Modelling Extreme Dependence for Multivariate Data

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Motivation

Investigate notion of 'multivariate' dependence and extreme multivariate dependence.

- univariate dependence = dependence between real r.v $\rightarrow$ copula framework applies.
- multivariate dependence = dependence between two random *vectors* / multivariate laws of probability.

As with bivariate copulas, marginals laws p and q are fixed ; but they are multivariate, i.e. laws on $\mathbf{R}^n$.
A *coupling* between p and q is a law of a couple (X, Y) with $X \sim p$, $Y \sim q$.
The set of all couplings between p and q is denoted $\Pi(p, q)$.

Introduction

In the univariate case, the strongest dependence between two random variables is given by upper Fréchet Copula:

$$C(u_1, u_2) = min(u_1, u_2)$$

A couple (X, Y) exhibiting upper Fréchet dependence maximizes the covariance

$$\mathbf{E}(XY) = \sup_{\substack{\tilde{X}\sim X \\ \tilde{Y}\sim Y}} \mathbf{E}(\tilde{X}\tilde{Y})$$

In higher dimensions, there is no notion of copula between multivariate vectors: no 'natural' notion of Fréchet multivariate dependence exists.

Introduction 2

One possible extension: *maximum correlation coupling* is the coupling π s.t.

$$\mathbf{E}_\pi(X'Y) = \sup_{\substack{\tilde{X}\sim X \\ \tilde{Y}\sim Y}} \mathbf{E}(\tilde{X}'\tilde{Y})$$

When p and q do not charge small sets, there exists a unique gradient of convex function $\nabla\varphi$ such that $Y = \nabla\varphi(X)\,a.s.$ In general, there exists a convex l.s.c function φ such that $Y \in \partial\varphi(X)$ a.s.

This is not fully satisfactory as it involves only component-wise covariances; the notion of cross dependence is not accounted for.

Our goal is to define a more general notion of extreme dependence that yields more extremally dependent couplings.

More on cross-covariance

As with the maximum correlation coupling, our solution involves second order cross-moments of X and Y: the object of interest is the cross-covariance matrix of (X, Y). As a single vector in $\mathbf{R}^{2n}$, its covariance matrix is

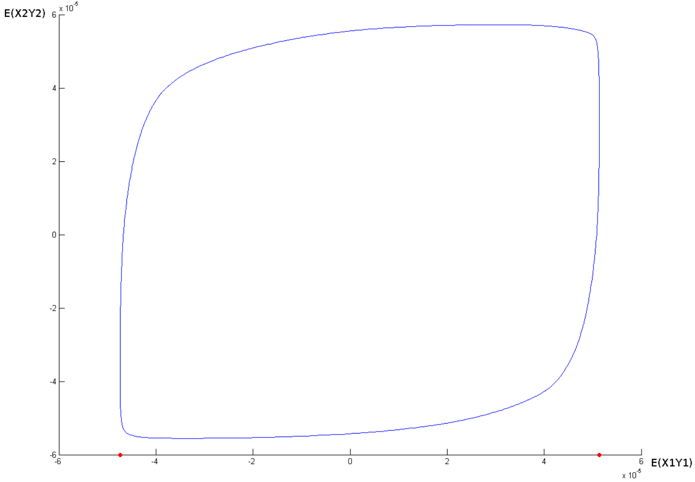
$$Cov((X,Y)) = \left(\begin{array}{c|c} Cov(X) & \mathbf{E}(XY') \\ \hline \mathbf{E}(XY')' & Cov(Y) \end{array} \right)$$

- The diagonal blocks are known (they do not depend on the coupling).
- For convenience we write $\sigma_{X,Y} = \mathbf{E}(XY') = (\mathbf{E}(X_i Y_j))_{i,j}$.
- Example: if $p = q = \mathcal{N}(0, Id_2)$, $X = Y \Rightarrow \sigma_{X,Y} = Id_2$.

Projecting couplings in the plane

- A simple example is to consider two bivariate laws $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ and $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$.
- One can project any coupling between X and Y in the plane by considering the coordinates $(\mathbf{E}(X_1 Y_1), \mathbf{E}(X_2 Y_2))$.
- We obtain the image of $\Pi(p, q)$: this is the set of attainable covariances, called the *covariogram*.

