Hedging under Arbitrage

Johannes Ruf

Columbia University, Department of Statistics

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Motivation

- Given: a frictionless market of stocks with continuous Markovian dynamics.
- **If there does not exist an equivalent local martingale measure can we have the concept of hedging?**
- Answer: Yes, if a square-integrable “market price of risk” exists.
- **If there exists an equivalent local martingale measure and a stock price process is a “strict local martingale” what is the cheapest way to hold this stock at time T?**
- Answer: Delta-hedging.
- **How can we compute hedging prices?**
- Answer: PDE techniques, (non-)equivalent changes of measures
- Techniques: Itô’s formula, PDE techniques to prove smoothness of hedging prices, Föllmer measure

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Two generic examples

- Reciprocal of the three-dimensional Bessel process (NFLVR):

$$d\tilde{S}(t) = -\,\tilde{S}^2(t)dW(t)$$

- Three-dimensional Bessel process:

$$dS(t) = \frac{1}{S(t)}dt + dW(t)$$

Strict local martingales

- A stochastic process $X(\cdot)$ is a *local martingale* if there exists a sequence of stopping times (τ_n) with $\lim_{n\to\infty} \tau_n = \infty$ such that $X^{\tau_n}(\cdot)$ is a martingale.
- Here, in our context, a local martingale is a nonnegative stochastic process $X(\cdot)$ which does not have a drift:

$$dX(t) = X(t)\textbf{something}dW(t).$$

- Strict local martingales (local martingales, which are not martingales) do only appear in continuous time.
- Nonnegative local martingales are supermartingales.

We assume a Markovian market model.

- Our time is finite: $T < \infty$. Interest rates are zero.
- The stocks $S(\cdot) = (S_1(\cdot), \ldots, S_d(\cdot))^{\mathsf{T}}$ follow

$$dS_i(t) = S_i(t) \left(\mu_i(t, S(t))dt + \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))dW_k(t) \right)$$

  with some measurability and integrability conditions.
- $\rightarrow$ Markovian
- but not necessarily complete ($K > d$ allowed).
- The covariance process is defined as

$$a_{i,j}(t, S(t)) := \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))\sigma_{j,k}(t, S(t)).$$

- The underlying filtration is denoted by $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t \le T}$.

An important guy: the market price of risk.

- A *market price of risk* is an $\mathbb{R}^K$-valued process $\theta(\cdot)$ satisfying

$$\mu(t, S(t)) = \sigma(t, S(t))\theta(t).$$

- We assume it exists and

$$\int_0^T \|\theta(t)\|^2 dt < \infty.$$

- The market price of risk is not necessarily unique.
- We will always use a Markovian version of the form $\theta(t, S(t))$. (needs argument!)

Related is the stochastic discount factor.

- The *stochastic discount factor* corresponding to θ is denoted by

$$Z^{\theta}(t) := \exp\left(-\int_0^t \theta^{\mathsf{T}}(u, S(u))dW(u) - \frac{1}{2}\int_0^t \|\theta(u, S(u))\|^2 du\right).$$

- It has dynamics

$$dZ^{\theta}(t) = -\theta^{\mathsf{T}}(t, S(t))Z^{\theta}(t)dW(t).$$

- If $Z^{\theta}(\cdot)$ is a martingale, that is, if $E[Z^{\theta}(T)] = 1$, then it defines a risk-neutral measure $\mathbb{Q}$ with $d\mathbb{Q} = Z^{\theta}(T)d\mathbb{P}$.
- