Multiscale Stochastic Volatility Models
Heston 1.5

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References:

- **Multiscale Stochastic Volatility for Equity, Interest-Rate and Credit Derivatives**
  J.-P. Fouque, G. Papanicolaou, R. Sircar and K. Sølna
  *Cambridge University Press (in press).*

- **A Fast Mean-Reverting Correction to Heston Stochastic Volatility Model**
  J.-P. Fouque and M. Lorig
  *SIAM Journal on Financial Mathematics (to appear).*
Outline

1. Multiscale Models
   - Motivation for Multiple Time Scales
   - Multiscale Dynamics
   - Singular-Regular Perturbations: Black-Scholes 2.0

2. Heston Model
   - Dynamics
   - Why We like Heston
   - Problems with Heston

3. Perturbation around Heston
   - Heston + Fast Factor
   - Option Pricing

4. Numerical Work
   - Multiscale Implied Volatility Surface
   - Multiscale Fit to Data
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Volatility Not Constant

Daily Log Returns

- S&P500
- Simulated GBM

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Skews of Implied Volatilities (SP500 on June 1, 2007)
What’s Wrong with Heston?

- Single factor of volatility running on single time scale not sufficient to describe dynamics of the volatility process.

- Not just Heston . . . Any one-factor stochastic volatility model has trouble fitting implied volatility levels across all strikes and maturities.

- Empirical evidence suggests existence of several stochastic volatility factors running on different time scales.
Evidence

- Fouque-Papanicolaou-Sircar 2000,
- Chernov-Gallant-Ghysels-Tauchen 2003,
- Fouque-Papanicolaou-Sircar-Solna 2003, Hillebrand 2003, Gatheral 2006,
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Multiscale SV Under Risk-Neutral

\[ dX_t = rX_t \, dt + f(Y_t, Z_t)X_t \, dW_t^{(0)*}, \]
\[ dY_t = \left( \frac{1}{\epsilon} \alpha(Y_t) - \frac{1}{\sqrt{\epsilon}} \beta(Y_t)\Lambda_1(Y_t, Z_t) \right) \, dt + \frac{1}{\sqrt{\epsilon}} \beta(Y_t) \, dW_t^{(1)*}, \]
\[ dZ_t = \left( \delta c(Z_t) - \sqrt{\delta} \, g(Z_t)\Lambda_2(Y_t, Z_t) \right) \, dt + \sqrt{\delta} \, g(Z_t) \, dW_t^{(2)*}. \]

Time scales: \( \epsilon \ll 1 \ll 1/\delta \)
Fast factor: \( Y \)
Slow factor: \( Z \)
Option pricing:

\[ P^{\epsilon,\delta}(t, X_t, Y_t, Z_t) = \mathbb{E}^* \left\{ e^{-r(T-t)} h(X_T) \mid X_t, Y_t, Z_t \right\}, \]
\[ \frac{\partial P^{\epsilon,\delta}}{\partial t} + \mathcal{L}(X,Y,Z)P^{\epsilon,\delta} - rP^{\epsilon,\delta} = 0. \]
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   - Why We like Heston
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Expand

$$P^{\varepsilon, \delta} = \sum_{i \geq 0} \sum_{j \geq 0} (\sqrt{\varepsilon})^i (\sqrt{\delta})^j P_{i,j},$$

$$P_0 = P_{BS}$$ computed at volatility $$\sigma(z):$$

$$\bar{\sigma}^2(z) = \langle f^2(\cdot, z) \rangle = \int f^2(y, z) \Phi_Y(dy).$$

$$\sqrt{\varepsilon} P_{1,0} = V_2 \varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3 \varepsilon \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right),$$

$$\sqrt{\delta} P_{0,1} = V_0 \delta \frac{\partial P_{BS}}{\partial \sigma} + V_1 \delta \frac{\partial}{\partial x} \left( \frac{\partial P_{BS}}{\partial \sigma} \right).$$

In terms of implied volatility:

$$I \approx \left( b^* + \tau b^\delta \right) + \left( a^\varepsilon + \tau a^\delta \right) \frac{\log(K/x)}{\tau}, \quad \tau = T - t.$$
Time Scales

The previous asymptotics is a perturbation around Black-Scholes leading to Black-Scholes 2.0 where $2.0 = 1 + 2x(0.5)$

Why not keeping one volatility factor on a time scale of order one? This would be a perturbation around a one-factor stochastic volatility model, but which one?

Heston of course!
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Heston Under Risk-Neutral Measure

\[
dX_t = rX_t \, dt + \sqrt{Z_t} \, X_t \, dW^X_t \\
dZ_t = \kappa (\theta - Z_t) \, dt + \sigma \sqrt{Z_t} \, dW^Z_t \\
d \langle W^X, W^Z \rangle_t = \rho \, dt
\]

- One-factor stochastic volatility model
- Square of volatility, \( Z_t \), follows CIR process
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Formulas!