Attainable covariances and Covariogram



More generally, the *covariogram* $\mathcal{C}(p, q)$ is the set $\{\sigma_\pi : \pi \in \Pi(p, q)\}$.

Extreme couplings induced from the covariogram

Definition of Extreme Couplings

A coupling π has extreme dependence if σ_π lies on the boundary $\partial\mathcal{C}(p,q)$ of the covariogram.

A variational characterization of extremal couplings

The following conditions are equivalent:
i) $(X,Y) \sim \pi \in \Pi(p,q)$ have extreme dependence;
ii) there exists $M \in \mathbf{M}_n(\mathbb{R})\backslash\{0\}$ such that

$$Tr\left(M'\sigma_\pi\right) = \sup_{\pi' \in \Pi(p,q)} Tr\left(M'\sigma_{\pi'}\right) \quad (1)$$

or equivalently $\mathbf{E}_\pi(X'MY) = \sup_{\pi' \in \Pi(p,q)} \mathbf{E}_{\pi'}(X'MY)$
iii) there exists $M \in \mathbf{M}_n(\mathbb{R})\backslash\{0\}$ and a convex function u on $\mathbb{R}^n$ such that $M.Y \in \partial u(X)$ holds almost surely.

$\rightarrow$ There are thus many extremally dependent couplings.

Conic orders on cross-covariance matrices

If C is a compact *basis* (convex set such that $0 \notin C$) in $M_n(\mathbf{R})$ then

$$K(C) = \{y \in \mathbf{M}_n(\mathbb{R}) | Tr(x'y) \geq 0, \forall x \in C\}$$

is a closed convex cone.
A conic strict (partial) order is defined on $\mathbf{M}_n(\mathbb{R})$ by

$$A \succ_C B \text{ if } A - B \in Int(K(C))$$

Example: Positive Definite Order
$C = \{S \in S_n^+(\mathbb{R}) | Tr(S) = 1\}$, $K(C)$ is the set of matrices M whose symmetric part, $\frac{M+M'}{2}$ is semi-definite positive.
Problem: Which couplings π yield a σ_π maximal for $\succ_C$?
We say that these couplings exhibit *positive extreme dependence with respect to* $\succ_C$.
$\rightarrow$ for instance the maximal correlation coupling has positive extreme dependence with respect to $\succ_C$ whenever $Id \in C$.

Main result on maximizing cross-covariance matrices

Variational characterization of positive extreme dependence

The following conditions are equivalent:
i) $(X, Y) \sim \pi \in \Pi(p, q)$ have extreme positive dependence with respect to $\succ_C$;
ii) there exists $M \in C$ such that

$$Tr\left(M'\sigma_\pi\right) = \sup_{\pi' \in \Pi(p,q)} Tr\left(M'\sigma_{\pi'}\right) \qquad (2)$$

or equivalently $\mathbf{E}_\pi(X'MY) = \sup_{\pi' \in \Pi(p,q)} \mathbf{E}_{\pi'}(X'MY)$;
iii) there exists $M \in C$ and a convex function u such that $M.Y \in \partial u(X)$ holds almost surely.

Example: $p = \mathcal{N}(0, I_2)$ and $q = \mathcal{N}(0,1) \otimes \mathcal{U}_{(0,1)}$.
$(X, (X_1, U))$, $U \sim \mathcal{U}_{(0,1)}$ independent from (X_1, X_2) is not the maximum correlation coupling but satisfies (2) with $M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Covariogram and positive extreme couplings

