Multiscale Stochastic Volatility Models
Heston 1.5

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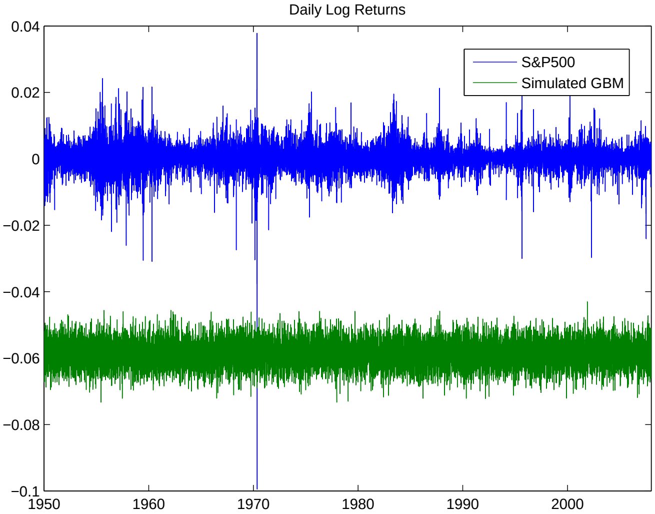
References:

- **Multiscale Stochastic Volatility for Equity, Interest-Rate and Credit Derivatives**
  J.-P. Fouque, G. Papanicolaou, R. Sircar and K. Sølna
  Cambridge University Press (in press).
- **A Fast Mean-Reverting Correction to Heston Stochastic Volatility Model**
  J.-P. Fouque and M. Lorig
  SIAM Journal on Financial Mathematics (to appear).

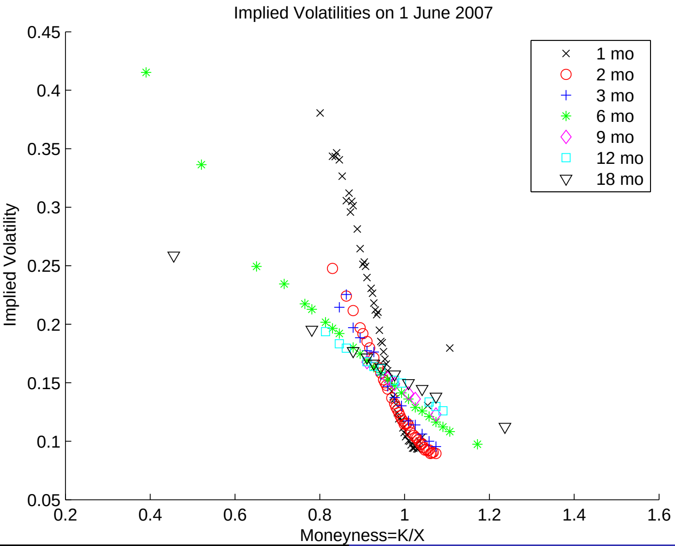
Outline

Outline

Volatility Not Constant



Skews of Implied Volatilities (SP500 on June 1, 2007)



What's Wrong with Heston?

- Single factor of volatility running on single time scale not sufficient to describe dynamics of the volatilty process.
- Not just Heston . . . Any one-factor stochastic volatility model has trouble fitting implied volatility levels accross all strikes and maturities.
- Emperical evidence suggests existence of several stochastic volatility factors running on different time scales
- $\longrightarrow$

Evidence

- Melino-Turnbull 1990, Dacorogna et al. 1997, Andersen-Bollerslev 1997,
- Fouque-Papanicolaou-Sircar 2000,
- Alizdeh-Brandt-Diebold 2001, Engle-Patton 2001, Lebaron 2001,
- Chernov-Gallant-Ghysels-Tauchen 2003,
- Fouque-Papanicolaou-Sircar-Solna 2003, Hillebrand 2003, Gatheral 2006,
- Christoffersen-Jacobs-Ornthanalai-Wang 2008.

