High order discretization schemes for stochastic volatility models.

Benjamin Jourdain

Joint work with Mohamed SBAI

Modeling and Managing Financial Risks

Outline of the talk

Stochastic volatility model

Asset price $(S_t)_{t\in[0,T]}$ solving

$$\begin{cases} dS_t = rS_t dt + f(Y_t)S_t(\rho dW_t + \sqrt{1-\rho^2}dB_t); \quad S_0 = s_0 > 0 \\ dY_t = b(Y_t)dt + \sigma(Y_t)dW_t; \quad Y_0 = y_0 \end{cases} . \qquad (1)$$

where

- r is the instantaneous interest rate,
- $(B_t)_{t\in[0,T]}$ and $(W_t)_{t\in[0,T]}$ are independent standard one-dimensional Brownian motions,
- $\rho \in [-1,1]$ is the correlation,
- $f, b, \sigma : \mathbb{R} \to \mathbb{R}$.

Much attention has been paid to the discretization of the Heston model ($f(y) = \sqrt{y}$, $b(y) = \kappa(\theta - y)$ and $\sigma(y) = \nu\sqrt{y}$).

Aim

$$\begin{cases} dS_t = rS_t dt + f(Y_t)S_t(\rho dW_t + \sqrt{1-\rho^2}dB_t) \\ dY_t = \sigma(Y_t)dW_t + b(Y_t)dt \end{cases}.$$

We are interested in the case of smooth coefficients $\sigma, b : \mathbb{R} \to \mathbb{R}$ where far less has been done.

- Kahl & Jäckel 2006 propose a scheme with a Milstein discretization of the integrals wrt dW_t and a trapezoidal discretization of the integral wrt dB_t $\to$ order of strong convergence 1/2 and, according to numerical experiments, smaller multiplicative constant than the Euler scheme.
- Our aim : take advantage of the structure of the model to construct performant schemes both for vanilla and path-dependent options.
  Keep the possibility to replace the discretization of Y by exact simulation in the Ornstein Uhlenbeck case.

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Transformation of the SDE (1)

Logarithmic change of variable for the asset : $X_t \stackrel{\text{def}}{=} \log(S_t)$ solves

$$dX_t = \left(r - \frac{1}{2}f^2(Y_t)\right)dt + f(Y_t)(\rho dW_t + \sqrt{1-\rho^2}dB_t).$$

Removal of the term $\rho f(Y_t)dW_t$: if f, σ are C^1 and σ does not vanish, for $F(y) \stackrel{\text{def}}{=} \int_{y_0}^{y} \frac{f}{\sigma}(z)dz$,

$$dF(Y_t) = f(Y_t)dW_t + \left[\frac{bf}{\sigma} + \frac{1}{2}(\sigma f' - f\sigma')\right](Y_t)dt.$$

Hence for $h(y) \stackrel{\text{def}}{=} r - \frac{1}{2}f^2(y) - \rho(\frac{bf}{\sigma} + \frac{1}{2}(\sigma f' - f\sigma'))(y)$,

$$\begin{cases} dX_t = \rho dF(Y_t) + h(Y_t)dt + \sqrt{1-\rho^2}f(Y_t)dB_t \\ dY_t = \sigma(Y_t)dW_t + b(Y_t)dt \end{cases}. \quad (2)$$

In the stochastic integral in dX indep. of $f(Y_t)$ and dB_t.

Vanilla option

We want to compute the price $\mathbb{E}(e^{-rT}g(S_T)) = \mathbb{E}(e^{-rT}g(e^{X_T}))$ of the option with

- maturity T
- payoff $g : \mathbb{R}_+^* \to \mathbb{R}_+$

Weak approximation problem

Recently schemes which do not involve iterated Brownian integrals and achieve an order of weak convergence greater than one have been developped.

Moment like families : Kusuoka 01 04, Ninomiya 03 03,...

Cubatures : Lyons & Victoir 04,...

Splitting and integration of ODEs : Ninomiya & Victoir 08, Ninomiya & Ninomiya 09, Tanaka & Kohatsu-Higa 09, Alfonsi 09,...

