SPEED OF CONVERGENCE OF THE THRESHOLD ESTIMATOR OF INTEGRATED VARIANCE

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THE FRAMEWORK

X = log price of a stock / index / foreign exchange rate, or
X = spot interest rate, univariate in this talk

AOA consistent model $\Rightarrow$ X SM

most SM mds used in finance: Itô SM

$$dX_t = a_t dt + \sigma_t dW_t + dJ_t, \quad t \in [0, T],\ X_0 \in \mathbb{R}$$

W standard Brownian motion
J pure jump SM with possibly IA jumps

J is said of finite activity (FA) if the paths jump finitely many times on each finite time interval, e.g. Compound proc. Poisson
J is said of infinite activity (IA) otherwise, e.g. α-stable proc.

$$J_t \equiv J_{1t} + \tilde{J}_{2t}$$

$$\doteq \int_0^t \int_{x \in \mathbb{R}, |\gamma_s(x)|>1} \gamma_s(x)\mu(ds, dx) + \int_0^t \int_{x \in \mathbb{R}, |\gamma_s(x)|\leq 1} \gamma_s(x)\tilde{\mu}(ds, dx),$$

$\tilde{\mu}(dt, dx) = \mu(dt, dx) - \nu_{\omega,t}(dx)dt$ compensated μ.
μ Poisson random measure, ν Lévy measure of J

X Ito SM: absolutely continuous characteristics/Lebesgue dt

Property: J_{1t} always FA, $\tilde{J}_{2t}$ possibly IA

Special case where J Lévy jumps: $\gamma_x(x) \equiv x,\ \nu_{\omega,t}(dx) \equiv \nu(dx)$

Observations: $\{x_0, X_{t_1}, ..., X_{t_{n-1}}, X_T\}$, $\{t_i = i\Delta\}_i$, partition of $[0, T]$, $T = n\Delta$ fixed

PROBLEM

Estimation of $IV \doteq \int_0^T \sigma_s^2 ds$

FIRST ISSUE: Model class uncertainty

has the drift part a specific feature (mean reverting/parametric ...)?

has the volatility coeff. e.g. exponential mean reverting dynamics?

have the jump sizes a specific law?

is the jump component necessary? numerous tests devised in the literature in this framework (starting from Barndorff-Nielsen & Shephard 2006) find empirical evidence of jumps in some assets

is the IA jump component necessary? Lee & Hannig (2010), Ait-Sahalia & Jacod (in press) find empirical evidence of IA jumps in some assets

is the Brownian component necessary? Cont & Mancini (in press), Ait-Sahalia & Jacod (2010) find empirical evidence of it in some assets, CGMY (2002) estimate W is absent in some others

NONPARAMETRIC ESTIMATORS are desirable

Our approach: *any* a, σ nonanticipative cadlag processes

these include most of the models used in finance (diffusions, jump-diffusions, stochastic volatility models with jumps, Lévy models, etc.)

exclude e.g. fractional BM (no SM), Multi fractal models (SM no Itô)

Fine measure of the amount of activity of the jump part J

For any Lévy process we have

$$\int_{|x|\leq 1} x^2 \nu(dx) < +\infty$$

however for powers $\eta < 2$ the integral can be ∞, meaning many jumps less than 1 in absolute value.

Blumenthal-Getoor index (BG) of J:

$$\alpha \doteq \inf\{\eta : \int_{|x|\leq 1} x^\eta \nu(dx) < +\infty\} \in [0, 2]$$

α measures the amount of jump activity

$\alpha > 0 \Rightarrow$ IA jumps, meaning $\int_{|x|\leq 1} \nu(dx) = +\infty$ jump frequency per unit time

The only Lévy process with FA jumps is compound Poisson

Examples

compound Poisson pr., Gamma pr., Variance Gamma pr. $\Rightarrow \alpha = 0$

α-stable pr. $\Rightarrow$ BG index $= \alpha$

NIG pr., Generalized Hyperbolic Lévy motion $\Rightarrow \alpha = 1$.

