Market Impact and Manipulation Strategies

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Modeling and managing financial risks

Introduction

It is well known that large orders executed on the market can notably modify asset prices. Also, executing large orders is more expensive and it is in general judicious to split the order into smaller ones. This raises the following problem :
Given a deadline $T > 0$, what is the optimal execution strategy to own X_0 shares at time T ?

Different models have been proposed in the literature to tackle this problem : Bertsimas and Lo (1998), Almgren and Chriss (1999), Obizhaeva and Wang (2005) to mention a few.

Agenda

- In this talk, we will present a market impact model derived from a simple Limit Order Book model in which we will solve the optimal execution problem.
- This problem is related to the absence of price manipulation strategies, which we will discuss.
- Last, we will investigate the same problem under a non-Markovian model that raises new questions on market viability.

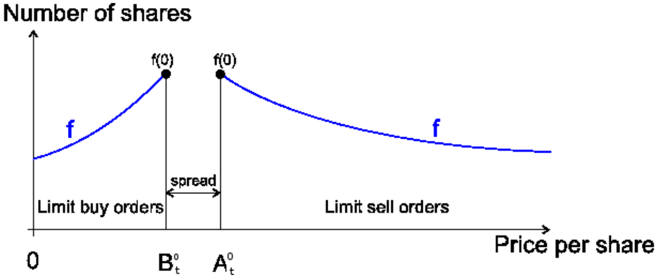
A simple Limit Order Book model

- We assume that there is one large trader that aim to buy X_0 shares.
- When the large trader is not active, we assume that the ask (resp. bid) price is given by $(A_t^0, t \geq 0)$ (resp. $(B_t^0, t \geq 0)$).
- We assume that $(A_t^0, t \geq 0)$ is a martingale and that $\forall t \geq 0, B_t^0 \leq A_t^0 a.s.$ (mg assumption on B_t^0 for a sell order).
- The LOB is model as follows : the number of sell orders between prices $A_t^0 + x$ and $A_t^0 + x + dx$ $(x \geq 0)$ is given by :

$$f(x)dx,$$

  and the number of buy orders between $B_t^0 + x$ and $B_t^0 + x + dx$ $(x < 0)$ is also $f(x)dx$. The function $f : \mathbb{R} \to \mathbb{R}_+^*$ is called the *shape function* of the LOB and is assumed to be continuous.

The LOB at time t without any trade from the large trader :



Model for large buy/sell order

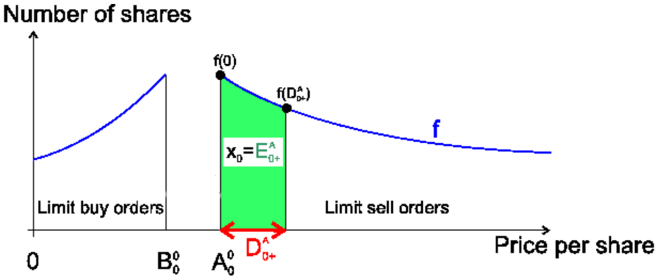
We will denote by $D_t^A \geq 0$ (resp. $D_t^B \leq 0$) the extra-shift on the ask (resp. bid) price by the large trader at time t. We assume that $D_0^A = D_0^B = 0$, and we set

$$A_t = A_t^0 + D_t^A \text{ (resp. } B_t = B_t^0 + D_t^B.\text{)}$$

This means that all the shares in the LOB between prices A_t^0 and A_t (resp. B_t and B_t^0) have been consumed by previous trades.
When the large trader buys $x_t > 0$ shares at time t, he will consume the cheapest one between A_t and A_{t+} where $\int_{D_t^A}^{D_{t+}^A} f(x)dx = x_t$: the ask price is shifted from $A_t = A_t^0 + D_t^A$ to $A_{t+} = A_t^0 + D_{t+}^A$ and the transaction cost is

$$\int_{D_t^A}^{D_{t+}^A} (x + A_t^0) f(x)dx = A_t^0 x_t + \int_{D_t^A}^{D_{t+}^A} x f(x)dx.$$

Similarly, a sell order of $-x_t > 0$ shares moves the bid price from B_t to B_{t+} where $\int_{D_t^B}^{D_{t+}^B} f(x)dx = x_t$.



