Markovian Order Book Modelling

Stability and Scaling Limits

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Outline

Objectives

1. Analyse stability of stochastic order book models.
2. Nature of the price process at large time scales.

Definitions

- The Order Book is the list of all buy and sell *limit orders*, with their corresponding prices and volumes, at a given instant of time.

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- There are 3 types of orders:
  1. *Limit order*: Specify a price at which one is willing to buy (sell) a certain number of shares;
  2. *Market order*: Immediately buy (sell) a certain number of shares at the best available opposite quote;
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  3. *Cancellation order*: Cancel an existing limit order.
- The price dynamics is the result of the interplay between the order book and the order flow.

Notations

- The order book is represented by a finite-size vector of quantities:

$$\mathbf{X}_t := (\mathbf{a}_t; \mathbf{b}_t) := (a_t^1, \ldots, a_t^K; b_t^1, \ldots, b_t^K);$$

- $\mathbf{a}_t$: ask side of the order book;
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- $\mathbf{a}_t$: ask side of the order book;
- $\mathbf{b}_t$: bid side of the order book;
- Δp: tick size;
- τ: unit volume;
- $P = \frac{P^{\text{Ask}} + P^{\text{Bid}}}{2}$: mid-price.

Model Setup

- The events affecting the order book are described by independent Poisson processes:
  - $M_t^{\pm}$: arrival of new market order $\hookrightarrow$ arrival rate $\lambda_M^{\pm}$;
  - $L_t^{\pm i}, i \in \{1, \ldots, K\}$: arrival of a limit order i ticks away from the best opposite quote $\hookrightarrow$arrival rate $\lambda_L^{i\pm}$;
  - $C_t^{\pm i}, i \in \{1, \ldots, K\}$: cancellation of a limit order i ticks away from the best opposite quote $\hookrightarrow$ arrival rate $\lambda_C^{i+} \dfrac{|X_i^t|}{\tau}$.
- $(\mathbf{X}_t)$ is a Markov process with state space $\mathbb{X} \subset \mathbb{Z}^{2K}$.

Reference Frame and Boundary Conditions

- Finite moving reference frame of size $2K$: Ask side ranges from 1 to K ticks away from the best available opposite quote. *Idem* for bid side of the order book.

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- Constant boundary conditions outside the moving frame: Every time the moving frame leaves a price level, the number of shares at that level is set to a^∞ (or b^∞, depending on the side of the order book).

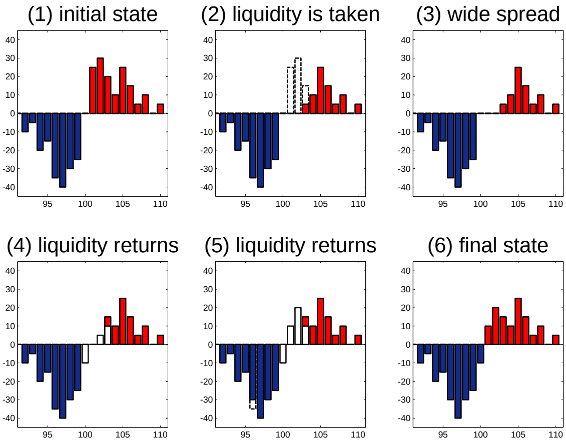

Figure 1: Illustration.

Order Book Dynamics

$$\begin{cases} da_t^i = -\left(\tau - \sum_{k=1}^{i-1} a^k\right)_+ dM_t^+ + \tau dL_t^{i+} - \tau dC_t^{i+} \\ \qquad + (J_M^-(\mathbf{a}) - \mathbf{a})_i dM_t^- + \sum_{i=1}^{K} (J_L^{i-}(\mathbf{a}) - \mathbf{a})_i dL_t^{i-} \\ \qquad + \sum_{i=1}^{K} (J_C^{i-}(\mathbf{a}) - \mathbf{a})_i dC_t^{i-}, \\ db_t^i = \text{similar expression}, \end{cases}$$

where $J_M^{\pm}$, $J_L^{i\pm}$, and $J_C^{i\pm}$ are shift operators corresponding to the effect of order arrivals on the reference frame.