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Splitting for (2)

If $Z_t \stackrel{\text{def}}{=} X_t - \rho F(Y_t)$, one has

$$\begin{cases} dY_t = \sigma(Y_t)dW_t + b(Y_t)dt \\ dZ_t = h(Y_t)dt + \sqrt{1-\rho^2}f(Y_t)dB_t \end{cases}.$$

Associated operator $\mathcal{L} = \underbrace{\frac{\sigma^2(y)}{2}\partial_{yy} + b(y)\partial_y}_{\mathcal{L}_Y} + \underbrace{\frac{(1-\rho^2)f^2(y)}{2}\partial_{zz} + h(y)\partial_z}_{\mathcal{L}_Z}$

where the coefficients of $\mathcal{L}_Z$ do not depend on $z \to$ for fixed y, exact simulation of the associated SDE possible

Strang splitting with weak order 2 for (Y, Z) : at each time-step of length T/N, one

- solves the SDE for Z with fixed Y up to time $T/2N$,
- integrate the SDE for Y with a scheme of weak order 2 on a time-interval with length T/N,
- solves the SDE for Z with fixed Y up to time $T/2N$.

Specific scheme with weak order 2 for Y

For $0 \leq k \leq N$, let $t_k \stackrel{\text{def}}{=} \frac{kT}{N}$.
We choose the Ninomiya-Victoir scheme for Y :

$$\begin{cases} \bar{Y}_0^N = y_0 \\ \forall 0 \leq k \leq N-1,\ \bar{Y}_{t_{k+1}}^N = e^{\frac{T}{2N}\widetilde{b}} e^{(W_{t_{k+1}} - W_{t_k})\sigma} e^{\frac{T}{2N}\widetilde{b}} (\bar{Y}_{t_{k+1}}^N) \end{cases}$$

where

- $\widetilde{b}(y) \stackrel{\text{def}}{=} b(y) - \frac{1}{2}\sigma\sigma'(y)$,
- for $v : \mathbb{R} \to \mathbb{R}$, $e^{tv}(y)$ denotes the solution $\xi(t)$ of the ODE

$$\begin{cases} \xi'(t) = v(\xi(t)) \\ \xi(0) = y \end{cases}.$$

If $\eta(z) \stackrel{\text{def}}{=} \int_0^z \frac{1}{v(x)} dx$, one has $e^{tv}(y) = \eta^{-1}(t + \eta(y))$.

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Convergence result

Theorem 1

Assume that

- $|\rho| \neq 1$,
- $F \in C_b^6, f \in C_b^4, h \in C_b^4$,
- $\sigma \in C^5$, $b \in C^4$, *with bounded derivatives,* σ *does not vanish,*
- $\inf_{\mathbb{R}} f^2 > 0$,
- g *is measurable and such that*
  $\exists c \geq 0,\ \exists \mu \in [0,2),\ \forall y > 0,\ |g(y)| \leq c e^{|\log(y)|^\mu}$.

Then there is a constant C *not depending on* N *such that*

$$\forall N \in \mathbb{N}^*,\ |\mathbb{E}(g(S_T)) - \mathbb{E}(g(e^{\bar{X}_T^N}))| \leq \frac{C}{N^2}.$$

Convergence for all measurable payoff functions g with polynomial growth.

Discretization bias

Let $g : C([0,T],\mathbb{R}) \to \mathbb{R}$ be a Lipschitz function (Asian or lookback option payoff) :

$$|\mathbb{E}(g((S_t)_{t\le T})) - \mathbb{E}(g((\widetilde{S}^N_t)_{t\le T}))| \le \mathbb{E}\left|g((S_t)_{t\le T}) - g((\widetilde{S}^N_t)_{t\le T})\right|$$
$$\le \|g\|_{\mathrm{Lip}}\mathbb{E}\left(\sup_{t\le T}|S_t - \widetilde{S}^N_t|\right).$$

$\le$: very rough. Preferably use Wasserstein metric $\mathcal{W}_1$,

$$|\mathbb{E}(g((S_t)_{t\le T})) - \mathbb{E}(g((\widetilde{S}^N_t)_{t\le T}))| \le \|g\|_{\mathrm{Lip}} \underbrace{\sup_{\|\gamma\|_{\mathrm{Lip}}\le 1} |\mathbb{E}(\gamma((S_t)_{t\le T})) - \mathbb{E}(\gamma((\widetilde{S}^N_t)_{t\le T}))}_{\mathcal{W}_1(\mathcal{L}(S),\mathcal{L}(\widetilde{S}^N))}$$

Dual formulation : $\mathcal{W}_1(\mathcal{L}(S),\mathcal{L}(\widetilde{S}^N)) = \inf_{\tilde{S} \overset{\mathcal{L}}{=} S} \mathbb{E}\left(\sup_{t\le T}|\tilde{S}_t - \widetilde{S}^N_t|\right)$.