The LOB just after a buy order of size $x_0 = \int_0^{D_{0+}^A} f(x)dx$ at time 0.

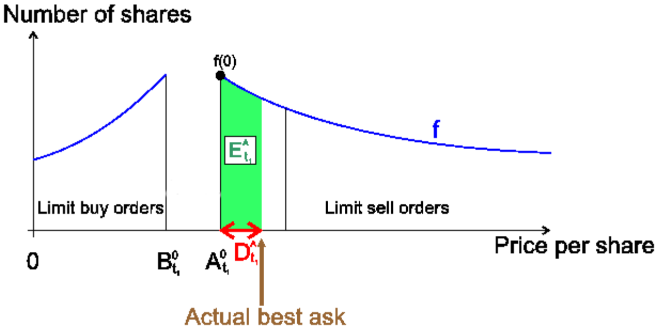
LOB dynamics without large trade

Now, we specify how new orders regenerate. We set $F(x) = \int_0^x f(u)du$ and denote $E_t^A = F(D_t^A)$ (resp. $E_t^B = F(D_t^B)$) the number of sell (resp. - up to the sign - buy) orders already eaten up at time t.
If the large investor is inactive on $[t, t+s[$, we assume :

- either **(Model 1) :** $E_{t+s}^A = e^{-\int_t^{t+s} \rho_u du} E_t^A$ (resp. $E_{t+s}^B = e^{-\int_t^{t+s} \rho_u du} E_t^B$).
- or **(Model 2) :** $D_{t+s}^A = e^{-\int_t^{t+s} \rho_u du} D_t^A$ (resp. $D_{t+s}^B = e^{-\int_t^{t+s} \rho_u du} D_t^B$).

$\rho_t > 0$ is assumed to be deterministic and is called the *resilience speed* of the LOB.
Rem : for $f(x) = q$ and $\rho_t = \rho$ both models coincide (Obizhaeva and Wang model).

Same example with no trade on $]0, t_1]$:



The cost minimization problem

At time t, a buy market order $x_t \geq 0$ moves D^A_t to D^A_{t+} s.t. $\int_{D^A_t}^{D^A_{t+}} f(x)\,dx = x_t$ and the cost is :

$$\pi_t(x_t) := \int_{D^A_t}^{D^A_{t+}} (A^0_t + x) f(x)\,dx = A^0_t x_t + \int_{D^A_t}^{D^A_{t+}} x f(x)\,dx.$$

Similarly, the cost of a sell order $x_t \leq 0$ is $\pi_t(x_t) := B^0_t x_t + \int_{D^B_t}^{D^B_{t+}} x f(x)\,dx$.

A admissible trading strategy is a sequence $\mathcal{T} = (\tau_0, \dots, \tau_N)$ of stopping times such that $0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_N = T$ and a sequence of adapted trades $(\xi_0, \dots, \xi_n)$ s.t. $\sum_{n=0}^N \xi_n = X_0$. The *average cost* $\mathcal{C}(\xi)$ to minimize is :

$$\boxed{\mathcal{C}(\xi) = \mathbb{E}\Big[\sum_{n=0}^{N} \pi_{\tau_n}(\xi_n)\Big].}$$

The simplified market impact model I

We introduce now a simplified model that sticks bid and ask sides.

- When the large trader is not active, we assume that the asset price is given by $(S_t^0, t \geq 0)$ which is a rightcontinuous martingale on a filtered probability space.
- We introduce a process $(E_t, t \geq 0)$ that describes the *volume impact* of the large trader and $(D_t, t \geq 0)$ that describes its *price impact*. Both processes are binded by the equation :

$$\int_0^{D_t} f(x)dx = E_t,$$

and the actual price process is defined by :

$$S_t = S_t^0 + D_t.$$

We moreover assume $E_0 = D_0 = 0$.

The simplified market impact model II

- If at time t, the large trader places an order $x_t \in \mathbb{R}^*$ (> 0 buy order, < 0 sell order), it has the cost :

$$\overline{\pi}_t(x_t) = S_t^0 x_t + \int_{D_t}^{D_{t+}} x f(x)\, dx.$$

  and we set :

$$E_{t+} = E_t + x_t.$$

- When the large trader is not active, we consider two models :
  - Model 1 with volume impact reversion : $dE_t = -\rho_t E_t dt$,
  - Model 2 with price impact reversion : $dD_t = -\rho_t D_t dt$, where ρ_t is a deterministic time-dependent function called the *resilience speed*.

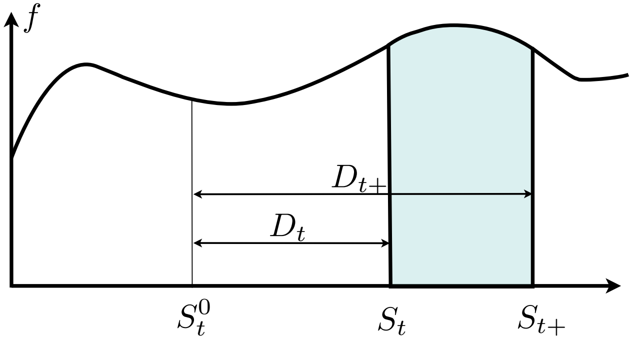


FIG.: The price impact of a buy market order of size $\xi_t > 0$ is defined by the equation $\xi_t = \int_{D_t}^{D_{t+}} f(x)\,dx = F(D_{t+}) - F(D_t)$.

Reduction to det. strategies I

For an admissible trading strategy $\mathcal{T} = (\tau_0, \dots, \tau_N)$ and $(\xi_0, \dots, \xi_n)$ s.t. $\sum_{n=0}^N \xi_n = X_0$, the average cost $\overline{\mathcal{C}}(\xi)$ to minimize is :

$$\boxed{\overline{\mathcal{C}}(\xi) = \mathbb{E}\Big[\sum_{n=0}^{N} \overline{\pi}_{\tau_n}(\xi_n)\Big].}$$

We have $\sum_{n=0}^N \overline{\pi}_{\tau_n}(\xi_n) = \sum_{n=0}^N S^0_{\tau_n}\xi_n + \sum_{n=0}^N \int_{D_{\tau_n}}^{D_{\tau_n+}} xf(x)\,dx$ and denote $X_t := X_0 - \sum_{\tau_k<t}\xi_k$ for $t \le T$ and $X_{\tau_{N+1}} := 0$ (bounded and predictable for admissible strategies).
$\sum_{n=0}^N S^0_{\tau_n}\xi_n = -\sum_{n=0}^N S^0_{\tau_n}(X_{\tau_{n+1}} - X_{\tau_n}) = X_0 S^0_0 + \sum_{n=1}^N X_{\tau_n}(S^0_{\tau_n} - S^0_{\tau_{n-1}})$,
and since in each model $i \in \{1,2\}$, there is a determinisitic function $C^{(i)}$ s.t. $\sum_{n=0}^N \int_{D_{t_n}}^{D_{t_n+}} xf(x)\,dx = C^{(i)}(\mathcal{T}, \xi_0, \dots, \xi_N)$ we get

$$\overline{\mathcal{C}}(\xi) = S^0_0 X_0 + \mathbb{E}\big[\,C^{(i)}(\xi_0, \dots, \xi_N, \mathcal{T})\,\big].$$

Reduction to det. strategies II

If there is a unique minimizer of $C^{(i)}$ in $\left\{ (x_0, \dots, x_N), (t_0, \dots, t_N) \in \mathbb{R}^{N+1} \mid \sum_{n=0}^{N} x_n = X_0,\ 0 = t_0 \leq t_1 \cdots \leq t_N = T \right\}$, the problem is solved and the optimal strategy is deterministic.

Link with the LOB model : with $A_t^0 = S_t^0$, $\mathcal{C}(\xi) \geq \overline{\mathcal{C}}(\xi)$ with $\mathcal{C}(\xi) = \overline{\mathcal{C}}(\xi)$ if $\xi_i \geq 0$ for all i. If the optimal strategy is made with only buy trades, it is also optimal for the corresponding bid/ask model.