Outline

Multiscale SV Under Risk-Neutral

$$
\begin{aligned}
dX_t &= rX_t\,dt + f(Y_t,Z_t)X_t\,dW_t^{(0)\star},\\
dY_t &= \left(\frac{1}{\varepsilon}\alpha(Y_t) - \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)\Lambda_1(Y_t,Z_t)\right)dt + \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)\,dW_t^{(1)\star},\\
dZ_t &= \left(\delta c(Z_t) - \sqrt{\delta}\,g(Z_t)\Lambda_2(Y_t,Z_t)\right)dt + \sqrt{\delta}\,g(Z_t)\,dW_t^{(2)\star}.
\end{aligned}
$$

Time scales: $\varepsilon \ll 1 \ll 1/\delta$
Fast factor: Y
Slow factor: Z
Option pricing:

$$P^{\varepsilon,\delta}(t,X_t,Y_t,Z_t) = \boldsymbol{E}^\star\left\{e^{-r(T-t)}h(X_T)\mid X_t,Y_t,Z_t\right\},$$

$$\frac{\partial P^{\varepsilon,\delta}}{\partial t} + \mathcal{L}_{(X,Y,Z)}P^{\varepsilon,\delta} - rP^{\varepsilon,\delta} = 0.$$

Outline

Expand

$$P^{\varepsilon,\delta} = \sum_{i\geq 0}\sum_{j\geq 0} (\sqrt{\varepsilon})^i\,(\sqrt{\delta})^j\,P_{i,j},$$

$P_0 = P_{BS}$ computed at volatility $\sigma(z)$:

$$\bar{\sigma}^2(z) = \langle f^2(\cdot, z)\rangle = \int f^2(y,z)\Phi_Y(dy).$$

$$\begin{aligned}\sqrt{\varepsilon}\,P_{1,0} &= V_2^{\varepsilon}\,x^2\frac{\partial^2 P_{BS}}{\partial x^2} + V_3^{\varepsilon}\,x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 P_{BS}}{\partial x^2}\right),\\ \sqrt{\delta}\,P_{0,1} &= V_0^{\delta}\,\frac{\partial P_{BS}}{\partial \sigma} + V_1^{\delta}\,\frac{\partial}{\partial x}\left(\frac{\partial P_{BS}}{\partial \sigma}\right).\end{aligned}$$

In terms of implied volatility:

$$I \approx \left(b^{\star} + \tau b^{\delta}\right) + \left(a^{\varepsilon} + \tau a^{\delta}\right)\frac{\log(K/x)}{\tau}, \quad \tau = T - t.$$

Time Scales

The previous asymptotics is a perturbation around Black-Scholes leading to Black-Scholes 2.0 where 2.0 $=$ 1$+$ 2x(0.5)

Why not keeping one volatility factor on a time scale of order one? This would be a perturbation around a one-factor stochastic volatility model, but which one?

Heston of course!

Outline

Heston Under Risk-Neutral Measure

$$
\begin{aligned}
dX_t &= rX_t dt + \sqrt{Z_t}\, X_t dW_t^x \\
dZ_t &= \kappa(\theta - Z_t)dt + \sigma\sqrt{Z_t}\, dW_t^z \\
d\left\langle W^x, W^z \right\rangle_t &= \rho dt
\end{aligned}
$$

- One-factor stochastic volatility model
- Square of volatility, Z_t, follows CIR process

Outline

Formulas!

- Explicit formulas for european options:

$$P_H(t,x,z) = e^{-r\tau}\frac{1}{2\pi}\int e^{-ikq}\widehat{G}(\tau,k,z)\widehat{h}(k)dk$$
$$q(t,x) = r(T-t)+\log x,$$
$$\widehat{h}(k) = \int e^{ikq}h(e^q)dq,$$
$$\widehat{G}(\tau,k,z) = e^{C(\tau,k)+zD(\tau,k)}$$

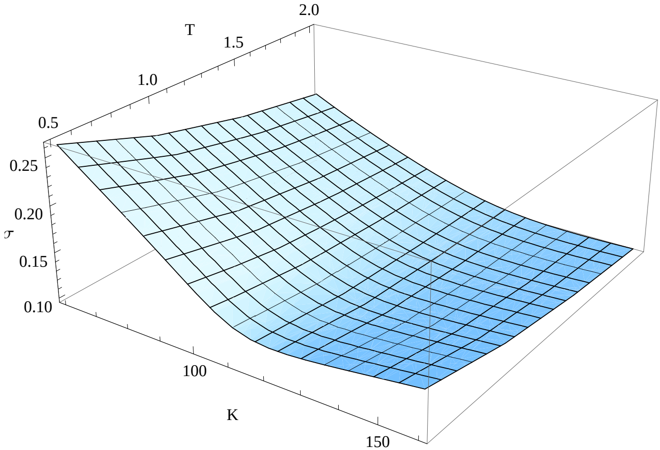
- $C(\tau,k)$ and $D(\tau,k)$ solve ODEs in $\tau = T-t$.