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Existing schemes

$$\begin{cases} dX_t = \left(r - \frac{1}{2}f^2(Y_t)\right) dt + f(Y_t)(\rho dW_t + \sqrt{1-\rho^2} dB_t) \\ dY_t = \sigma(Y_t) dW_t + b(Y_t) dt \end{cases}.$$

Milstein scheme : strong order of convergence 1 but the commutativity condition writes $\sigma f' = 0$ i.e. holds for deterministic volatility

Cruzeiro Malliavin & Thalmaier 2004 : under ellipticity → scheme with order one of convergence for $\mathcal{W}_1$ (very clever rotation of the Brownian motion). But

- if Y is OU, not possible to preserve order one convergence for $\mathcal{W}_1$ when replacing Y^{CMT} by Y in the evolution of X^{CMT},
- in the perspective of statistical Romberg extrapol. (Kebaier 05) or multi-level Monte Carlo (Giles & al 07 08 09), no coupling with strong order 1 between the schemes with N and $2N$ steps.

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Our scheme

We look for a simple scheme with order one of convergence for $\mathcal{W}_1$ and overcoming the previous restrictions.

Milstein scheme for Y : $\widetilde{Y}^N_{t_0} = y_0$ and for $0 \leq k \leq N-1$

$$\widetilde{Y}^N_{t_{k+1}} = \widetilde{Y}^N_{t_k} + b(\widetilde{Y}^N_{t_k})\frac{T}{N} + \sigma(\widetilde{Y}^N_{t_k})\Delta W_{k+1} + \frac{1}{2}\sigma\sigma'(\widetilde{Y}^N_{t_k})\left(\Delta W^2_{k+1} - \frac{T}{N}\right).$$

$$dX_t = \rho dF(Y_t) + h(Y_t)dt + \sqrt{1-\rho^2} f(Y_t)dB_t$$

$$\mathrm{Var}\left(\int_{t_k}^{t_{k+1}} f(Y_s)dB_s \middle| W\right) = \int_{t_k}^{t_{k+1}} f^2(Y_s)ds \simeq \frac{f^2(Y_{t_k})T}{N} + \sigma f^{2\prime}(Y_{t_k})\int_{t_k}^{t_{k+1}} W_s - W_{t_k} ds$$

$$\widetilde{X}^N_{t_{k+1}} = \widetilde{X}^N_{t_k} + \rho\left(F(\widetilde{Y}^N_{t_{k+1}}) - F(\widetilde{Y}^N_{t_k})\right) + h(\widetilde{Y}^N_{t_k})\frac{T}{N}$$

$$+ \sqrt{1-\rho^2}\sqrt{\left(f^2(\widetilde{Y}^N_{t_k}) + \frac{N\sigma f^{2\prime}(\widetilde{Y}^N_{t_k})}{T}\int_{t_k}^{t_{k+1}}(W_s - W_{t_k})ds\right) \vee \underline{f}^2}\ \Delta B_{k+1}$$

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$$+ \sqrt{1-\rho^2}\sqrt{\left(f^2(\widetilde{Y}^N_{t_k}) + \frac{N\sigma f^{2\prime}(\widetilde{Y}^N_{t_k})}{T}\int_{t_k}^{t_{k+1}}(W_s - W_{t_k})ds\right) \vee \underline{f^2}}\ \Delta B_{k+1}$$

Convergence

Theorem 2

Assume that $\forall N$, $\mathbb{R}^{2(N+1)}$ *is endowed with the supremum norm,*