- Explicit formulas for european options:

\[ P_H(t, x, z) = e^{-r\tau} \frac{1}{2\pi} \int e^{-ikq} \hat{G}(\tau, k, z) \hat{h}(k) dk \]

\[ q(t, x) = r(T - t) + \log x, \]

\[ \hat{h}(k) = \int e^{ikq} h(e^q) dq, \]

\[ \hat{G}(\tau, k, z) = e^{C(\tau, k) + zD(\tau, k)} \]

- \( C(\tau, k) \) and \( D(\tau, k) \) solve ODEs in \( \tau = T - t \).
ODEs

\[
\begin{align*}
\frac{dD}{d\tau}(\tau, k) &= \frac{1}{2} \sigma^2 D^2(\tau, k) - (\kappa + \rho \sigma i k) D(\tau, k) + \frac{1}{2} (-k^2 + ik), \\
D(0, k) &= 0.
\end{align*}
\]

\[
\begin{align*}
\frac{dC}{d\tau}(\tau, k) &= \kappa \theta D(\tau, k), \\
C(0, k) &= 0,
\end{align*}
\]
Pretty Pictures! Heston captures well-documented features of implied volatility surface: smile and skew
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Captures Some ... Not All Features of Smile

Simultaneous Fit Across Expirations Is Poor

- Particular difficulty fitting short expirations [Gatheral 2006]
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Multiscale Under Risk-Neutral Measure

\[ dX_t = rX_t \, dt + \sqrt{Z_t} \, f(Y_t) \, X_t \, dW^X_t \]

\[ dY_t = \frac{Z_t}{\varepsilon} (m - Y_t) \, dt + \nu \sqrt{2} \, \frac{Z_t}{\varepsilon} \, dW^Y_t \]

\[ dZ_t = \kappa (\theta - Z_t) \, dt + \sigma \sqrt{Z_t} \, dW^Z_t \]

\[ d\langle W^i, W^j \rangle_t = \rho_{ij} \, dt \quad i,j \in \{x, y, z\} \]

- Volatility controlled by product \( \sqrt{Z_t} \, f(Y_t) \)
- \( Y_t \) modeled as OU process running on time-scale \( \varepsilon/Z_t \)
- Note: \( f(y) = 1 \Rightarrow \) model reduces to Heston
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Option Pricing PDE

- Price of European Option Expressed as

\[ P_t = \mathbb{E} \left[ e^{-r(T-t)} h(X_T) \bigg| X_t, Y_t, Z_t \right] =: P^\varepsilon(t, X_t, Y_t, Z_t) \]

- Using Feynman-Kac, derive following PDE for \( P^\varepsilon \)

\[ \mathcal{L}^\varepsilon P^\varepsilon(t, x, y, z) = 0, \]

\[ \mathcal{L}^\varepsilon = \frac{\partial}{\partial t} + \mathcal{L}(x, y, z) - r, \]

\[ P^\varepsilon(T, x, y, z) = h(x) \]
Some Book-Keeping of $\mathcal{L}^\varepsilon$

- $\mathcal{L}^\varepsilon$ has convenient form ($z$ factorizes)

\[
\mathcal{L}^\varepsilon = \frac{Z}{\varepsilon} \mathcal{L}_0 + \frac{Z}{\sqrt{\varepsilon}} \mathcal{L}_1 + \mathcal{L}_2,
\]

\[
\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m - y) \frac{\partial}{\partial y}
\]

\[
\mathcal{L}_1 = \rho_{yz} \sigma \nu \sqrt{2} \frac{\partial^2}{\partial y \partial z} + \rho_{xy} \nu \sqrt{2} f(y) x \frac{\partial^2}{\partial x \partial y}
\]

\[
\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2} f^2(y) z x^2 \frac{\partial^2}{\partial x^2} + r \left( x \frac{\partial}{\partial x} - . \right)
\]

\[
+ \frac{1}{2} \sigma^2 z \frac{\partial^2}{\partial z^2} + \kappa (\theta - z) \frac{\partial}{\partial z} + \rho_{xz} \sigma f(y) z x \frac{\partial^2}{\partial x \partial z}.
\]
Expand

- Perform singular perturbation with respect to $\varepsilon$

$$P^\varepsilon = P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2 + \ldots$$

- Look for $P_0$ and $P_1$ functions of $t$, $x$, and $z$ only
- Find

$$P_0(t, x, z) = P_H(t, x, z)$$

with effective square volatility $\langle f^2 \rangle Z_t = Z_t$, and effective correlation $\rho \rightarrow \rho_{xz} \langle f \rangle$
More Formulas!

\[ P_1(t, x, z) = \frac{e^{-r\tau}}{2\pi} \int e^{-ikq} \left( \kappa \theta \hat{f}_0(\tau, k) + z \hat{f}_1(\tau, k) \right) \]
\[ \times \hat{G}(\tau, k, z) \hat{h}(k) dk, \]
\[ \hat{f}_0(\tau, k) = \int_0^\tau \hat{f}_1(t, k) dt, \]
\[ \hat{f}_1(\tau, k) = \int_0^\tau b(s, k) e^{A(\tau, k, s)} ds. \]

\[ b(\tau, k) = - \left( V_1 D(\tau, k) (-k^2 + ik) + V_2 D^2(\tau, k) (-ik) \right. \]
\[ \left. + V_3 (ik^3 + k^2) + V_4 D(\tau, k) (-k^2) \right). \]