ODEs

$$
\begin{aligned}
\frac{dD}{d\tau}(\tau,k) &= \frac{1}{2}\sigma^2 D^2(\tau,k) - (\kappa + \rho\sigma i k)\, D(\tau,k) + \frac{1}{2}\left(-k^2 + ik\right), \\
D(0,k) &= 0.
\end{aligned}
$$

$$
\begin{aligned}
\frac{dC}{d\tau}(\tau,k) &= \kappa\theta D(\tau,k), \\
C(0,k) &= 0,
\end{aligned}
$$

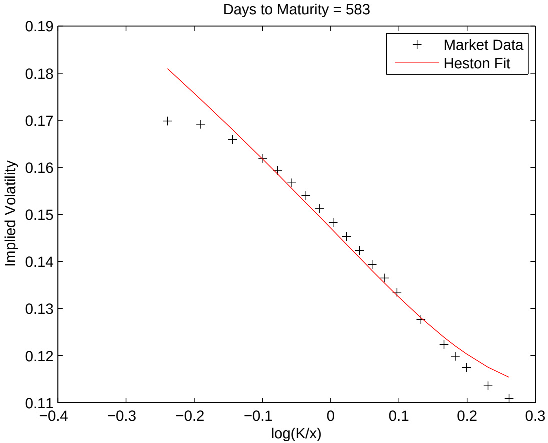
Pretty Pictures! Heston captures well-documented features of implied volatility surface: smile and skew



Outline

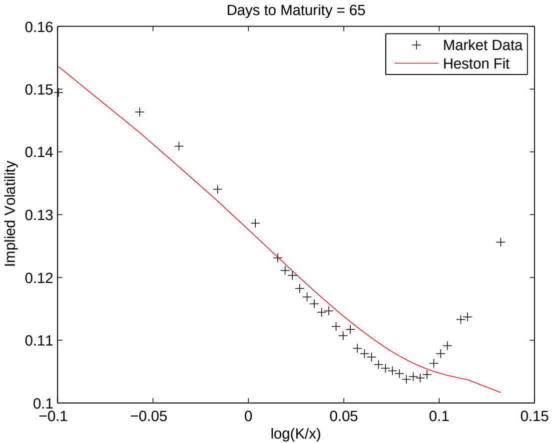
Captures Some . . . Not All Features of Smile

- Misprices far ITM and OTM European options [Fiorentini-Leon-Rubio 2002, Zhang-Shu 2003 ...]



Simultaneous Fit Across Expirations Is Poor

- Particular difficulty fitting short expirations [Gatheral 2006]



Outline

Multiscale Under Risk-Neutral Measure

$$dX_t = rX_t dt + \sqrt{Z_t}\, f(Y_t) X_t dW_t^x$$

$$dY_t = \frac{Z_t}{\varepsilon}(m - Y_t)dt + \nu\sqrt{2}\sqrt{\frac{Z_t}{\varepsilon}}dW_t^y$$

$$dZ_t = \kappa(\theta - Z_t)dt + \sigma\sqrt{Z_t}\, dW_t^z$$

$$d\left\langle W^i, W^j \right\rangle_t = \rho_{ij} dt \qquad i,j \in \{x,y,z\}$$

- Volatility controled by product $\sqrt{Z_t}\, f(Y_t)$
- Y_t modeled as OU process running on time-scale ε / Z_t
- Note: $f(y) = 1 \Rightarrow$ model reduces to Heston

Outline

Option Pricing PDE

- Price of European Option Expressed as

$$P_t = \mathbb{E}\left[e^{-r(T-t)}h(X_T)\middle| X_t, Y_t, Z_t\right] =: P^\varepsilon(t, X_t, Y_t, Z_t)$$

- Using Feynman-Kac, derive following PDE for P^ε

$$\mathcal{L}^\varepsilon P^\varepsilon(t,x,y,z) = 0,$$

$$\mathcal{L}^\varepsilon = \frac{\partial}{\partial t} + \mathcal{L}_{(X,Y,Z)} - r\,,$$

$$P^\varepsilon(T,x,y,z) = h(x)$$

Some Book-Keeping of $\mathcal{L}^\varepsilon$

- $\mathcal{L}^\varepsilon$ has convenient form (z factorizes)

$$\mathcal{L}^\varepsilon = \frac{z}{\varepsilon}\mathcal{L}_0 + \frac{z}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2,$$

$$\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m - y)\frac{\partial}{\partial y}$$

$$\mathcal{L}_1 = \rho_{yz}\sigma\nu\sqrt{2}\,\frac{\partial^2}{\partial y \partial z} + \rho_{xy}\nu\sqrt{2}\,f(y)x\frac{\partial^2}{\partial x \partial y}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}f^2(y)zx^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right) + \frac{1}{2}\sigma^2 z\frac{\partial^2}{\partial z^2} + \kappa(\theta - z)\frac{\partial}{\partial z} + \rho_{xz}\sigma f(y)zx\frac{\partial^2}{\partial x \partial z}.$$