- $b \in C_b^3$ *and* $\sigma \in C_b^4$ *with* $\inf_{y\in\mathbb{R}} \sigma(y) > 0$,
- $f \in C_b^4$ *with* $\underline{f^2} \stackrel{\text{def}}{=} \inf_{y\in\mathbb{R}} f^2(y) > 0$.

$$\exists C > 0,\ \forall N \in \mathbb{N}^*,\ \mathcal{W}_1\left(\mathcal{L}((X_{t_k}, Y_{t_k})_{k\le N}), \mathcal{L}((\tilde{X}^N_{t_k}, \tilde{Y}^N_{t_k})_{k\le N})\right) \le \frac{C}{N}.$$

Moreover, it is possible to couple the schemes with N *and* $2N$ *steps by simulating* $(X^N_{t_k}, \tilde{X}^{2N}_{t_k})_{k\le N}$ *with* $(X^N_{t_k})_{k\le N} \stackrel{\mathcal{L}}{=} (\tilde{X}^N_{t_k})_{k\le N}$ *and*

$$\forall p \ge 1,\ \exists C \ge 0,\ \forall N \in \mathbb{N}^*, \mathbb{E}\left(\max_{0\le k\le N} |X^N_{t_k} - \tilde{X}^{2N}_{t_k}|^{2p}\right) \le \frac{C}{N^{2p}}.$$

The coupling is usefull in the perspective of multi-level Monte Carlo.

Coupling of the schemes with N and $2N$ steps

Let $\delta = \frac{T}{2N}$ be the step size of the scheme with $2N$ steps.

$$\widetilde{X}^{2N}_{(j+1)\delta} = \widetilde{X}^{2N}_{j\delta} + \rho\left(F(\widetilde{Y}^{2N}_{(j+1)\delta}) - F(\widetilde{Y}^{2N}_{j\delta})\right) + h(\widetilde{Y}^{2N}_{j\delta})\delta$$

$$+\sqrt{1-\rho^2}\sqrt{\underbrace{\left(f^2(\widetilde{Y}^{2N}_{j\delta}) + \frac{\sigma f^{2\prime}(\widetilde{Y}^{2N}_{j\delta})}{\delta}\int_{j\delta}^{j\delta+\delta}(W_s - W_{j\delta})ds\right)\vee \underline{f}^2}_{v_j^{2N}}}\ \underbrace{(B_{(j+1)\delta} - B_{j\delta})}_{\Delta B_j^{2N}}$$

Because of the independence of $\widetilde{Y}^{2N}$ and B, $(\widetilde{X}^N_{t_k})_{k\le N} \overset{\mathcal{L}}{=} (X^N_{t_k})_{k\le N}$ where

$$X^N_{t_{k+1}} = X^N_{t_k} + \rho\left(F(\widetilde{Y}^N_{t_{k+1}}) - F(\widetilde{Y}^N_{t_k})\right) + h(\widetilde{Y}^N_{t_k})\frac{T}{N}$$

$$+\sqrt{1-\rho^2}\sqrt{v_k^N}\times\underbrace{\sqrt{2}\frac{\sqrt{v^{2N}_{2k}}\Delta B^{2N}_{2k} + \sqrt{v^{2N}_{2k+1}}\Delta B^{2N}_{2k+1}}{\sqrt{v^{2N}_{2k}+v^{2N}_{2k+1}}}}_{\sim\mathcal{N}_1(0,T/N)\text{ indep of }W}$$

Framework

Numerical experiments are performed with Scott's model ($f(y) = e^y$, Y OU)

$$\begin{cases} dS_t = rS_t dt + e^{Y_t} S_t(\rho dW_t + \sqrt{1-\rho^2} dB_t) \\ dY_t = \kappa(\theta - Y_t)dt + \nu dW_t \\ \Rightarrow f(y) = e^y,\ b(y) = \kappa(\theta - y) \text{ and } \sigma(y) = \nu \end{cases}$$

with the parameters found in Kahl & Jäckel 2006 :

- $s_0 = 100$, $y_0 = \log(0.25)$,
- $r = 0.05$
- $\kappa = 1$, $\theta = 0$, $\nu = \frac{7\sqrt{2}}{20}$,
- $\rho = -0.2$,
- $T = 1$.