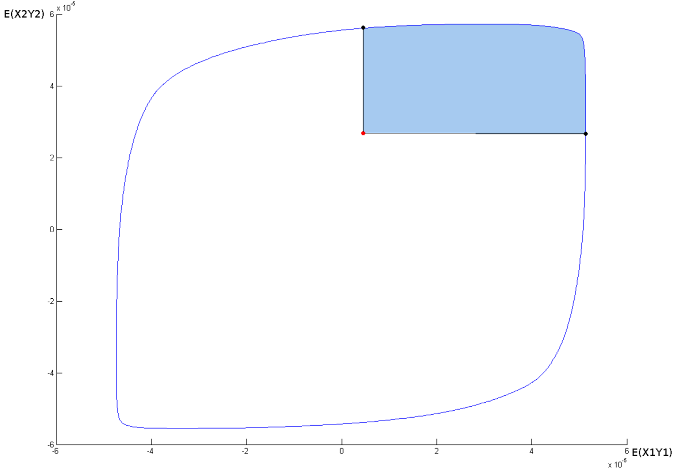


Figure: Positive orthant order

Covariogram and positive extreme couplings (2)

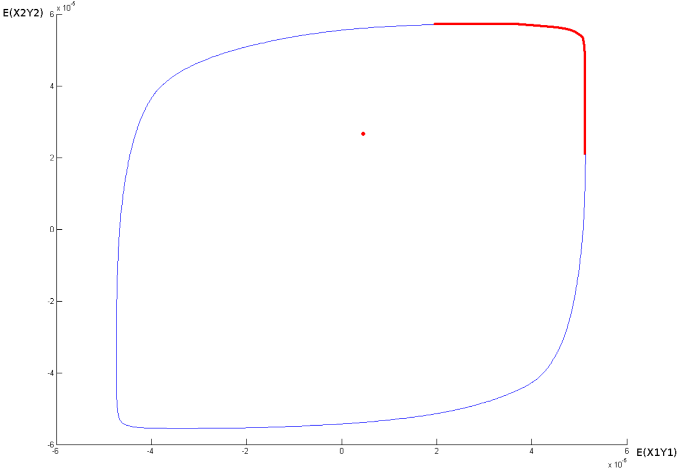


Figure: Location of positive extreme couplings

Finding extreme couplings: entropic relaxation

The problem $\mathbf{E}_\pi(X'MY) = \sup_{\pi' \in \Pi(p,q)} \mathbf{E}_{\pi'}(X'MY)$ is relaxed by entropic penalization:

$$W(M,T) = \max_{\pi} \mathbf{E}(X'MY) + TEnt(\pi) \tag{3}$$

$Ent(\pi)$ is the *entropy* of π, defined as $-\mathbf{E}_\pi\big(\log(\pi(X,Y))\big)$.
Homogeneity in (M,T): we set the temperature at 1. We then aim at finding $\hat{M}$ s.t

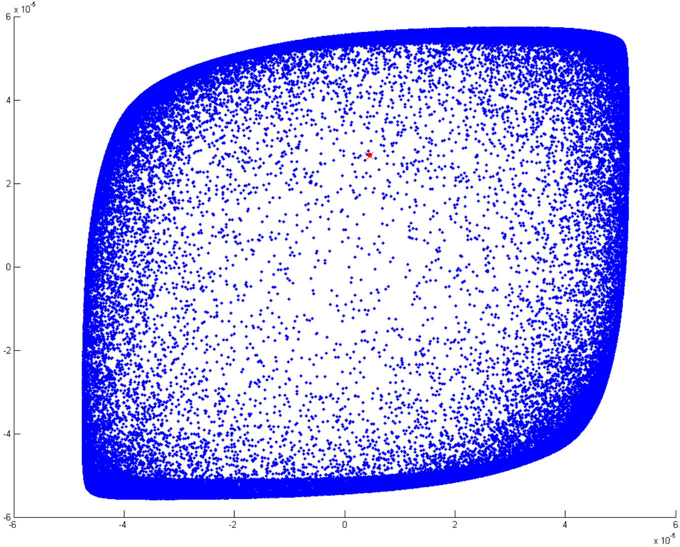
$$\sigma_{\hat{\pi}} = \sigma_{\pi(\widehat{M})} \text{ where } \pi(M) \text{ is a solution of solution of } (3)$$

$\hat{M}$ is called the *affinity matrix* of $\hat{\pi}$.
As $\nabla_M W(M,1) = \sigma_{\pi(M)}$, it thus amounts to solve :

$$\min_{M} W(M,1) - \sigma_{\hat{\pi}} \cdot M \tag{4}$$

which is a *convex minimization problem*.

