High order discretization schemes for stochastic volatility models.

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Joint work with Mohamed SBAI

Modeling and Managing Financial Risks
Outline of the talk

1. Introduction
2. Vanilla options
3. Path dependent options
4. Numerical results
Stochastic volatility model

Asset price \((S_t)_{t\in[0,T]}\) solving

\[
\begin{align*}
ds_t &= rS_t dt + f(Y_t) S_t (\rho dW_t + \sqrt{1 - \rho^2} dB_t); \quad S_0 = s_0 > 0 \\
dY_t &= b(Y_t) dt + \sigma(Y_t) dW_t; \quad Y_0 = y_0
\end{align*}
\]

where

- \(r\) is the instantaneous interest rate,
- \((B_t)_{t\in[0,T]}\) and \((W_t)_{t\in[0,T]}\) are independent standard one-dimensional Brownian motions,
- \(\rho \in [-1, 1]\) is the correlation,
- \(f, b, \sigma : \mathbb{R} \rightarrow \mathbb{R}\).

Much attention has been paid to the discretization of the Heston model \((f(y) = \sqrt{y}, \ b(y) = \kappa(\theta - y) \text{ and } \sigma(y) = \nu \sqrt{y})\).
Aim

\[
\begin{aligned}
    dS_t &= rS_t dt + f(Y_t)S_t(\rho dW_t + \sqrt{1-\rho^2} dB_t) \\
    dY_t &= \sigma(Y_t) dW_t + b(Y_t) dt
\end{aligned}
\]

We are interested in the case of smooth coefficients \(\sigma, b : \mathbb{R} \to \mathbb{R}\) where far less has been done.

- Kahl & Jäckel 2006 propose a scheme with a Milstein discretization of the integrals wrt \(dW_t\) and a trapezoidal discretization of the integral wrt \(dB_t\) → order of strong convergence 1/2 and, according to numerical experiments, smaller multiplicative constant than the Euler scheme.

- Our aim: take advantage of the structure of the model to construct performant schemes both for vanilla and path-dependent options. Keep the possibility to replace the discretization of \(Y\) by exact simulation in the Ornstein Uhlenbeck case.
High order discretization schemes for stochastic volatility models.

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- **Our aim**: take advantage of the structure of the model to construct performant schemes both for vanilla and path-dependent options. Keep the possibility to replace the discretization of \( Y \) by exact simulation in the Ornstein Uhlenbeck case.
Transformation of the SDE (1)

Logarithmic change of variable for the asset: $X_t \overset{\text{def}}{=} \log(S_t)$ solves

$$dX_t = \left( r - \frac{1}{2}f^2(Y_t) \right) dt + f(Y_t)(\rho dW_t + \sqrt{1 - \rho^2} dB_t).$$

Removal of the term $\rho f(Y_t)dW_t$: if $f, \sigma$ are $C^1$ and $\sigma$ does not vanish, for $F(y) \overset{\text{def}}{=} \int_{y_0}^{y} \frac{f(z)}{\sigma} dz$,

$$dF(Y_t) = f(Y_t)dW_t + \left[ \frac{bf}{\sigma} + \frac{1}{2}(\sigma f' - f \sigma') \right] (Y_t) dt.$$

Hence for $h(y) \overset{\text{def}}{=} r - \frac{1}{2}f^2(y) - \rho\left( \frac{bf}{\sigma} + \frac{1}{2}(\sigma f' - f \sigma') \right)(y)$,

$$\begin{cases} dX_t = \rho dF(Y_t) + h(Y_t) dt + \sqrt{1 - \rho^2} f(Y_t) dB_t \\ dY_t = \sigma(Y_t)dW_t + b(Y_t) dt \end{cases}.$$

In the stochastic integral in $dX$ indep. of $f(Y_t)$ and $dB_t$. (2)
High order discretization schemes for stochastic volatility models.

Vanilla options

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4. Numerical results
Vanilla option

We want to compute the price $\mathbb{E}(e^{-rT}g(S_T)) = \mathbb{E}(e^{-rT}g(e^{X_T}))$ of the option with

- maturity $T$
- payoff $g : \mathbb{R}_+^* \rightarrow \mathbb{R}_+$

Weak approximation problem

Recently schemes which do not involve iterated Brownian integrals and achieve an order of weak convergence greater than one have been developed.