CGMY model $\Rightarrow$ BG index $= Y$

$\alpha < 1 \Rightarrow J$ finite variation (fV), meaning $\int_{|x|\leq 1} |x|\nu(dx) < +\infty$
$\alpha > 1 \Rightarrow J$ infinite variation (iV)

Generalizations of BG index for SM have been devised in the recent literature (Woerner 2006, Ait-Sahalia & Jacod 2009, Todorov & Tauchen 2010)

Notation.

$\Delta_i Z \doteq Z_{t_i} - Z_{t_{i-1}}$ increment of Z on $]t_{i-1}, t_i]$

$\Delta J_t \doteq J_t - J_{t-}$ size of the jump (eventually) occurred at time t

$IV = \int_0^T \sigma_u^2 du$ integrated variance

$IQ \doteq \int_0^T \sigma_u^4 du$ integrated quarticity

ESTIMATING IV

When no jumps

$$dX_t = a_t dt + \sigma_t dW_t$$

then as $\Delta \to 0$

$$\sum_{i=1}^{n} (\Delta_i X)^2 \xrightarrow{P} \int_0^T \sigma_u^2 du.$$

However when

$$dX_t = a_t dt + \sigma_t dW_t + dJ_t$$

then

$$\sum_i (\Delta_i X)^2 \xrightarrow{P} \int_0^T \sigma_u^2 du + \sum_{t \le T} (\Delta J_t)^2.$$

How to disentangle diffusion part / jump part?

MOTIVATION

$\sum_i (\Delta_i X)^2$ is a measure of the *global* risk affecting the asset. Separating is needed for:

1. Hedging

Bjork, Kabanov & Runggaldier (1997), Andersen, Bollerslev & Diebold (2007): show that, in the dynamics of a pf, Brownian risk and jump risk are amplified by different coeff. $\Rightarrow$ need of capturing them separately

consequently

* different risk premiums for W and J risks: $I\hat{V}$ allows premiums assessment Wright & Zhou (2007)
* portfolio selection (Mykland & Zhang, 2006)
* derivatives pricing (Duffie, Pan & Singleton, 2000)

2. Model selection

* $\hat{IV}$ used for testing the presence of jumps (Barndorff-Nielsen & Shephard (2006))
* $\hat{IV}$ used for testing the presence of $\sigma.W$ (Cont & Mancini, in press)

3. Volatility forecasting

* including separate IV and $[J]_T$ contributions in econometric models for X evolution improves the forecasting ability (Andersen, Bollerslev & Diebold, 2007, Corsi, Pirino & Renò, 2010)

LITERATURE

Non-parametric $\widehat{IV}$ estimators based on discrete observations in our framework

In the presence of only FA jumps

* Quantile based Bipower variation (Christensen, Oomen & Podolskij, 2010)

* MinRV, MedRV (Andersen, Dobrev & Schaumburg, 2009)

* Realized outlyingness weighted variation (Boudt, Croux & Laurent, 2010)

* Range based estimation (Christensen & Podolskij, 2009)

* Generalized range (Dobrev, 2007)

* Duration based estimation (Andersen, Dobrev & Schaumburg, 2009)

* Wavelet method (Fan & Wang, 2008)

In the presence of also IA jumps

bipower or **multipower variations** of Barndorff-Nielsen & Shephard (2006)

threshold estimator of Mancini (2001)

threshold-bipower estimator: Corsi Pirino Renò (2010), Vetter (2010)

The only efficient estimator (minimal asymptotic variance) in the presence of IA fV jumps is the Threshold estimator

Simulations

MODEL 1. FA J, stochastic σ correlated with W:

$$dX_t = \sigma_t dW_t^{(1)} + dJ_t,$$

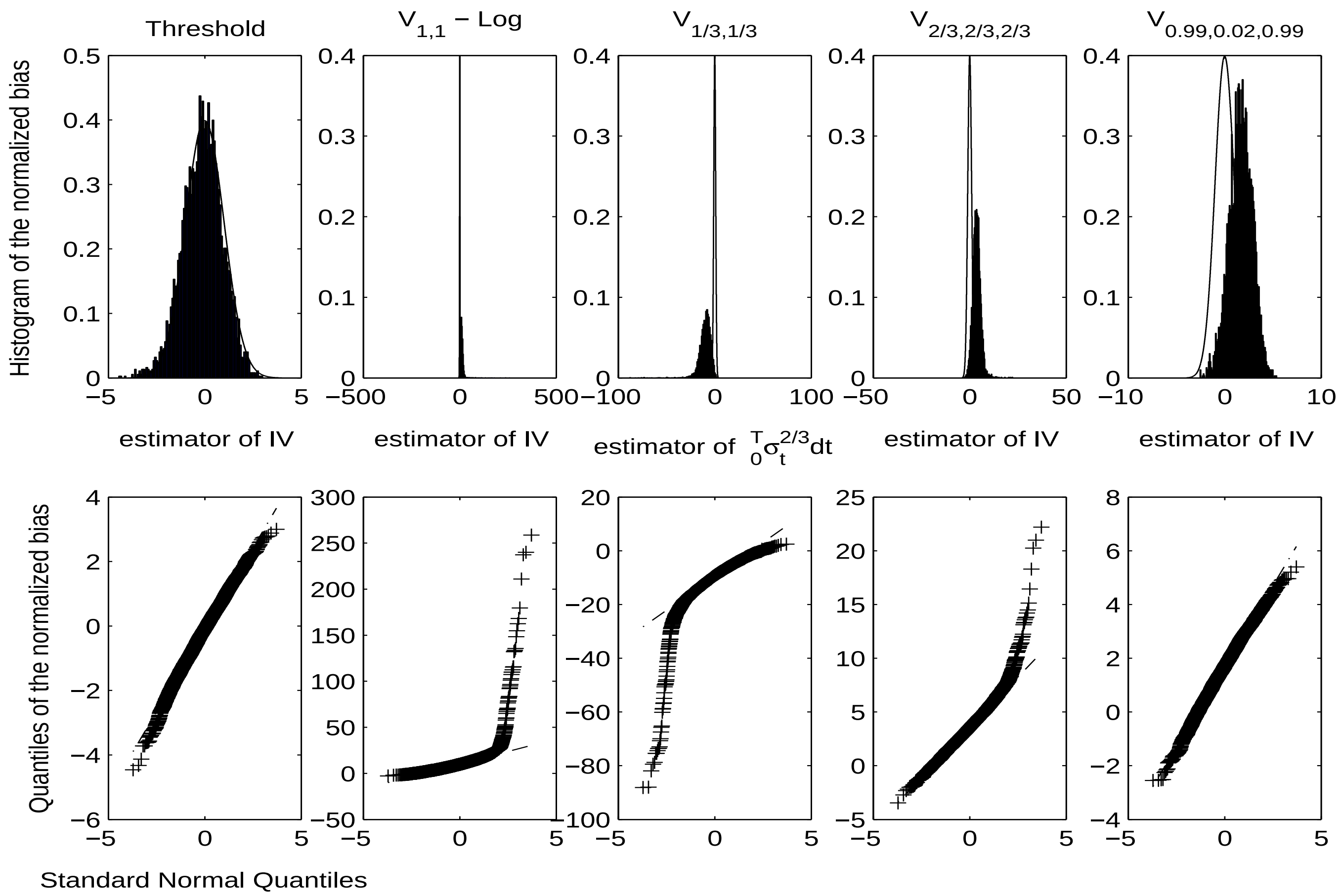
$J_t = \sum_{j=1}^{N_t} Z_j, \quad Z_j \sim \mathcal{N}(0, 0.6^2)$, N Poisson, $\lambda = 5$

$\sigma_t = e^{H_t}, \quad dH_t = -k(H_t - \bar{H})dt + \nu dW_t^{(2)}, \quad d < W^{(1)}, W^{(2)} >_t = \rho dt.$