Infinitesimal Generator

- The infinitesimal generator of $(\mathbf{X}_t)_{t\geq 0}$ is the operator L, defined to act on sufficiently regular functions $f : \mathbb{R}^n \to \mathbb{R}$, by

$$\mathcal{L}f(\mathbf{x}) = \lim_{t\downarrow 0} \frac{\mathbb{E}[f(\mathbf{X}_t)|\mathbf{X}_0 = \mathbf{x}] - f(\mathbf{x})}{t}.$$

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- We have:

$$\begin{aligned}
\mathcal{L}f\,(\mathbf{a};\mathbf{b}) = {} & \lambda_M^{+}\Big(f\left((a^i - (\tau - A^{i-1})_+)_+; J_M^{+}(\mathbf{b})\right) - f\Big) \\
& + \sum_{i=1}^{K} \lambda_L^{i+}(f\left(a^i + \tau; J_L^{i+}(\mathbf{b})\right) - f) \\
& + \sum_{i=1}^{K} \lambda_C^{i+}\frac{a_i}{\tau}(f\left(a^i - \tau; J_C^{i+}(\mathbf{b})\right) - f) \\
& + \text{similar terms for the events affecting the bid side.}
\end{aligned}$$

Stability of the Order Book

- A Markov Process $(\mathbf{X}_t)_{t\geq 0}$ is ergodic if an invariant probability distribution π exists and

  $$\lim_{t\to\infty} \|Q^t(\mathbf{x},.) - \pi\| = 0, \forall \mathbf{x} \in \mathbb{X},$$

  where $\|.\|$ is the *total variation norm* defined by

  $$\|\mu\| := \sup_{A\in\mathcal{B}(\mathbb{X})} |\mu(A)| - \inf_{A\in\mathcal{B}(\mathbb{X})} |\mu(A)|.$$

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Theorem 1 [CST10], [AJ11]

If $\underline{\lambda_C} = \min_{1\leq i\leq K}\{\lambda_C^{\pm i}\} > 0$, then $(\mathbf{X}_t) = (\mathbf{a}_t;\mathbf{b}_t)$ is an *ergodic* Markov process. In particular $(\mathbf{X}_t)$ has a *stationary distribution* π.

Stability of the Order Book

Proof. Let $V(\mathbf{a};\mathbf{b}) = \tau + \sum_{i=1}^{K} a^i + \sum_{i=1}^{K} |b^i|$.

$$\begin{aligned}
\mathcal{L}V(\mathbf{a};\mathbf{b}) &\le \sum_{i=1}^{K}(\lambda_L^{i+} + \lambda_L^{i-})\tau - (\lambda_M^{+} + \lambda_M^{-})\tau - \sum_{i=1}^{K}(\lambda_C^{i+} a^i + \lambda_C^{i-}|b^i|) \\
&\quad + \sum_{i=1}^{K} \lambda_L^{i-}(i_S - i)_+ a^\infty + \sum_{i=1}^{K} \lambda_L^{i+}(i_S - i)_+ |b^\infty| \\
&\le K(\overline{\lambda_L^{+}} + \overline{\lambda_L^{-}})\tau - (\lambda_M^{+} + \lambda_M^{-})\tau - \underline{\lambda_C} f(\mathbf{a};\mathbf{b}) \\
&\quad + K(K+1)(\overline{\lambda_L^{-}} a^\infty + \overline{\lambda_L^{+}} |b^\infty|) \\
\mathcal{L}V(\mathbf{x}) &\le -cV(\mathbf{x}) + d, \qquad \text{(GDC)}
\end{aligned}$$

where $\overline{\lambda_L^{\pm}} = \max_{1\le i\le K}\{\lambda_L^{i\pm}\}$ and $\underline{\lambda_C} = \min_{1\le i\le K}\{\lambda_C^{i\pm}\} > 0$.

Stability of the Order Book

Corollary 1 [AJ11]

The spread process $S_t := Q_t^A - Q_t^B = \Psi(\mathbf{X}_t)$, is ergodic—expected as (S_t) is bounded by $K + 1$.