Moment like families: Kusuoka 01 04, Ninomiya 03 03,...
Cubatures: Lyons & Victoir 04,...
Splitting and integration of ODEs: Ninomiya & Victoir 08, Ninomiya & Ninomiya 09, Tanaka & Kohatsu-Higa 09, Alfonsi 09,...
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Splitting for (2)

If $Z_t \overset{\text{def}}{=} X_t - \rho F(Y_t)$, one has

\[
\begin{align*}
    dY_t &= \sigma(Y_t)dW_t + b(Y_t)dt \\
    dZ_t &= h(Y_t)dt + \sqrt{1 - \rho^2 f(Y_t)}dB_t
\end{align*}
\]

Associated operator $L = \frac{\sigma^2(y)}{2} \partial_{yy} + b(y)\partial_y + \frac{(1 - \rho^2)f^2(y)}{2} \partial_{zz} + h(y)\partial_z$

where the coefficients of $L_Z$ do not depend on $z \rightarrow$ for fixed $y$, exact simulation of the associated SDE possible

Strang splitting with weak order 2 for $(Y, Z)$: at each time-step of length $T/N$, one

- solves the SDE for $Z$ with fixed $Y$ up to time $T/2N$,
- integrate the SDE for $Y$ with a scheme of weak order 2 on a time-interval with length $T/N$,
- solves the SDE for $Z$ with fixed $Y$ up to time $T/2N$. 
Specific scheme with weak order 2 for $Y$

For $0 \leq k \leq N$, let $t_k \overset{\text{def}}{=} \frac{kT}{N}$.

We choose the Ninomiya-Victoir scheme for $Y$:

$$
\begin{cases}
\bar{Y}_0^N = y_0 \\
\forall 0 \leq k \leq N - 1, \quad \bar{Y}_{t_{k+1}}^N = e^{\frac{T}{2N}\tilde{b}} e^{(W_{t_{k+1}} - W_{t_k})\sigma} e^{\frac{T}{2N}\tilde{b}} (\bar{Y}_{t_{k+1}}^N)
\end{cases}
$$

where

- $\tilde{b}(y) \overset{\text{def}}{=} b(y) - \frac{1}{2}\sigma\sigma'(y)$,

- for $\nu : \mathbb{R} \rightarrow \mathbb{R}$, $e^{\nu}(y)$ denotes the solution $\xi(t)$ of the ODE

$$
\begin{cases}
\xi'(t) = \nu(\xi(t)) \\
\xi(0) = y
\end{cases}
$$

If $\eta(z) \overset{\text{def}}{=} \int_0^z \frac{1}{\nu(x)} dx$, one has $e^{\nu}(y) = \eta^{-1}(t + \eta(y))$. 

Specific scheme with weak order 2 for $\gamma$

For $0 \leq k \leq N$, let $t_k \overset{\text{def}}{=} \frac{kT}{N}$.

We choose the Ninomiya-Victoir scheme for $\gamma$:

\[
\begin{cases}
\tilde{Y}_0^N = y_0 \\
\forall 0 \leq k \leq N - 1, \quad \tilde{Y}_t^N_{ik+1} = e^{T_{2N} \tilde{b}} e^{(W_{t_{k+1}} - W_{t_k})\sigma} e^{T_{2N} \tilde{b}} (\tilde{Y}_t^N_{ik+1})
\end{cases}
\]

where

- $\tilde{b}(y) \overset{\text{def}}{=} b(y) - \frac{1}{2} \sigma \sigma'(y)$,

- for $v : \mathbb{R} \rightarrow \mathbb{R}$, $e^{tv}(y)$ denotes the solution $\xi(t)$ of the ODE

\[
\begin{cases}
\xi'(t) = v(\xi(t)) \\
\xi(0) = y
\end{cases}
\]

If $\eta(z) \overset{\text{def}}{=} \int_0^z \frac{1}{v(x)} dx$, one has $e^{tv}(y) = \eta^{-1}(t + \eta(y))$. 
Convergence result

**Theorem 1**

Assume that

- $|\rho| \neq 1$,
- $F \in C^6_b, f \in C^4_b, h \in C^4_b$,
- $\sigma \in C^5, b \in C^4$, with bounded derivatives, $\sigma$ does not vanish,
- $\inf_{\mathbb{R}} f^2 > 0$,
- $g$ is measurable and such that
  \[ \exists c \geq 0, \exists \mu \in [0, 2), \forall y > 0, |g(y)| \leq ce^{\log(y)|^\mu}. \]

Then there is a constant $C$ not depending on $N$ such that

\[ \forall N \in \mathbb{N}^*, \left| \mathbb{E}(g(S_T)) - \mathbb{E}(g(e^{\bar{X}_T}^N)) \right| \leq \frac{C}{N^2}. \]

Convergence for all measurable payoff functions $g$ with polynomial growth.
High order discretization schemes for stochastic volatility models.