$\rho = -0.7$ (SVJ1F model of Huang & Tauchen, 2005)

a path of σ within $[0, T]$ varies most between 10% and 50%

relative amplitudes of the jumps of S, in absolute value, most between 0.01 and 0.60



$$\frac{I\hat{V} - IV}{\sqrt{\Delta}\sqrt{A\hat{V}ar}}$$

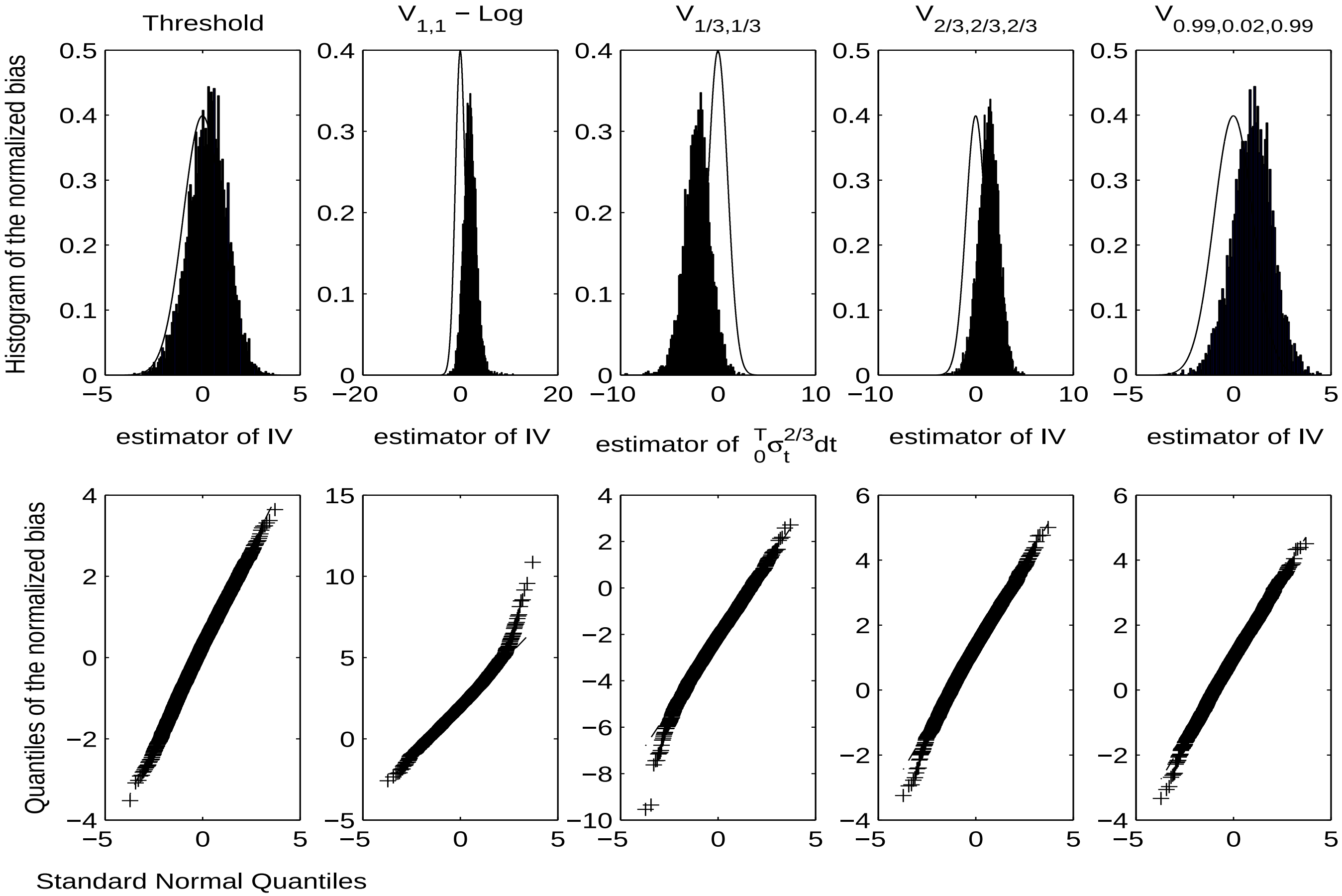
histograms and QQ-plots: $n = 1000$, $T = 1$, $\Delta = 1/n$, $r(\Delta) = \Delta^{0.99}$

MODEL 2. IA and fV J, constant σ

$$dX_t = 0.3dW_t + dJ_t,$$

$J_t = cG_t + \eta B_{G_t}$ Variance Gamma process: B std BM $\perp$ G Gamma process