$\implies$ Thus, we will only work in the sequel in this simplified model.

Assumptions and notations

We consider a general LOB shape function f, and assume from now that $\lim_{x\uparrow\infty} F(x) = \infty$ and $\lim_{x\downarrow -\infty} F(x) = -\infty$.
For a fixed N, we set $\alpha = \frac{1}{N}\int_0^T \rho_u du$ and consider the time grid that is regular w.r.t. the resilience :

$$\mathcal{T}^* = (t_0^*, \dots, t_N^*), \text{ where } \int_{t_{i-1}^*}^{t_i^*} \rho_u du = \alpha.$$

We also set $a^* = \exp(-\alpha)$.
We now look at the optimal strategy on the homogeneous grid $\mathcal{T}^*$.

Optimal strategy on $\mathcal{T}^*$ for model 1

Suppose $h_1(u) := F^{-1}(u) - a^* F^{-1}(a^* u)$ one-to-one. Then there exists a unique optimal strategy $\xi^{(1)} = (\xi_0^{(1)}, \dots, \xi_N^{(1)})$.
$\xi_0^{(1)}$: unique solution of the equation

$$F^{-1}\left(X_0 - N\xi_0^{(1)}(1-a^*)\right) = \frac{h_1(\xi_0^{(1)})}{1-a^*},$$

the intermediate orders are given by

$$\xi_1^{(1)} = \cdots = \xi_{N-1}^{(1)} = \xi_0^{(1)}(1-a^*),$$

the final order is determined by

$$\xi_N^{(1)} = X_0 - \xi_0^{(1)} - (N-1)\xi_0^{(1)}(1-a^*).$$

It is deterministic and s.t. $\xi_n^{(1)} > 0$ for all n.

Optimal strategy on $\mathcal{T}^*$ for model 2

Suppose $h_2(x) := x\frac{f(x)-(a^*)^2 f(a^* x)}{f(x)-a^* f(a^* x)}$ one-to-one, and $\lim_{|x|\to\infty} x^2 \inf_{y\in[a^* x,x]} f(y) = \infty$. Then there exists a unique optimal strategy $\xi^{(2)} = (\xi_0^{(2)}, \dots, \xi_N^{(2)})$.

$\xi_0^{(2)}$: unique solution of the equation

$$F^{-1}\left(X_0 - N\big[\xi_0^{(2)} - F\big(a^* F^{-1}(\xi_0^{(2)})\big)\big]\right) = h_2\big(F^{-1}(\xi_0^{(2)})\big),$$

the intermediate orders are given by

$$\xi_1^{(2)} = \cdots = \xi_{N-1}^{(2)} = \xi_0^{(2)} - F\big(a^* F^{-1}(\xi_0^{(2)})\big)$$

the final order is determined by

$$\xi_N^{(2)} = X_0 - N\xi_0^{(2)} + (N-1)F\big(a^* F^{-1}(\xi_0^{(2)})\big).$$

It is deterministic and s.t. $\xi_n^{(2)} > 0$ for all n.

Comments

- Optimal strategies have a clear interpretation in both models : the first trade shifts the ask price to the best trade-off between price and attracting new orders.
- One can show that h_1 is one-to-one if f is increasing on $\mathbb{R}_-$ and decreasing on $\mathbb{R}_+$. There is no such simple characterization for h_2.
- In the case $f(x) = q$ (block-shaped LOB), both theorems give the following optimal strategy :

$$\xi_0^* = \xi_N^* = \frac{X_0}{(N-1)(1-a^*)+2} \quad \text{and} \quad \xi_1^* = \cdots = \xi_{N-1}^* = \frac{X_0 - 2\xi_0^*}{N-1}.$$

It does not depend on q.

