Markovian Order Book Modelling

Stability and Scaling Limits

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Objectives

2. Nature of the price process at large time scales.
The Order Book is the list of all buy and sell *limit orders*, with their corresponding prices and volumes, at a given instant of time.
Definitions

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There are 3 types of orders:

1. *Limit order*: Specify a price at which one is willing to buy (sell) a certain number of shares;
2. *Market order*: Immediately buy (sell) a certain number of shares at the best available opposite quote;
3. *Cancellation order*: Cancel an existing limit order.
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The price dynamics is the result of the interplay between the order book and the order flow.
The order book is represented by a finite-size vector of quantities:

\[ X_t := (a_t; b_t) := (a_t^1, \ldots, a_t^K; b_t^1, \ldots, b_t^K); \]

- \( a_t \): ask side of the order book;
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- \( a_t \): ask side of the order book;
- \( b_t \): bid side of the order book;
- \( \Delta p \): tick size;
- \( \tau \): unit volume;
- \( P = \frac{P_{\text{Ask}} + P_{\text{Bid}}}{2} \): mid-price.
The events affecting the order book are described by independent Poisson processes:

- $M_t^\pm$: arrival of a new market order $\leftrightarrow$ arrival rate $\lambda_M^\pm$;
- $L_t^\pm i, i \in \{1, \ldots, K\}$: arrival of a limit order $i$ ticks away from the best opposite quote $\leftrightarrow$ arrival rate $\lambda_L^{i\pm}$;
- $C_t^\pm i, i \in \{1, \ldots, K\}$: cancellation of a limit order $i$ ticks away from the best opposite quote $\leftrightarrow$ arrival rate $\lambda_C^i |X_i^t| / \tau$.

$(X_t)$ is a Markov process with state space $X \subset \mathbb{Z}^{2K}$. 
Finite moving reference frame of size $2K$: Ask side ranges from 1 to $K$ ticks away from the best available opposite quote. *Idem* for bid side of the order book.
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Constant boundary conditions outside the moving frame: Every time the moving frame leaves a price level, the number of shares at that level is set to $a^\infty$ (or $b^\infty$, depending on the side of the order book).
Figure 1: Illustration.
\[
\begin{align*}
    da^i_t &= -\left( \tau - \sum_{k=1}^{i-1} a^k \right) dM^+_t + \tau dL^+_t - \tau dC^+_t \\
    &\quad + (J^{-}_M(a) - a)_i dM^-_t + \sum_{i=1}^{K} (J^{-}_L(a) - a)_i dL^-_t \\
    &\quad + \sum_{i=1}^{K} (J^{-}_C(a) - a)_i dC^-_t,
\end{align*}
\]

\[
db^i_t = \text{similar expression,}
\]

where \( J^\pm_M, J^{i\pm}_L, \) and \( J^{i\pm}_C \) are shift operators corresponding to the effect of order arrivals on the reference frame.
The infinitesimal generator of \((X_t)_{t \geq 0}\) is the operator \(L\), defined to act on sufficiently regular functions \(f : \mathbb{R}^n \to \mathbb{R}\), by

\[
\mathcal{L}f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}[f(X_t)|X_0 = x] - f(x)}{t}.
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We have:

\[
\mathcal{L}f(a; b) = \lambda^+_M(f((a^i - (\tau - A^{i-1})_+)_+; J^+_M(b)) - f) \\
+ \sum_{i=1}^{K} \lambda^+_L(f(a^i + \tau; J^+_L(b)) - f) \\
+ \sum_{i=1}^{K} \lambda^+_C \frac{a^i}{\tau} (f(a^i - \tau; J^+_C(b)) - f)
\]

+ similar terms for the events affecting the bid side.
A Markov Process \((X_t)_{t \geq 0}\) is ergodic if an invariant probability distribution \(\pi\) exists and

\[
\lim_{t \to \infty} \|Q^t(x, .) - \pi\| = 0, \forall x \in X,
\]

where \(\|\cdot\|\) is the total variation norm defined by

\[
\|\mu\| := \sup_{A \in \mathcal{B}(X)} |\mu(A)| - \inf_{A \in \mathcal{B}(X)} |\mu(A)|.
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and \((Q^t)\) are the transition functions of the Markov process.
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**Theorem 1 [CST10], [AJ11]**

If \(\lambda_C = \min_{1 \leq i \leq K} \{\lambda_C^{\pm i}\} > 0\), then \((X_t) = (a_t; b_t)\) is an ergodic Markov process. In particular \((X_t)\) has a stationary distribution \(\pi\).
Proof. Let $V(a; b) = \tau + \sum_{i=1}^{K} a^i + \sum_{i=1}^{K} |b^i|$. 

\[
\mathcal{L}V(a; b) \leq \sum_{i=1}^{K} (\lambda_{L}^{i+} + \lambda_{L}^{i-})\tau - (\lambda_{M}^{+} + \lambda_{M}^{-})\tau - \sum_{i=1}^{K} (\lambda_{C}^{i+} a^i + \lambda_{C}^{i-} |b^i|) \\
+ \sum_{i=1}^{K} \lambda_{L}^{i-} (i_s - i)_+ a^\infty + \sum_{i=1}^{K} \lambda_{L}^{i+} (i_s - i)_+ |b^\infty| \\
\leq K(\lambda_{L}^{\max} + \lambda_{L}^{\min})\tau - (\lambda_{M}^{+} + \lambda_{M}^{-})\tau - \lambda_{C} f(a; b) \\
+ K(K + 1)(\lambda_{L}^{\max} a^\infty + \lambda_{L}^{\max} |b^\infty|) \\
\mathcal{L}V(x) \leq -cV(x) + d, \quad (\text{GDC})
\]

where $\lambda_{L}^{\pm} = \max_{1 \leq i \leq K} \{\lambda_{L}^{i\pm}\}$ and $\lambda_{C} = \min_{1 \leq i \leq K} \{\lambda_{C}^{i\pm}\} > 0$. 