Rate of Convergence to the Stationary State

- Actually, $(\mathbf{X}_t)$ is V-uniformly ergodic.
- Denote, as above, $||.||$ the total-variation norm:

$$||\mu|| := \sup_{A \in \mathcal{B}(\mathcal{S})} |\mu(A)| - \inf_{A \in \mathcal{B}(\mathcal{S})} |\mu(A)|,$$

- And define the V-norm distance between Q_1, Q_2 by:

$$|||Q_1 - Q_2|||_V := \sup_{\mathbf{x} \in \mathbb{X}} \frac{||Q_1(\mathbf{x}, .) - Q_2(\mathbf{x}, .)||_V}{V(\mathbf{x})},$$

and the outer-product:

$$[1 \otimes \pi](\mathbf{x}, A) := \pi(A), \mathbf{x} \in \mathbb{X}, A \in \mathcal{B}(\mathbb{X}).$$

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Theorem 2 [MT93], [AJ11]

There exist $\beta < 1$ and $B < \infty$ such that

$$|||Q^t - 1 \otimes \pi|||_V \leq B\beta^{-t}.$$

Proof. A by-product of the proof of theorem 1.

Large-scale Limit of the Price Process

In what follows, I will focus on the embedded *discrete-time* Markov chain:

$$(\mathbf{X}_n)_{n\in\mathbb{N}},$$

defined by its transition probabilities u_{ij}:

$$u_{ij} = \begin{cases} = -\frac{q_{ij}}{q_{ii}} = \frac{q_{ij}}{\sum_{k\neq i} q_{ik}} & \text{if } i \neq j \\ 0 & \text{otherwise} \end{cases},$$

where q_{ij} are the transition rates of $(\mathbf{X}_t)$.

Large-scale Limit of the Price Process

Lemma 1 [MT93]

$(\mathbf{X}_n)$ is $V-$geometrically mixing, that is, there exist $B > 0$ and $0 < \beta < 1$ such that for all $g^2, h^2 \leq V$ and $l, n \in \mathbb{Z}$:

$$\left|\mathbb{E}_x\left[g(\mathbf{X}_n)h(\mathbf{X}_{n+l})\right] - \mathbb{E}_x\left[g(\mathbf{X}_n)\right]\mathbb{E}_x\left[h(\mathbf{X}_{n+l})\right]\right| \leq B\beta^l\left(1+\rho^n V(x)\right).$$

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Lemma 2 [MT93]

Let $\bar{h} = h - \pi(h), \bar{g} = g - \pi(g)$. The *stationary* version of $(\mathbf{X}_t)$ satisfies a geometric mixing condition, that is, there exists $B' > 0$ such that for all $l, n \in \mathbb{Z}$:

$$\mathbb{E}_\pi\left[\bar{g}(\mathbf{X}_n)\bar{h}(\mathbf{X}_{n+l})\right] \leq B'\beta^l.$$

Large-scale Limit of the Price Process

Theorem 3 [AJ11]

The rescaled centered price process

$$\widetilde{P}(t) = \lim_{n \to \infty} \frac{P(\lfloor nt \rfloor) - \mathbb{E}\left[P(\lfloor nt \rfloor)\right]}{\sqrt{n}}$$

is a *Brownian motion* in the limit of n going to infinity.

Large-scale Limit of the Price Process

Proof.

1. Using Lemma 2 (geometric-mixing), there exists $(\delta(l))_{l\in\mathbb{N}}$ such that

$$\sup_{n\in\mathbb{N},|g|,|h|\leq 1} |\mathbb{E}[g(\mathbf{X}_n)h(\mathbf{X}_{n+l})] - \mathbb{E}[g(\mathbf{X}_n)]\mathbb{E}[h(\mathbf{X}_{n+l})]| \leq \delta(l),$$

   and

$$\sum_l \delta(l) < \infty;$$

2. The price increments[1] $\delta P_n = \Phi(\mathbf{X}_n, \mathbf{X}_{n-1}, \epsilon_n)$ are *weakly dependent* and have *finite variance*;
3. The Functional CLT holds.

[1] $\Phi : \mathbb{Z}^{2K} \times \mathbb{Z}^{2K} \times \{-1, 1\} \to \pm\{0, \Delta, 2\Delta, \ldots, K\Delta\}$.

Summary

- A Markov-chain order book model was described;
- The order book is “stable” (ergodic);
- The convergence to the stationary state happens geometrically fast (under a certain norm);
- The large-scale limit of the price process is a Brownian motion.