$b = Var(G_1) = 0.23$, $c = -0.2$, $\eta = 0.2$, Madan (2001) model



$$\frac{I\hat{V} - IV}{\sqrt{\Delta}\sqrt{A\hat{V}ar}}$$

histograms and QQ-plots

Threshold method basic tool: Identification of the jump times

1) FINITE JUMP ACTIVITY: $J_t \equiv J_{1t} = \sum_{j=0}^{N_t} \gamma_{\tau_j}$

KEY THEOREM

$r(\Delta)$ deterministic function of the step Δ such that

$$\lim_{\Delta \to 0} r(\Delta) = 0, \text{ and } \lim_{\Delta \to 0} \frac{\Delta \log \frac{1}{\Delta}}{r(\Delta)} = 0, \text{ then}$$

P-a.s. $\exists \bar{\Delta}(\omega) > 0$ s.t. $\forall \Delta \leq \bar{\Delta}(\omega)$ we have $\forall i = 1, ..., n,$

$$I_{\{\Delta_i N = 0\}}(\omega) = I_{\{(\Delta_i X)^2 \leq r(\Delta)\}}(\omega)$$

a jump occurred iff $I_{\{(\Delta_i X)^2 > r(\Delta)\}}(\omega)$.

Why? (idea)

the increments of a BM tend a.s. to zero as the deterministic function $\sqrt{2\Delta \ln \frac{1}{\Delta}}$:

$$\text{a.s.} \qquad \lim_{\Delta \to 0} \sup_{i \in \{1,\dots,n\}} \frac{|\Delta_i W|}{\sqrt{2\Delta \log \frac{1}{\Delta}}} \leq 1.$$

The stochastic integral $\sigma.W$ is a time changed BM $\Rightarrow$

$$\text{a.s.} \qquad \sup_{i \in \{1,\dots,n\}} \frac{|\Delta_i \sigma.W|}{\sqrt{2\Delta \log \frac{1}{\Delta}}} \leq M(\omega) < \infty, \qquad M(\omega) = \sup_{s \in [0,T]} |\sigma(\omega)| + 1$$

drift part negligible

$$\Downarrow$$

a.s. for small Δ, if $(\Delta_i X)^2$ is larger than $r(\Delta) > 2\Delta \ln \frac{1}{\Delta}$, it is likely that some jumps occurred.

2) INFINITE ACTIVITY JUMPS

$$J_t \equiv J_{1t} + \tilde{J}_{2t}$$

$$\doteq \int_0^t \int_{x \in \mathbb{R}, |\gamma(x)|>1} \gamma(x)\mu(ds, dx) + \int_0^t \int_{x \in \mathbb{R}, |\gamma(x)| \leq 1} \gamma(x)\tilde{\mu}(ds, dx),$$

we have

$$I_{\{(\Delta_i X)^2 \leq r(\Delta)\}} \approx I_{\{\Delta_i J_1 = 0, (\Delta_i \tilde{J}_2)^2 \leq 4r(\Delta)\}}$$

$I_{\{(\Delta_i X)^2 > r(\Delta)\}}$ accounts for
the FA jumps and the IA jumps bigger than $2\sqrt{r(\Delta)}$

Estimate of IV, general SM jumps

$$\hat{IV}_n = \sum_{i=1}^{n} (\Delta_i X)^2 I_{\{(\Delta_i X)^2 \le r(\Delta)\}}$$

THEOREM

As soon as $\int_{x\in\mathbb{R}} 1 \wedge \gamma^2(x,\omega,t)dx$ is locally bounded, then as $\Delta \to 0$

$$\hat{IV} \xrightarrow{P} \int_0^T \sigma_t^2 dt.$$

Remarks.