Path dependent options

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4 Numerical results
Discretization bias

Let \( g : C([0, T], \mathbb{R}) \rightarrow \mathbb{R} \) be a Lipschitz function (Asian or lookback option payoff):

\[
|\mathbb{E}(g((S_t)_{t \leq T})) - \mathbb{E}(g((\tilde{S}^N_t)_{t \leq T}))| \leq \mathbb{E}\left| g((S_t)_{t \leq T}) - g((\tilde{S}^N_t)_{t \leq T}) \right| \\
\leq \|g\|_{Lip} \mathbb{E}\left( \sup_{t \leq T} |S_t - \tilde{S}^N_t| \right).
\]

\( \leq \) : very rough. Preferably use Wasserstein metric \( \mathcal{W}_1 \),

\[
|\mathbb{E}(g((S_t)_{t \leq T})) - \mathbb{E}(g((\tilde{S}^N_t)_{t \leq T}))| \leq \|g\|_{Lip} \sup_{\|\gamma\|_{Lip} \leq 1} \mathbb{E}(\gamma((S_t)_{t \leq T})) - \mathbb{E}(\gamma((\tilde{S}^N_t)_{t \leq T}))
\]

Dual formulation: \( \mathcal{W}_1(\mathcal{L}(S), \mathcal{L}(\tilde{S}^N)) = \inf_{\tilde{S} \leq S} \mathbb{E}\left( \sup_{t \leq T} |\tilde{S}_t - \tilde{S}^N_t| \right) \).
Discretization bias

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$$\left| \mathbb{E}(g((S_t)_{t \leq T})) - \mathbb{E}(g((\tilde{S}^N_t)_{t \leq T})) \right| \leq \mathbb{E}\left| g((S_t)_{t \leq T}) - g((\tilde{S}^N_t)_{t \leq T}) \right|$$

$$\leq \|g\|_{\text{Lip}} \mathbb{E}\left( \sup_{t \leq T} |S_t - \tilde{S}^N_t| \right).$$

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$$\left| \mathbb{E}(g((S_t)_{t \leq T})) - \mathbb{E}(g((\tilde{S}^N_t)_{t \leq T})) \right| \leq \|g\|_{\text{Lip}} \sup_{\|\gamma\|_{\text{Lip}} \leq 1} \mathbb{E}(\gamma((S_t)_{t \leq T})) - \mathbb{E}(\gamma((\tilde{S}^N_t)_{t \leq T}))$$

$$\mathcal{W}_1(\mathcal{L}(S), \mathcal{L}(\tilde{S}^N))$$

Dual formulation: $\mathcal{W}_1(\mathcal{L}(S), \mathcal{L}(\tilde{S}^N)) = \inf_{\tilde{S} \leq S} \mathbb{E}\left( \sup_{t \leq T} |\tilde{S}_t - \tilde{S}^N_t| \right).$
High order discretization schemes for stochastic volatility models.

Path dependent options

Existing schemes

\[
\begin{aligned}
    dX_t &= (r - \frac{1}{2} f^2(Y_t)) \, dt + f(Y_t)(\rho dW_t + \sqrt{1 - \rho^2} dB_t) \\
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\end{aligned}
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**Milstein scheme**: strong order of convergence 1 but the commutativity condition writes \( \sigma f' = 0 \) i.e. holds for deterministic volatility

Cruzeiro Malliavin & Thalmaier 2004: under ellipticity \( \rightarrow \) scheme with order one of convergence for \( W_1 \) (very clever rotation of the Brownian motion). But

- if \( Y \) is OU, not possible to preserve order one convergence for \( W_1 \) when replacing \( Y^{CMT} \) by \( Y \) in the evolution of \( X^{CMT} \),
- in the perspective of statistical Romberg extrapol. (Kebaier 05) or multi-level Monte Carlo (Giles & al 07 08 09), no coupling with strong order 1 between the schemes with \( N \) and 2\( N \) steps.
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Our scheme

We look for a simple scheme with order one of convergence for $\mathcal{W}_1$ and overcoming the previous restrictions.

Milstein scheme for $\tilde{Y}^N$: $\tilde{Y}^{N}_{t_0} = y_0$ and for $0 \leq k \leq N - 1$

$$\tilde{Y}^{N}_{t_{k+1}} = \tilde{Y}^{N}_{t_k} + b(\tilde{Y}^{N}_{t_k}) \frac{T}{N} + \sigma(\tilde{Y}^{N}_{t_k}) \Delta W_{k+1} + \frac{1}{2} \sigma \sigma'(\tilde{Y}^{N}_{t_k}) \left( \Delta W_{k+1}^2 - \frac{T}{N} \right).$$

$$dX_t = \rho dF(Y_t) + h(Y_t)dt + \sqrt{1 - \rho^2} f(Y_t)dB_t$$

$$\text{Var}(\int_{t_k}^{t_{k+1}} f(Y_s)dB_s | W) = \int_{t_k}^{t_{k+1}} f^2(Y_s)ds \approx \frac{f^2(Y_{t_k})T}{N} + \sigma f^2(Y_{t_k}) \int_{t_k}^{t_{k+1}} W_s - W_{t_k} ds$$

$$\tilde{X}^{N}_{t_{k+1}} = \tilde{X}^{N}_{t_k} + \rho \left( F(\tilde{Y}^{N}_{t_{k+1}}) - F(\tilde{Y}^{N}_{t_k}) \right) + h(\tilde{Y}^{N}_{t_k}) \frac{T}{N} + \sqrt{1 - \rho^2} \left( \frac{f^2(Y_{t_k})}{T} + \frac{N \sigma f^2(Y_{t_k})}{T} \int_{t_k}^{t_{k+1}} (W_s - W_{t_k}) ds \right) \vee f^2 \Delta B_{k+1}.$$
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$$+ \sqrt{1 - \rho^2} \sqrt{\left( f^2(\tilde{Y}^N_{t_k}) + \frac{N \sigma f^2'(\tilde{Y}^N_{t_k})}{T} \int_{t_k}^{t_{k+1}} (W_s - W_{t_k}) ds \right) \vee f^2 \Delta B_{k+1}}$$
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dX_t = \rho dF(Y_t) + h(Y_t) dt + \sqrt{1 - \rho^2} f(Y_t) dB_t
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\tilde{X}^N_{t_{k+1}} = \tilde{X}^N_{t_k} + \rho \left( F(\tilde{Y}^N_{t_{k+1}}) - F(\tilde{Y}^N_{t_k}) \right) + h(\tilde{Y}^N_{t_k}) \frac{T}{N}
\]

\[
+ \sqrt{1 - \rho^2} \sqrt{f^2(\tilde{Y}^N_{t_k}) + \frac{N \sigma f^2(\tilde{Y}^N_{t_k})}{T} \int_{t_k}^{t_{k+1}} (W_s - W_{t_k}) ds} \vee f^2 \Delta B_{k+1}
\]
Convergence

**Theorem 2**

Assume that $\forall N$, $\mathbb{R}^{2(N+1)}$ is endowed with the supremum norm,

- $b \in C^3_b$ and $\sigma \in C^4_b$ with $\inf_{y \in \mathbb{R}} \sigma(y) > 0$,
- $f \in C^4_b$ with $f^2 \overset{\text{def}}{=} \inf_{y \in \mathbb{R}} f^2(y) > 0$.

$$\exists C > 0, \forall N \in \mathbb{N}^*, \mathcal{W}_1 \left( \mathcal{L}((X_{t_k}, Y_{t_k})_{k \leq N}), \mathcal{L}((\widetilde{X}^N_{t_k}, \widetilde{Y}^N_{t_k})_{k \leq N}) \right) \leq \frac{C}{N}.$$  

Moreover, it is possible to couple the schemes with $N$ and $2N$ steps by simulating $(X^N_{t_k}, \widetilde{X}^{2N}_{t_k})_{k \leq N}$ with $(X^N_{t_k})_{k \leq N} \overset{\mathcal{L}}{=} (\widetilde{X}^N_{t_k})_{k \leq N}$ and

$$\forall p \geq 1, \exists C \geq 0, \forall N \in \mathbb{N}^*, \mathbb{E} \left( \max_{0 \leq k \leq N} |X^N_{t_k} - \widetilde{X}^{2N}_{t_k}|^{2p} \right) \leq \frac{C}{N^{2p}}.$$  

The coupling is useful in the perspective of multi-level Monte Carlo.
Coupling of the schemes with $N$ and $2N$ steps

Let $\delta = \frac{T}{2N}$ be the step size of the scheme with $2N$ steps.