- \( A(\tau, k, s) \) solves ODE in \( \tau \)
Goup Parameters: the $V$’s

- $P_1$ is linear function of four constants

\[
V_1 = \rho_{yz} \sigma \nu \sqrt{2} \langle \phi' \rangle, \\
V_2 = \rho_{xz} \rho_{yz} \sigma^2 \nu \sqrt{2} \langle \psi' \rangle, \\
V_3 = \rho_{xy} \nu \sqrt{2} \langle f \phi' \rangle, \\
V_4 = \rho_{xy} \rho_{xz} \sigma \nu \sqrt{2} \langle f \psi' \rangle.
\]

- $\psi(y)$ and $\phi(y)$ solve Poisson equations

\[
\mathcal{L}_0 \phi = \frac{1}{2} (f^2 - \langle f^2 \rangle), \\
\mathcal{L}_0 \psi = f - \langle f \rangle.
\]

- Each $V_i$ has unique effect on implied volatility
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Effect of $V_1$ and $V_2$ on Implied Volatility

- $V_i = 0$ corresponds to Heston
Effect of $V_3$ and $V_4$ on Implied Volatility

- $V_i = 0$ corresponds to Heston
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   - Multiscale Dynamics
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   - Dynamics
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Captures More Features of Smile

- Better fit for far ITM and OTM European options (SPX, \( T=121 \) days, on May 17, 2006)
Simultaneous Fit Across Expirations Is Improved

- Vast improvement for short expirations
- 65 days to maturity
Accuracy of Approximation: Smooth Payoffs Usual technique for smooth payoffs (Schwartz space of rapidly decaying functions):

- Write a PDE for the residual (model price minus next-order approximated price)
- Use probabilistic representation for parabolic PDEs with source (small of order $\varepsilon$) and terminal condition (small of order $\varepsilon$)
- Deduce residual is small of order $\varepsilon$

Well, not exactly because moments of $Y_t$ need to be controlled uniformly in $\varepsilon$
Moments of $Y$

$$dY_t = \frac{Z_t}{\varepsilon} (m - Y_t) dt + \nu \sqrt{2} \sqrt{\frac{Z_t}{\varepsilon}} dW^Y_t$$

$$Y_t = m + (y - m) e^{-\frac{1}{\varepsilon} \int_0^t Z_s ds} + \frac{\nu \sqrt{2}}{\sqrt{\varepsilon}} e^{-\frac{1}{\varepsilon} \int_0^t Z_u du} \int_0^t e^{\frac{1}{\varepsilon} \int_s^t Z_u du} \nu \sqrt{Z_s} dW^Y_s$$

We show that for any given $\alpha \in (1/2, 1)$ there is a constant $C$ such that.

$$\mathbb{E}|Y_t| \leq C \varepsilon^{\alpha - 1},$$

and therefore an accuracy of $\varepsilon^{1-}$
Numerical Illustration for Call Options Monte Carlo simulation for a European call option: standard Euler scheme, time step of $10^{-5}$ years, $10^6$ sample paths with $\varepsilon = 10^{-3}$ so that $\sqrt{\varepsilon} V_3 = 0.0303$ is of the same order as $V_3^\varepsilon$, the largest of the $V_i^\varepsilon$'s obtained in the data calibration example. The parameters used in the simulation are:

$$
\begin{align*}
x &= 100, \quad z = 0.24, \quad r = 0.05, \quad \kappa = 1, \quad \theta = 1, \quad \sigma = 0.39, \quad \rho_{xz} = -0.35, \\
y &= 0.06, \quad m = 0.06, \quad \nu = 1, \quad \rho_{xy} = -0.35, \quad \rho_{yz} = 0.35, \\
\tau &= 1, \quad K = 100,
\end{align*}
$$

and $f(y) = e^{y-m-\nu^2}$ so that $\langle f^2 \rangle = 1$.

| $\varepsilon$ | $\sqrt{\varepsilon} P_1$ | $P_0 + \sqrt{\varepsilon} P_1$ | $\hat{P}_{MC}$ | $\hat{\sigma}_{MC}$ | $|P_0 + \sqrt{\varepsilon} P_1 - \hat{P}_{MC}|$ |
|--------------|--------------------------|---------------------------|----------------|----------------------|---------------------------------------------|
| $0$          | $0$                      | $21.0831$                 |               | $-$                  | $-$                                         |
| $10^{-3}$    | $-0.2229$                | $20.8602$                 | $20.8595$     | $0.0364$             | $0.0007$                                    |
Summary

- **Heston model** provides easy way to calculate option prices in stochastic volatility setting, but fails to capture some features of implied volatility surface.
- **Multiscale model** offers improved fit to implied volatility surface while maintaining convenience of option pricing formulas.

Present work with Matt Lorig to appear in the *SIAM Journal on Financial Mathematics*

Work in preparation for *Risk*: formulas for **fast and slow time scales** around Heston (Heston 2.0).