Expand

- Perform singular perturbation with respect to ε

$$P^{\varepsilon} = P_0 + \sqrt{\varepsilon} P_1 + \varepsilon P_2 + \dots$$

- Look for P_0 and P_1 functions of t, x, and z only
- Find

$$\boxed{P_0(t,x,z) = P_H(t,x,z)}$$

with effective square volatility $\langle f^2 \rangle Z_t = Z_t$,
and effective correlation $\rho \to \rho_{xz} \langle f \rangle$

More Formulas!

$$P_1(t,x,z) = \frac{e^{-r\tau}}{2\pi} \int e^{-ikq} \left(\kappa\theta \widehat{f}_0(\tau,k) + z\widehat{f}_1(\tau,k) \right) \times \widehat{G}(\tau,k,z)\widehat{h}(k)dk,$$

$$\widehat{f}_0(\tau,k) = \int_0^\tau \widehat{f}_1(t,k)dt,$$

$$\widehat{f}_1(\tau,k) = \int_0^\tau b(s,k)e^{A(\tau,k,s)}ds.$$

$$b(\tau,k) = -\left(V_1 D(\tau,k)\left(-k^2+ik\right) + V_2 D^2(\tau,k)\left(-ik\right) + V_3\left(ik^3+k^2\right) + V_4 D(\tau,k)\left(-k^2\right)\right).$$

- $A(\tau,k,s)$ solves ODE in τ

Goup Parameters: the V's

- P_1 is linear function of four constants

$$V_1 = \rho_{yz}\sigma\nu\sqrt{2}\left\langle\phi'\right\rangle,$$
$$V_2 = \rho_{xz}\rho_{yz}\sigma^2\nu\sqrt{2}\left\langle\psi'\right\rangle,$$
$$V_3 = \rho_{xy}\nu\sqrt{2}\left\langle f\phi'\right\rangle,$$
$$V_4 = \rho_{xy}\rho_{xz}\sigma\nu\sqrt{2}\left\langle f\psi'\right\rangle.$$

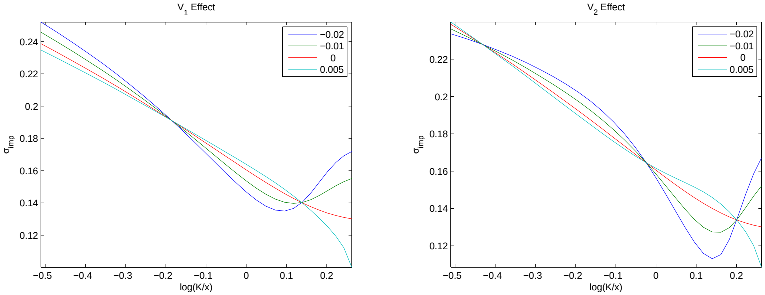
- $\psi(y)$ and $\phi(y)$ solve Poisson equations

$$\mathcal{L}_0\phi = \frac{1}{2}\left(f^2 - \left\langle f^2\right\rangle\right),$$
$$\mathcal{L}_0\psi = f - \left\langle f\right\rangle.$$