$$
\tilde{X}^{2N}_{(j+1)\delta} = \tilde{X}^{2N}_j + \rho \left( F(\tilde{Y}^{2N}_{(j+1)\delta}) - F(\tilde{Y}^{2N}_{j\delta}) \right) + h(\tilde{Y}^{2N}_{j\delta}) \delta \\
+ \sqrt{1 - \rho^2} \sqrt{f^2(\tilde{Y}^{2N}_{j\delta}) + \frac{\sigma f^2(\tilde{Y}^{2N}_{j\delta})}{\delta} \int_{j\delta}^{j\delta+\delta} (W_s - W_{j\delta}) ds} \sqrt{\Delta B^{2N}_j} \sim \mathcal{N}_1(0, T/N) \text{ indep of } W
$$

Because of the independence of $\tilde{Y}^{2N}$ and $B$, $(\tilde{X}^N_{t_k})_{k \leq N} \overset{\mathcal{L}}{=} (X^N_{t_k})_{k \leq N}$ where

$$
X^N_{t_{k+1}} = X^N_{t_k} + \rho \left( F(\tilde{Y}^N_{t_{k+1}}) - F(\tilde{Y}^N_{t_k}) \right) + h(\tilde{Y}^N_{t_k}) \frac{T}{N} \\
+ \sqrt{1 - \rho^2} \sqrt{\nu^N_{2k}} \Delta B^{2N}_{2k} + \sqrt{\nu^N_{2k+1}} \Delta B^{2N}_{2k+1}
$$
1 Introduction

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4 Numerical results
Framework

Numerical experiments are performed with Scott’s model ($f(y) = e^y$, $Y$ OU)

$$
\begin{align*}
    dS_t &= rS_t dt + e^{Y_t} S_t (\rho dW_t + \sqrt{1 - \rho^2} dB_t) \\
    dY_t &= \kappa (\theta - Y_t) dt + \nu dW_t \\
    \Rightarrow f(y) &= e^y, \quad b(y) = \kappa (\theta - y) \text{ and } \sigma(y) = \nu
\end{align*}
$$

with the parameters found in Kahl & Jäckel 2006:

- $s_0 = 100$, $y_0 = \log(0.25)$,
- $r = 0.05$
- $\kappa = 1$, $\theta = 0$, $\nu = \frac{7\sqrt{2}}{20}$,
- $\rho = -0.2$,
- $T = 1$. 
Coupling at terminal time

Figure: $\log \left( \mathbb{E} \left( \left( e^{X_T^N} - e^{X_T^{2N}} \right)^2 \right) \right)$ in function of $\log(N)$. Except for CMT, $X_T^N$ and $X_T^{2N}$ are generated using the same single normal r.v. for the integral wrt $B$. 
Coupling at terminal time

<table>
<thead>
<tr>
<th></th>
<th>OU Improved</th>
<th>WeakTraj 1</th>
<th>Weak 2</th>
<th>IJK</th>
<th>Euler</th>
<th>CMT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope</td>
<td>-2.97</td>
<td>-2.02</td>
<td>-1.98</td>
<td>-1.95</td>
<td>-1.34</td>
<td>-1.08</td>
</tr>
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</table>

Table: Slope of the regression of \( \log (E \left( (e^{X^N_T} - e^{X^{2N}_T})^2 \right)) \) in terms of \( \log(N) \)

- order 3/2 for OU Improved,
- order 1 for WeakTraj 1, Weak 2 and Kahl Jäckel scheme IJK,
- order 1/2 CMT, slightly better for Euler.
Multi-level computation for the ATM Call

Figure: Computation time in function of the required precision $\varepsilon$
Strong convergence with coupling

**Figure:** $\log \left( \mathbb{E} \left( \max_{0 \leq k \leq N} \left( e^{X_{t_k}^N} - e^{X_{t_k}^{2N}} \right)^2 \right) \right)$ in function of $\log(N)$. Modified increments of $B$ for the scheme with $N$ steps.
Strong convergence with coupling

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<td>-0.95</td>
<td>-0.85</td>
</tr>
</tbody>
</table>

Table: Slope of the regression of $\log \left( \mathbb{E} \left( \max_{0 \leq k \leq N} \left( e^{X_{1k}^N} - e^{X_{2k}^{2N}} \right)^2 \right) \right)$ in terms of $\log(N)$. Modified increments of $B$ for the scheme with $N$ steps.

- order 1 for OU Improved and WeakTraj 1,
- order 1/2 for Weak 2, IJK and Euler,
- No coupling possible for CMT.
Multi-level computation for the Lookback Call

Figure: Computation time multiplied by the square of the required precision $\varepsilon$ in function of $\varepsilon$ (payoff $S_T - \min_{t \in [0,T]} S_t$)