_k) c_s V^k_s \frac{1}{Y^k_s} \sum_{l=1}^{k-1} Y^l_s \gamma(s, T_l).$$

**Step 1:** Introduce the process $M = (L^k, H^l, H^k) \in (0, \infty)^3$ and the function $C^2 \ni f : (0, \infty)^3 \to \mathbb{R}$ by

$$f(x) = \frac{\delta x_1}{1 + \delta x_1} \frac{\delta x_2}{1 + \delta x_2} \frac{1 + \delta x_3}{\delta x_3}.$$  \hspace{1cm} (15)

Thus

$$f(M_s) = V^k_s Y^l_s \frac{1}{Y^k_s}.$$ \hspace{1cm} (16)
Approximations methods
Log-normal approximation

Log-normal approximation III

Apply Itô’s formula to \( f \) (and after some calculations):

\[
 f(M_t) = f(M_0) + \int_0^t \Gamma_s(H(s), L(s))ds + \int_0^t \Delta_s(H(s), L(s))dW_s, \tag{17}
\]

where

\[
 \Gamma_s(H(s), L(s)) = \sum_{i \leq 3} \partial_i f(M_s) M_s A^i_s + \frac{1}{2} \sum_{i,j \leq 3} \partial_{ij} f(M_s) M_s^i M_s^j B^i_s B^j_s, \tag{18}
\]

and

\[
 \Delta_s(H(s), L(s)) = \sum_{i \leq 3} \partial_i f(M_s) M_s B^i_s. \tag{19}
\]

Here \( A^i = \) function of \( (H, L) \) and \( B^i = \) deterministic.
Log-normal approximation IV

**Step 2:** Apply Picard iteration to \( f(M) \).

The first Picard iterate is

\[
f^{(1)}(M_t) = f(M_0) + \int_0^t \Gamma_s(H(0), L(0))ds + \int_0^t \Delta_s(H(0), L(0))dW_s,
\]

where

\[
\Gamma_s(H(0), L(0)) = \sum_{i \leq 3} \partial_i f(M_0) M_0^i A_0^i + \frac{1}{2} \sum_{i,j \leq 3} \partial_{ij} f(M_0) M_0^i M_0^j B_s^i B_s^j,
\]

and

\[
\Delta_s(H(0), L(0)) = \sum_{i \leq 3} \partial_i f(M_0) M_0^i B_s^i.
\]

Hence \( f^{(1)}(M_t) \) is normally distributed.
Log-normal approximation V

**Step 3:** Approximating the terms

\[
c\lambda_k \sum_{l=1}^{k-1} V_s^k Y_s^l \frac{1}{Y_s^k} \gamma_l = c\lambda_k \sum_{l=1}^{k-1} f_l(M_s)\gamma_l
\]

\[
\approx c\lambda_k \sum_{l=1}^{k-1} f_l^{(1)}(M_s)\gamma_l =: \sum_{l=1}^{k-1} f_l^{(1)}(M_s)\xi_l, \quad (20)
\]

and applying integration by parts

\[
\int_0^t \xi_l(s)f_l^{(1)}(M_s)ds = \int_0^t f_l^{(1)}(M_s)d\Xi_l(s)
\]

\[
= f_l(M_0)\Xi_l(t) + \int_0^t (\Xi_l(t) - \Xi_l(s))\Gamma_s(H(0), L(0))ds
\]

\[
+ \int_0^t (\Xi_l(t) - \Xi_l(s))\Delta_s(H(0), L(0))dW_s. \quad (21)
\]
Step 4: Finally, we arrive at the log-normal approximation \((G = \log H)\)

\[
G(t, T_k) \approx G(0, T_k)
\]

\[
- \int_0^t \left( \frac{1}{2} c_s \gamma^2(s, T_k) + \sum_{l=k+1}^{N} \Xi_l \cdot E_s^0(L^l; H, L) 
- \sum_{l=1}^{k} \Xi_l \cdot E_s^0(H^l; H, L) 
- \sum_{l=1}^{k-1} \Xi_l \cdot \Gamma_s^0(H, L) \right) ds
\]

\[
+ \int_0^t \left( \sqrt{c_s} \gamma(s, T_k) + \sum_{l=k+1}^{N} \Xi_l \cdot Z_s^0(L^l; H, L) 
- \sum_{l=1}^{k} \Xi_l \cdot E_s^0(H^l; H, L) 
- \sum_{l=1}^{k-1} \Xi_l \cdot \Delta_s^0(H, L) \right) dW_s.
\]
Data for the example

1. Tenor structure: 10 years, semi-annual ($N = 20$)
2. Flat volatilities: $\lambda(\cdot, T_i) = 25\%$, $\gamma(\cdot, T_i) = 8\%$
3. Flat term structure of interest rates:
   \[ \bar{B}(0, T_i) = \exp(-0.04 T_i) \quad \text{and} \quad B(0, T_i) = \exp(-0.02 T_i) \]
4. Brownian motion
5. Credit derivatives:
   - Reference risk: defaultable FRA, CDS
   - Counterparty risk: vulnerable call
Defaultable FRA

Reference risk

Bank A  Asset  Bank B

Defaultable FRA

- Payoff: \( \overline{L}(T_k, T_k) - K \)
- Model-free price: \( \overline{L}(\cdot, T_k) \in \mathcal{M}(\mathbb{P}_{T_{k+1}}) \)
D-FRA: numerical results

Terminal Measure ATM Def. FRAs $N=20, \gamma_i=0.08, \lambda_i=0.25, L(0,T_i)=0.0201, L^0(0,T_i)=0.040403$

- Full Sim
- Frozen Drift Sim.
- LOGN Sim.
Credit default swap (CDS)

- Credit event: default of fixed coupon bond C
- In case of default: A receives $1 - \pi (1 + c)$ from B
- Time-0 value of payments: $S \sum_{k=1}^{n} \overline{B}(0, T_{k-1})$
- CDS rate

$$S = \frac{1 - \pi (1 + c)}{\sum_{k=1}^{n} \overline{B}(0, T_{k-1})} \sum_{k=1}^{n} \left( \overline{B}(0, T_{k}) \delta \mathbb{E}_{\overline{F}_{T_{k}}} [H(T_{k-1}, T_{k-1})] \right).$$
CDS: numerical results

Terminal Measure CDS spreads $N=20$, $\gamma_i=0.08$, $\lambda_i=0.25$, $L(0,T_i)=0.0201$, $L^0(0,T_i)=0.040403$
Vulnerable call option

Call option written by default-prone issuer:

\[ C_{T_k} 1_{\{\tau > T_k\}} + q C_{T_k} 1_{\{\tau \leq T_k\}} , \quad (23) \]

where \( C_{T_k} = (B(T_k, T_m) - K)^+ \), \( q = \) recovery rate
V-Call: numerical results
We have presented...

- approximation methods for LIBOR models
- closed, or semi-closed, form solutions
- applicable for many driving processes
- empirical evidence supportive – more work needed
Summary and Outlook

1. We have presented . . .
   - approximation methods for LIBOR models
   - closed, or semi-closed, form solutions
   - applicable for many driving processes
   - empirical evidence supportive – more work needed

2. Z. Grbac, A. Papapantoleon, D. Skovmand
   *Valuation of derivatives with reference and counterparty default risk in LIBOR models.*
   In preparation, 2011

Thank you for your attention!