Example

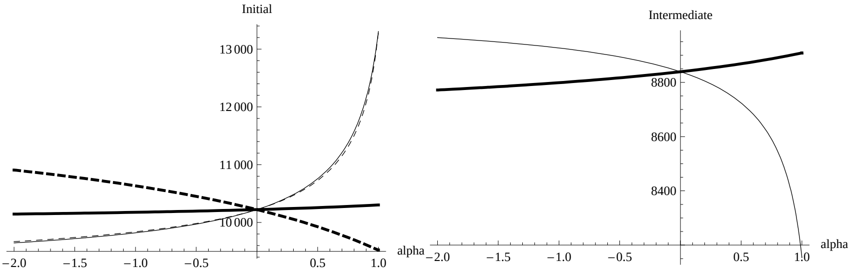


FIG.: The plots show the optimal strategies for $f(x) = q/(|x|+1)^{\alpha}$. We set $X_0 = 100,000$ and $q = 5,000$ shares, $\rho = 20$, $T = 1$ and $N = 10$. In the left figure we see $\xi_0^{(1)}, \xi_N^{(1)}$ (thick lines) and $\xi_0^{(2)}, \xi_N^{(2)}$. The figure on the right hand side shows $\xi_1^{(1)}$ (thick line) and $\xi_1^{(2)}$.

The continuous time limit (T fixed, $N \to +\infty$)

- **Model 1 :** If $F^{-1}(X_0 - \int_0^T \rho_u du x) = h_1^\infty(x) := F^{-1}(x) + \frac{x}{f(F^{-1}(x))}$ has a unique solution $\xi_0^{(1),\infty}$, the optimal strategy consists in an initial block order of $\xi_0^{(1),\infty}$ shares at time 0, continuous buying at the rate $\rho_t \xi_0^{(1),\infty}$ during $]0,T[$, and a final block order of $\xi_T^{(1),\infty} = X_0 - \xi_0^{(1),\infty}(1 + \int_0^T \rho_u du)$ shares at time T.
  This result has been recently extended by Predoiu, Shaikhet and Shreve in a model where $F(x) = \mu([0,x))$ (positive measure) and $dE_t = -h(E_t)dt$ instead of $dE_t = -\rho_t E_t dt$
- **Model 2 :** Idem with an initial trade solution of $F^{-1}(X_0 - \int_0^T \rho_u du F^{-1}(x) f(F^{-1}(x))) = h_2^\infty(F^{-1}(x))$ where $h_2^\infty(x) := x(1 + \frac{f(x)}{f(x)+xf'(x)})$, continuous buying rate $\rho_t F^{-1}(\xi_0^{(2),\infty}) f(F^{-1}(\xi_0^{(2),\infty}))$ on $]0,T[$, and a final block order $\xi_T^{(2),\infty} := X_0 - \xi_0^{(2),\infty} - \int_0^T \rho_u du F^{-1}(\xi_0^{(2),\infty}) f(F^{-1}(\xi_0^{(2),\infty}))$.

Time-grid optimization in Model 1

Assumption : In Model 1, we assume that f is nondecreasing on $\mathbb{R}_-$ and nonincreasing on $\mathbb{R}_+$ or that $f(x) = \lambda|x|^\alpha$, $\lambda, \alpha > 0$.

Proposition 1

Suppose that an admissible sequence of trading times $\mathcal{T} = (t_0, t_1, \dots, t_N)$ is given. There exists a $\mathcal{T}$-admissible trading strategy $\boldsymbol{\xi}^{(1),\mathcal{T}}$, unique (up to equivalence), that minimizes the cost among all $\mathcal{T}$-admissible trading strategies. Moreover, it consists only of nontrivial buy orders, i.e., $\xi_i^{(1),\mathcal{T}} > 0$ $\mathbb{P}$-a.s. for all i up to equivalence.

Theorem 2

There is a unique optimal strategy $(\boldsymbol{\xi}^{(1)}, \mathcal{T}^*)$ consisting of homogeneous time spacing $\mathcal{T}^*$ and the deterministic trading strategy $\boldsymbol{\xi}^{(1)}$ defined in slide 17.