Filled covariogram



Algorithm : Iterative Proportional Fitting Procedure

The solution π of (3) can be shown to take the form:

$$\log \pi(x,y) \quad = x'My + u(x) + v(y), \quad u \in L^1(dp), v \in L^1(dq)$$

u and v must be adjusted so that $\pi \in \Pi(p,q)$.
This is the purpose of IPFP (Deming & Stephan 1940, Von Neumann 1950). Recursion scheme :

$$\begin{cases} e^{u_{n+1}(x)} & = \quad \dfrac{p(x)}{\int e^{x'My+v_n(y)}dy} \\ e^{v_{n+1}(x)} & = \quad \dfrac{q(y)}{\int e^{x'My+u_{n+1}(x)}dx} \end{cases}$$

- $\pi_{2n} \propto e^{x'My+u_n(x)+v_n(y)}$ has first marginal p
- $\pi_{2n+1} \propto e^{x'My+u_n(x)+v_{n+1}(y)}$ has second marginal q
- $\pi_n \to \pi \in \Pi(p,q)$ in total variation

Examples with sector indices

We consider the time series of daily returns on S&P 500 and DJ EUROSTOXX subsectors: construction, health care and financials.

We introduce $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$, $Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$ with,

$$\begin{aligned} X_1 = &\quad \text{return on the S\&P 500 construction sector} \\ X_2 = &\quad \text{return on the S\&P 500 health care sector} \\ X_3 = &\quad \text{return on the S\&P 500 financial sector} \end{aligned}$$

Y is defined in the same manner for the DJ Eurostoxx.

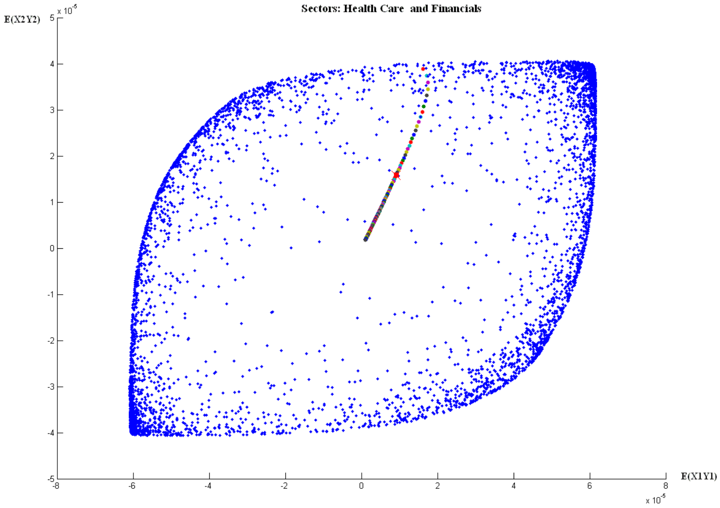
Multivariate discrete laws are defined from historical data X_t and Y_t:

$$p = \frac{1}{N}\sum_{t=1}^{N}\delta_{X_t} \quad , \quad q = \frac{1}{N}\sum_{t=1}^{N}\delta_{Y_t}$$

For historical data spanning 5 years between september 2004 and september 2009 one gets the following results :

# of components	2	3
$\widehat{M}$	$\begin{pmatrix} 0.23 & -0.14 \\ -0.10 & 0.40 \end{pmatrix}$	$\begin{pmatrix} 0.25 & -0.139 & -0.37 \\ -0.39 & 0.44 & -0.80 \\ -0.57 & -0.15 & 0.86 \end{pmatrix}$
error $= \frac{\lvert\lvert\sigma_{\hat{M}} - \sigma_{\hat{\pi}}\rvert\rvert}{\lvert\lvert\sigma_{\hat{\pi}}\rvert\rvert}$	$\approx 0.1\%$	< 0.2 %