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Corollary 1 [AJ11]

The spread process $S_t := Q_t^A - Q_t^B = \Psi(X_t)$, is ergodic—expected as $(S_t)$ is bounded by $K + 1$. 
Actually, \((X_t)\) is \(V\)-uniformly ergodic.

Denote, as above, \(\|.\|\) the total-variation norm:

\[
\|\mu\| := \sup_{A \in \mathcal{B}(S)} |\mu(A)| - \inf_{A \in \mathcal{B}(S)} |\mu(A)|,
\]

And define the \(V\)-norm distance between \(Q_1, Q_2\) by:

\[
\|\|Q_1 - Q_2\|\|_V := \sup_{x \in X} \frac{\|Q_1(x, .) - Q_2(x, .)\|_V}{V(x)},
\]

and the outer-product:

\[
[1 \otimes \pi](x, A) := \pi(A), x \in X, A \in \mathcal{B}(X).
\]
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**Theorem 2 [MT93], [AJ11]**

There exist \(\beta < 1\) and \(B < \infty\) such that

\[
\|\|Q^t - 1 \otimes \pi\|\|_V \leq B\beta^{-t}.
\]
In what follows, I will focus on the embedded *discrete-time* Markov chain:

\[(X_n)_{n \in \mathbb{N}},\]

defined by its transition probabilities \(u_{ij}:\)

\[
u_{ij} = \begin{cases} 
- \frac{q_{ij}}{q_{ii}} = \frac{q_{ij}}{\sum_{k \neq i} q_{ik}} & \text{if } i \neq j \\
0 & \text{otherwise}
\end{cases}
\]

where \(q_{ij}\) are the transition rates of \((X_t).\)
Lemma 1 [MT93]

\((X_n)\) is \(V\)-geometrically mixing, that is, there exist \(B > 0\) and \(0 < \beta < 1\) such that for all \(g^2, h^2 \leq V\) and \(l, n \in \mathbb{Z}\):

\[
\left| \mathbb{E}_x [g(X_n)h(X_{n+l})] - \mathbb{E}_x [g(X_n)] \mathbb{E}_x [h(X_{n+l})] \right| \leq B \beta^l (1 + \rho^n V(x)) .
\]
Lemma 1 [MT93]

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$$\left| \mathbb{E}_x \left[ g(X_n)h(X_{n+l}) \right] - \mathbb{E}_x \left[ g(X_n) \right] \mathbb{E}_x \left[ h(X_{n+l}) \right] \right| \leq B \beta^l (1 + \rho^n V(x)).$$

Lemma 2 [MT93]

Let $\bar{h} = h - \pi(h), \bar{g} = g - \pi(g)$. The stationary version of $(X_t)$ satisfies a geometric mixing condition, that is, there exists $B' > 0$ such that for all $l, n \in \mathbb{Z}$:

$$\mathbb{E}_\pi \left[ \bar{g}(X_n)\bar{h}(X_{n+l}) \right] \leq B' \beta^l.$$
Theorem 3 [AJ11]

The rescaled centered price process

$$\tilde{P}(t) = \lim_{n \to \infty} \frac{P(\lfloor nt \rfloor) - \mathbb{E}[P(\lfloor nt \rfloor) \)}{\sqrt{n}}$$

is a Brownian motion in the limit of $n$ going to infinity.
Large-scale Limit of the Price Process

Proof.

1. Using Lemma 2 (geometric-mixing), there exists $(\delta(l))_{l \in \mathbb{N}}$ such that

$$
\sup_{n \in \mathbb{N}, |g|, |h| \leq 1} |\mathbb{E}[g(X_n)h(X_{n+1})] - \mathbb{E}[g(X_n)]\mathbb{E}[h(X_{n+1})]| \leq \delta(l),
$$

and

$$
\sum_{l} \delta(l) < \infty;
$$

2. The price increments $^{1} \delta P_n = \Phi(X_n, X_{n-1}, \epsilon_n)$ are weakly dependent and have finite variance;

3. The Functional CLT holds.

$^{1}\Phi : \mathbb{Z}^{2K} \times \mathbb{Z}^{2K} \times \{-1, 1\} \rightarrow \pm \{0, \Delta, 2\Delta, \ldots, K\Delta\}$. 
A Markov-chain order book model was described;
The order book is “stable” (ergodic);
The convergence to the stationary state happens geometrically fast (under a certain norm);
The large-scale limit of the price process is a Brownian motion.
Thank you for your attention.


Figure 2: Average depth profile.
Figure 3: Price sample path (~ 100 events).
Figure 4: Price sample path (~ 10,000 events).