Time-grid optimization in Model 2

Assumption : In Model 2, we assume that $f(x) = \lambda|x|^\alpha$, $\lambda, \alpha > 0$ or that f is $\mathcal{C}^2$ on $\mathbb{R}\backslash\{0\}$, $\nearrow$ on $\mathbb{R}_-$ and $\searrow$ on $\mathbb{R}_+$, and :

$$x \mapsto xf'(x)/f(x) \text{ is } \nearrow \text{ on } \mathbb{R}_-, \searrow \text{ on } \mathbb{R}_+, \text{ and } (-1,0]\text{-valued},$$

$$1 + x\frac{f'(x)}{f(x)} + 2x^2\left(\frac{f'(x)}{f(x)}\right)^2 - x^2\frac{f''(x)}{f(x)} \geq 0 \qquad \text{for all } x \geq 0.$$

Analogous proposition and

Theorem 3

Under the above assumption, there is a unique optimal strategy $(\boldsymbol{\xi}^{(2)}, \mathcal{T}^*)$, consisting of homogeneous time spacing $\mathcal{T}^*$ and the deterministic trading strategy $\boldsymbol{\xi}^{(2)}$ defined in slide 18.

Example : $f(x) = q/(1+\lambda|x|)^\alpha$ satisfy this condition.

Price manipulation strategies I

A *round trip* is an admissible strategy $(\overline{\boldsymbol{\xi}}, \overline{\mathcal{T}})$ such that $\sum_{i=0}^{N} \overline{\xi}_i = 0$.
A *price manipulation strategy* (Huberman and Stanzl) is a round trip $(\overline{\boldsymbol{\xi}}, \overline{\mathcal{T}})$ s.t. $\mathcal{C}(\overline{\boldsymbol{\xi}}, \overline{\mathcal{T}}) < 0$.

Corollary 4

Under the respective assumptions, any nontrivial round trip has a strictly positive average cost in Model 1 and 2. In particular, there are no price manipulation strategies.

Price manipulation strategies II

This is in contrast with the result by Gatheral :

$$S_t = S_t^0 + \int_0^t \varphi(\dot{x}_s) e^{-\rho(t-s)} ds$$

has no PMS iff φ is linear.
As a comparison, the continuous version of our Model 1 is for a constant resilience ρ :

$$S_t = S_t^0 + F^{-1}\left(\int_0^t \dot{x}_s e^{-\rho(t-s)} ds\right).$$

The model

We consider a block-shape LOB so that the price impact is proportional to the trade size.
When the strategy $\boldsymbol{\xi} = (\xi_{t_0}, \xi_{t_1}, \dots, \xi_{t_N})$ is applied, the price at time t is

$$S_t = S_t^0 + \sum_{t_n < t} \xi_{t_n} G(t - t_n), \tag{1}$$

where G is a nonincreasing function on the time axis $[0, \infty)$, the *resilience function*.
Three types of price impact :

- The *instantaneous impact* is $\xi_{t_n}(G(0) - G(0+))$, where $G(0+)$ denotes the righthand limit of G at $t = 0$.
- The *permanent impact* is $\xi_{t_n} G(\infty)$, where $G(\infty) := \lim_{t \nearrow \infty} G(t)$.
- The remaining part, $\xi_{t_n}(G(0+) - G(\infty))$, is called the *transient impact*.

The cost function

$\mathcal{C}(\boldsymbol{\xi}) := \mathbb{E}\Big[\sum_{n=0}^{N}\int_{S_{t_n}}^{S_{t_n+}} yG(0)^{-1}\,dy\Big] = \frac{1}{2G(0)}\mathbb{E}\Big[\sum_{n=0}^{N}\left(S_{t_n+}^2 - S_{t_n}^2\right)\Big].$
Since $S_{t_n+}^2 - S_{t_n}^2 = 2S_{t_n}\xi_{t_n} + \xi_{t_n}^2$, we get

$$\mathcal{C}(\boldsymbol{\xi}) = X_0 S_0 + \mathbb{E}[\,C(\boldsymbol{\xi})\,],$$

with
$C(\boldsymbol{x}) := \frac{1}{2}\sum_{i,j=0}^{N} x_i x_j G(|t_i - t_j|) = \frac{1}{2}\langle \boldsymbol{x}, M\boldsymbol{x}\rangle,\ \ \boldsymbol{x} = (x_0, \dots, x_N) \in \mathbb{R}^{N+1}.$
The function G is said *positive definite* if $C(.) \geq 0$ and is *strictly definite positive* when $C(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq 0$.
When G is **strictly definite positive**, the optimal strategy on $(t_0, \dots, t_N)$ is :

$$\boldsymbol{x}^* = \frac{X_0}{\mathbf{1}^\top M^{-1}\mathbf{1}} M^{-1}\mathbf{1}.$$

and **there is no Price manipulation strategies**.