- Each V_i has unique effect on implied volatility

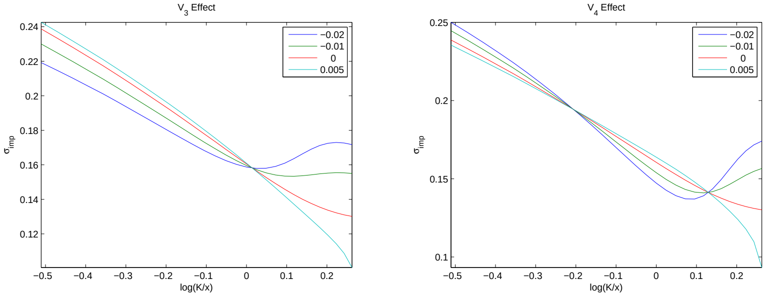
Outline

Effect of V_1 and V_2 on Implied Volatility



- $V_i = 0$ corresponds to Heston

Effect of V_3 and V_4 on Implied Volatility

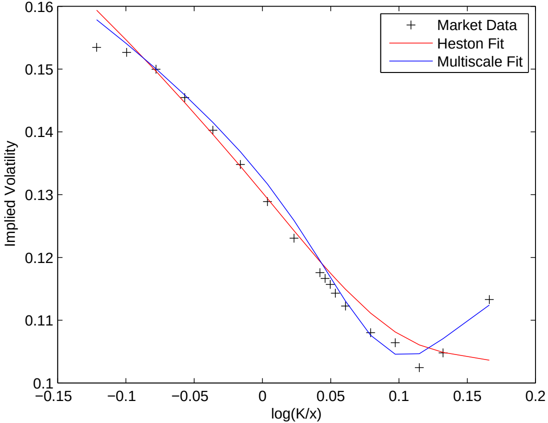


- $V_i = 0$ corresponds to Heston

Outline

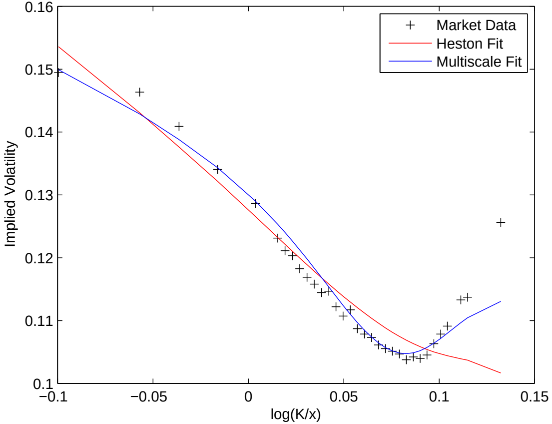
Captures More Features of Smile

- Better fit for far ITM and OTM European options (SPX, T=121 days, on May 17, 2006)



Simultaneous Fit Across Expirations Is Improved

- Vast improvement for short expirations
- 65 days to maturity



Accuracy of Approximation: Smooth Payoffs Usual technique for smooth payoffs (Schwartz space of rapidly decaying functions):

- Write a PDE for the residual (model price minus next-order approximated price)
- Use probabilistic representation for parabolic PDEs with source (small of order ε) and terminal condition (small of order ε)
- Deduce residual is small of order ε

Well, not exactly because moments of Y_t need to be controlled uniformly in ε

Moments of Y

$$dY_t = \frac{Z_t}{\varepsilon}(m - Y_t)dt + \nu\sqrt{2}\sqrt{\frac{Z_t}{\varepsilon}}dW_t^y$$

$$Y_t = m + (y - m)e^{-\frac{1}{\varepsilon}\int_0^t Z_s ds} + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}e^{-\frac{1}{\varepsilon}\int_0^t Z_u du}\int_0^t e^{\frac{1}{\varepsilon}\int_0^s Z_u du}\nu\sqrt{Z_s}\,dW_s^y$$

We show that for any given $\alpha \in (1/2, 1)$ there is a constant C such that.

$$\mathbb{E}|Y_t| \leq C\,\varepsilon^{\alpha-1},$$

and therefore an accuracy of ε^{1-}

Numerical Illustration for Call Options Monte Carlo simulation for a European call option: standard Euler scheme, time step of 10^{-5} years, 10^6 sample paths with $\varepsilon = 10^{-3}$ so that $\sqrt{\varepsilon} V_3 = 0.0303$ is of the same order as V_3^ε, the largest of the V_i^ε's obtained in the data calibration example. The parameters used in the simulation are:

$$x = 100, z = 0.24, r = 0.05, \kappa = 1, \theta = 1, \sigma = 0.39, \rho_{xz} = -0.35,$$
$$y = 0.06, m = 0.06, \nu = 1, \rho_{xy} = -0.35, \rho_{yz} = 0.35,$$
$$\tau = 1, K = 100,$$

and $f(y) = e^{y-m-\nu^2}$ so that $\langle f^2 \rangle = 1$.

ε	$\sqrt{\varepsilon} P_1$	$P_0 + \sqrt{\varepsilon} P_1$	$\widehat{P}_{MC}$	$\widehat{\sigma}_{MC}$	$\lvert P_0 + \sqrt{\varepsilon} P_1 - \widehat{P}_{MC}$
0	0	21.0831	-	-	-
10^{-3}	-0.2229	20.8602	20.8595	0.0364	0.0007

Summary

- Heston model provides easy way to calculate option prices in stochastic volatility setting, but fails to capture some features of implied volatility surface
- Multiscale model offers improved fit to implied volatility surface while maintaining convenience of option pricing forumulas

Present work with Matt Lorig to appear in the *SIAM Journal on Financial Mathematics*

Work in preparation for *Risk*: formulas for **fast and slow time scales** around Heston (Heston 2.0).