- $r(\Delta) = \Delta^\beta$ for any $\beta \in]0,1[$ satisfies the conditions on $r(\Delta)$

- not evenly spaced observations: $(\Delta_i X)^2$ vs $r(\max_i \Delta t_i)$ or, equivalently, $(\Delta_i X)^2$ vs $r(\Delta t_i)$

- When J FA, we estimate the jump times

$$\hat{N}_t^{(n)} \doteq \sum_{i:\ t_i \le t} I_{\{(\Delta_i X)^2 > r(\Delta)\}},$$

  consistent as $T \to \infty$ and $\Delta \to 0$, and jump sizes

$$\hat{\gamma}^{(i)} \doteq \Delta_i X I_{\{(\Delta_i X)^2 > r(\Delta)\}}$$

  $\forall i$ is a consistent estimate of $\gamma^{(i)}$, the first jump size (if $\Delta_i N \ge 1$) on $]t_{i-1}, t_i]$

Speed of convergence of $\hat{IV}_n$

J Lévy process, jump measure $\mu(dx, dt)$, Lévy measure $\nu(dx)$

Assume *Let* $\exists \alpha \in [0, 2]$:

$$\int_{|x| \leq \varepsilon} x^2 \nu(dx) \sim \varepsilon^{2-\alpha}, \text{ as } \varepsilon \to 0,$$

Then α *BG index of J*

compound Poisson, Gamma, VG, NIG, Stable, CGMY processes satisfy the condition

Assume $\sigma \not\equiv 0$, $r(\Delta) = \Delta^\beta$, $\beta \in]0, 1[$

then

• if $\alpha < 1$ then for β sufficiently large $\left(\beta > \frac{1}{2-\alpha} \in [1/2, 1[\right)$

$$\frac{\widehat{IV} - \int_0^T \sigma_t^2 dt}{\sqrt{2\Delta \widehat{IQ}}} \xrightarrow{d} \mathcal{N}(0, 1)\,;$$

where for any $\alpha \in [0, 2]$

$$\widehat{IQ} \doteq \frac{1}{3} \frac{\sum_i (\Delta_i X)^4 I_{\{(\Delta_i X)^2 \leq r(\Delta)\}}}{\Delta} \xrightarrow{P} IQ = \int_0^T \sigma_t^4 dt.$$

Thus $AVar = 2IQ$

• if $\alpha \geq 1$ then, for any $\beta \in]0, 1[$,

$$\frac{\widehat{IV} - \int_0^T \sigma_t^2 dt}{\sqrt{2\Delta \widehat{IQ}}} \xrightarrow{P} +\infty,$$

Remark. Consistent result with Jacod (2008) where J more general jump component but σ Itô SM

However: Ait-Sahalia & Jacod (2008) find Fisher information for IV in the case J Lévy and argue that minimal converging rate to estimate IV still is $\sqrt{\Delta}$ when J has iV

Thus: for $\alpha > 1$ inefficient estimator

How far is the Threshold estimator form efficiency?

Assumption J is symmetric α-stable.

Theorem Take $r(\Delta) = c\Delta^{\beta}$, $\beta \in]0,1[$, $c \in \mathbb{R}$. Then as $\Delta \to 0$

$$\hat{IV} - IV \stackrel{P}{\sim} \sqrt{\Delta} Z_{\Delta} + r(\Delta)^{1-\alpha/2}, \tag{1}$$

where $Z_{\Delta} \stackrel{st}{\to} \mathcal{N}$, and $\mathcal{N}$ denotes a standard normal random variable.

Remark. The term $\sqrt{\Delta} Z_{\Delta}$ is due to the presence of W within X

$r(\Delta)^{1-\alpha/2}$ is led by the sum of the jumps of X smaller in absolute value than $\sqrt{r(\Delta)}$.

COMPLETE PICTURE

$$\frac{\hat{IV}_\Delta - IV}{\sqrt{2\Delta\ IQ}} \xrightarrow{st} \mathcal{N} \qquad \text{if } \sigma \not\equiv 0 \text{ and } \alpha < 1, \beta > \frac{1}{2-\alpha}$$

$$\hat{IV}_\Delta - IV \overset{P}{\sim} r(\Delta)^{1-\alpha/2} \qquad \begin{array}{l} \text{if } \sigma \equiv 0 \text{ or} \\ \text{if } \sigma \not\equiv 0 \text{ and } \alpha < 1, \beta \leq \frac{1}{2-\alpha} \text{ or} \\ \text{if } \sigma \not\equiv 0 \text{ and } \alpha \geq 1 \end{array}$$