Bochner's theorem (1932)

A continuous resilience function G is positive definite if and only if the function $x \to G(|x|)$ is the Fourier transform of a positive finite Borel measure μ on $\mathbb{R}$. If, in addition, the support of μ is not discrete, then G is strictly positive definite.

In particular, when G is convex and nonconstant, it is strictly positive definite (Caratheodory (1907), Toeplitz (1911) and Young (1913)).

Example : $G(t) = (1+t)^{-0.4}$

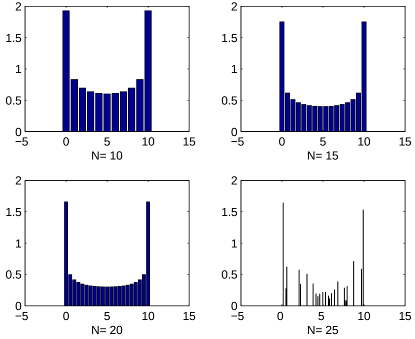


FIG.: Optimal strategies for power-law resilience $G(t) = (1+t)^{-0.4}$ and various values of N. For $N = 25$ we use randomly chosen trading times.

Example : $G(t) = e^{-t^2}$

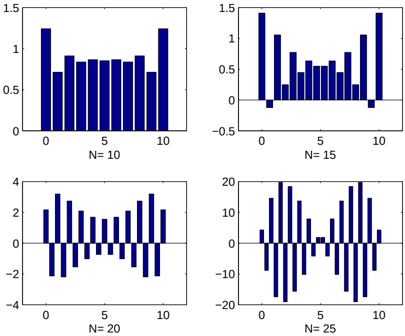


FIG.: Optimal strategies for Gaussian resilience $G(t) = e^{-t^2}$.

Example : $G(t) = 1/(1+t^2)$

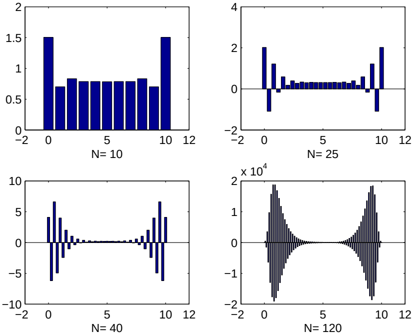


FIG.: Optimal strategies for $G(t) = 1/(1+t^2)$.

Transaction-triggered price manipulations

These examples motivate the following definition :
A market impact model admits *transaction-triggered price manipulation* if the expected execution costs of a sell (buy) program can be decreased by intermediate buy (sell) trades.

- Weaker notion of manipulation strategy :

$$\text{No TTPMS} \implies \text{No PMS}.$$

- In the previous LOB model with exponential resilience, there is no TTPMS since the optimal strategy has only positive trades.

Theorem 5

For a convex resilience function G there are no transaction-triggered price manipulation strategies. If G is even strictly convex, then all trades in an optimal execution strategy are strictly positive for a buy program and strictly negative for a sell program.

We also have the following partial converse to the preceding theorem.

Proposition 6

Suppose that

$$\text{there are } s,\ t > 0,\ s \neq t, \text{ such that} \qquad G(0) - G(s) < G(t) - G(t+s). \quad (2)$$

Then the model admits transaction-triggered price manipulation strategies.

Sum up

- We have proposed a simple LOB model with a general shape function and exponenial resilience.
- In that model, there is under general conditions a unique optimal strategy to buy X_0 shares that consists in deterministic buy trades. In particular there is no PMS.
- We have looked at a simple model with a block shape LOB and a general resilience function.
- We have introduced the notion of TTPMS and shown that convex resilience functions exclude this kind of manipulation strategies