Example of trajectory



Analysis of the optimal coupling

The empirical coupling $\hat{\pi}$ is thus associated with a $\pi(\widehat{M}, T = 0)$ which maximizes

$$\mathbf{E}_{\pi}(X'\hat{M}Y), \ \pi \in \Pi(p, q)$$

Singular value decomposition on $\hat{M}$ yields $\hat{M} = USV'$ where U, V are unitary and S diagonal nonnegative.
$(\tilde{X}, \tilde{Y}) = (\sqrt{S}U'X, \sqrt{S}VY)$ has maximum correlation.
Ex: with 3 components, we obtain

$$\begin{aligned}
\tilde{X} &= \begin{pmatrix} -0.42X_1+0.95X_2-0.019X_3 \\ -0.64X_1-0.27X_2+0.26X_3 \\ 0.11X_1+0.06X_2+0.35X_3 \end{pmatrix} \\
\tilde{Y} &= \begin{pmatrix} -0.30Y_1+0.99Y_2-0.13Y_3 \\ -0.67Y_1-0.16Y_2+0.28Y_3 \\ 0.12Y_1+0.08Y_2+0.34Y_3 \end{pmatrix}
\end{aligned}$$

$\rightarrow$ two *indices* most correlated to one another.

Application to dependence stress testing

The dependence between two set of assets X and Y, with cross-covariance matrix σ, can be stressed:

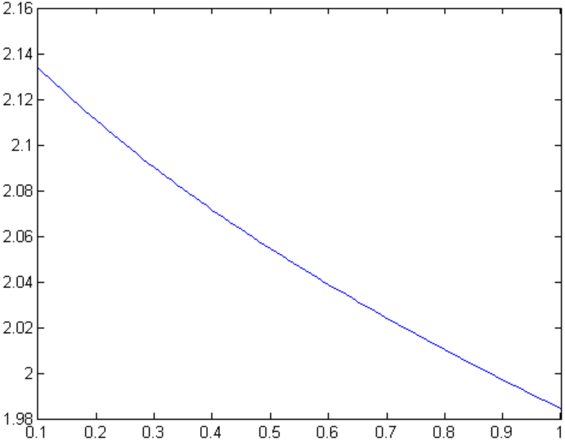
1. based on empirical data, the dependence between X and Y is stressed by building a sequence σ_T with $T \to 0$.
2. by considering the maximum correlation coupling
3. by considering the cross-corrrelation matrices $corr_\rho = \begin{pmatrix} \rho & \dots & \rho \\ \vdots & & \vdots \\ \rho & \dots & \rho \end{pmatrix}$ yielding cross-covariance matrices σ_ρ and letting $\rho \to 1$.

Example: An investor solves the mean-variance allocation problem

$$max_\omega \mu'\omega - \frac{\lambda}{2}\omega'\Sigma\omega$$

yielding $w_{opt} = \frac{1}{\lambda}\Sigma^{-1}\mu$. The matrix Σ is stressed into $\tilde{\Sigma}$, generating the unexpected variance $w'_{opt}\tilde{\Sigma}w_{opt}$.

Covariance stress



Plot of $T \mapsto w'_{opt}\Sigma_T w_{opt}$

Results

1. The investor faces a 7.5% increase of the variance as $T \to 0$.
2. Maximum Correlation coupling: the variance is 12% *lower* than the expected variance $w'_{opt}\Sigma w_{opt}$.
3. σ_ρ: admissibility problem. The resulting covariance matrix $\begin{pmatrix} Cov(X) & \sigma_\rho \\ \sigma'_\rho & Cov(Y) \end{pmatrix}$ *is not necessarily semidefinite positive*. In this example, $\rho_{max} = 74\%$ and it yields a variance that is 13% lower than the maximal variance found with the first method.

$\to$ The maximum correlation coupling might not be a proper means to increase dependence as it disregards the cross dependence.
$\to$ The proposed method has the advantage of providing admissible covariance matrices.
The same ideas apply to pricing of multi-underlyings european options.

Conclusion

- We proposed a new notion of extreme dependence in the multivariate case.
- It is linked to the maximization of cross-covariance matrices with respect to conic orders.
- To every coupling between multivariate laws (historical, simulated ...) we can associate an extremal coupling.
- It yields a natural construction of indices of maximum correlation.
- It can be applied to build scenarios where the dependence becomes extreme.