Thank you for your attention.

References

F. Abergel and A. Jedidi.
Forthcoming.
2011.

P. Billingsley.
Convergence of Probability Measures.
Wiley, 1999.

R. Cont, S. Stoikov, and R. Talreja.
A stochastic model for order book dynamics.
Operations Research, 2010.

S. Meyn and R.L. Tweedie.
Markov Chains and Stochastic Stability.
Cambridge University Press, 1993.

I. Muni-Toke.
Market making behaviour in an order book model and its impact on the spread.
Econophysics of Order-driven Markets, Springer-Verlag, 2010.

E. Smith, D. Farmer, L. Guillemot, and S. Krishnamurthy.
Statistical theory of the continuous double auction.
Quantitative Finance, 2003.

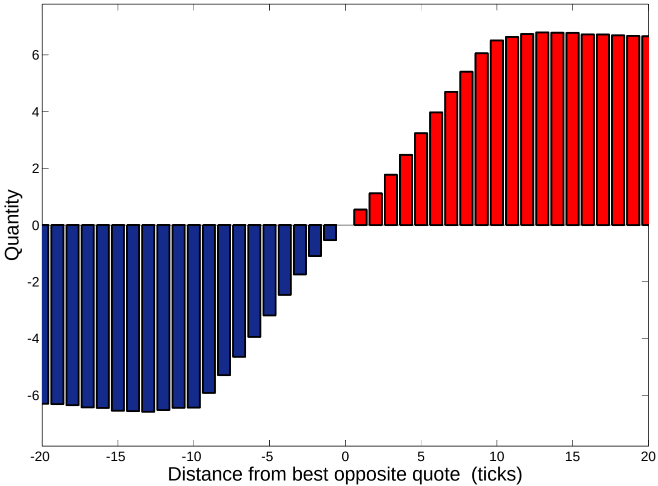

Figure 2: Average depth profile.

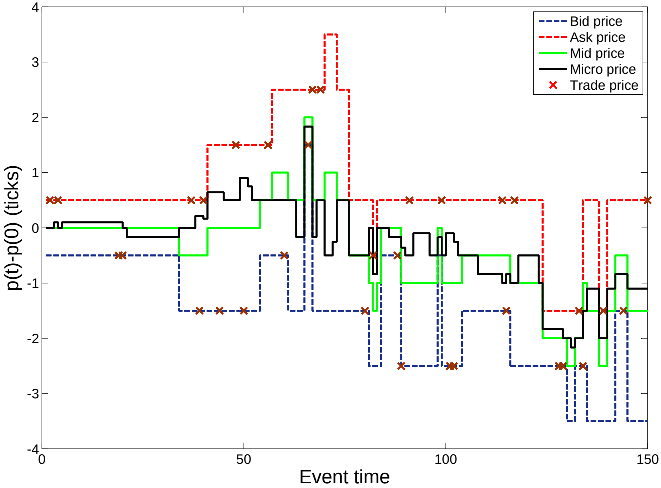

Figure 3: Price sample path ($\sim$ 100 events).

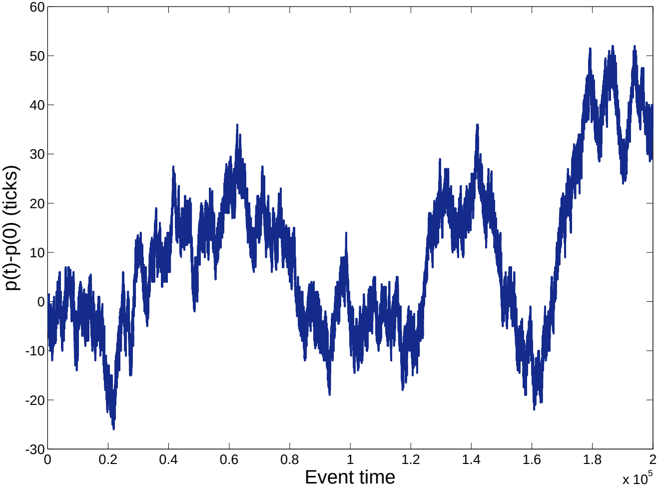

Figure 4: Price sample path ($\sim 10,000$ events).