Coupling at terminal time

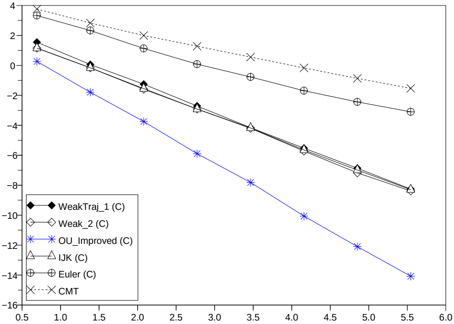

Figure: $\log\left(\mathbb{E}\left(\left(e^{X_T^N} - e^{X_T^{2N}}\right)^2\right)\right)$ in function of $\log(N)$. Except for CMT, X_T^N and X_T^{2N} are generated using the same single normal r.v. for the integral wrt B.

Coupling at terminal time

OU Improved	WeakTraj 1	Weak 2	IJK	Euler	CMT
-2.97	-2.02	-1.98	-1.95	-1.34	-1.08

Table: Slope of the regression of $\log\left(\mathbb{E}\left(\left(e^{X_T^N} - e^{X_T^{2N}}\right)^2\right)\right)$ in terms of $\log(N)$

- order $3/2$ for OU Improved,
- order 1 for WeakTraj 1, Weak 2 and Kahl Jäckel scheme IJK,
- order $1/2$ CMT, slightly better for Euler.

Multi-level computation for the ATM Call

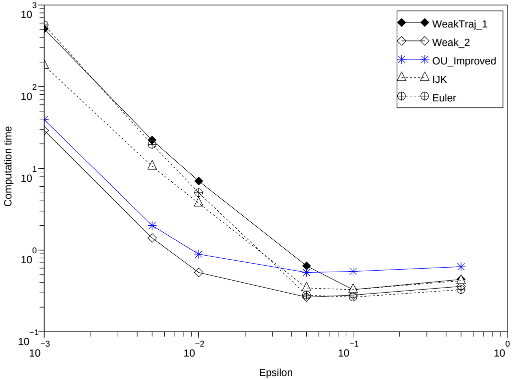

Figure: Computation time in function of the required precision ε

Strong convergence with coupling

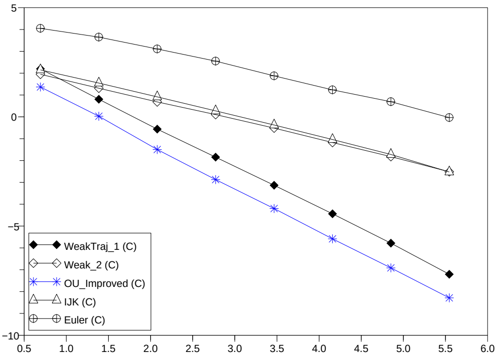

Figure: $\log\left(\mathbb{E}\left(\max_{0\leq k\leq N}\left(e^{X^N_{t_k}} - e^{X^{2N}_{t_k}}\right)^2\right)\right)$ in function of $\log(N)$. Modified increments of B for the scheme with N steps.

Strong convergence with coupling

OU Improved	WeakTraj 1	Weak 2	IJK	Euler
-1.99	-1.92	-0.91	-0.95	-0.85

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- order 1 for OU Improved and WeakTraj 1,
- order 1/2 for Weak 2, IJK and Euler,
- No coupling possible for CMT.

Multi-level computation for the Lookback Call

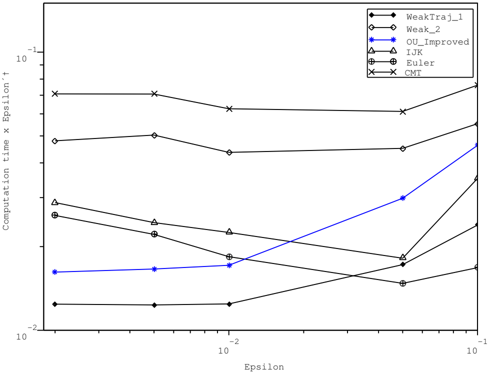

Figure: Computation time multiplied by the square of the required precision ε in in function of ε (payoff $S_T - \min_{t\in[0,T]} S_t$)