OPEN PROBLEM

finding non-parametric efficient estimator of IV in the presence of iV jumps

It is important because:
* we have empirical findings that iV jumps can occur
* much more precise risks estimates would be available

IF NEEDED

bipower variation of X

$$V_{r,s}(X) := \sum_{i=2}^{n} |\Delta_i X|^r |\Delta_{i-1} X|^s$$

THEOREM (Woerner, 2006)

if $\alpha < 1$; X has no drift part; σ is càdlàg, a.s. strictly positive, has paths regular enough and is independent of W; if $r, s > 0$, $\max(r, s) < 1$ and $r + s > \alpha/(2 - \alpha)$ then as $\Delta \to 0$

$$\frac{\Delta^{1-r/2-s/2} V_{r,s}(X) - \mu_s \mu_r \int_0^T \sigma_u^{r+s} du}{\sqrt{\Delta}\sqrt{C_{BPV} \int_0^T \sigma_u^{2r+2s} du}} \xrightarrow{d} \mathcal{N}(0, 1),$$

where $C_{BPV} = \mu_{2r}\mu_{2s} + 2\mu_r\mu_s\mu_{r+s} - 3\mu_r^2\mu_s^2$, $\mu_r = E[|Z|^r]$, $Z \sim \mathcal{N}(0, 1)$

e.g. $r = s = 1 \Rightarrow C_{BPV} = 2.6$

$\inf_{r,s \leq 2} C_{BPV} = 2$ when $r = 0, s = 2$ (but then BPV does not estimate IV in the presence of jumps) *multipower variation*

$$V_{r_1,..,r_k}(X) := \sum_{i=k}^{n} |\Delta_i X|^{r_1} |\Delta_{i-1} X|^{r_2} ... |\Delta_{i-k} X|^{r_k}.$$

THEOREM

if $\alpha < 1$; if $r_i > 0$ for all $i = 1..k$, $\max_i r_i < 1$ and $\sum_i r_i > \alpha/(2-\alpha)$ then as $\Delta \to 0$

$$\frac{\Delta^{1-\sum_i r_i/2}\ V_{r_1,..,r_k}(X) - \mu_{r_1}\mu_{r_2}\cdots\mu_{r_k}\int_0^T \sigma_u^{\sum_i r_i} du}{\sqrt{\Delta}\sqrt{C_{MPV}\int_0^T \sigma_u^{2\sum_i r_i} du}} \xrightarrow{d} \mathcal{N}(0,1),$$

$$C_{MPV} = \prod_{p=1}^{k} \mu_{2r_p} + 2\sum_{i=1}^{k-1}\prod_{p=1}^{i}\mu_{r_p}\prod_{p=k-i+1}^{k}\mu_{r_p}\prod_{p=1}^{k-i}\mu_{r_p+r_{p+i}} - (2k-1)\prod_{p=1}^{k}\mu_{r_p}^2.$$

The integrals at denominators can be estimated using in turn the multipower variations.

Statistics of the considered normalized biases: MD1

pct is the percentage of the 5000 realizations for which the normalized bias is in absolute value larger than 1.96 (asymptotic value 0.05).

mean and *StDev* are the mean and the standard deviation of the 5000 values assumed by the normalized bias of each estimator (asymptotic values 0 and 1).

	Threshold	$V_{1,1}$-Log	$V_{1/3,1/3}$	$V_{2/3,2/3,2/3}$	$V_{0.99,0.02,0.99}$
pct	0.0558	0.9586	0.4122	0.8056	0.9624
mean	-0.1260	-10.1758	1.6686	3.6479	11.3959
StDev	1.0235	6.9555	1.1895	2.0914	12.1050

	MD2				
	Threshold	$V_{1,1}$-Log	$V_{1/3,1/3}$	$V_{2/3,2/3,2/3}$	$V_{0.99,0.02,0.99}$
pct	0.0536	0.5402	0.1606	0.2760	0.5126
mean	0.2570	-2.1264	0.9677	1.3172	2.0289
StDev	0.9850	1.3110	1.0143	1.0433	1.3096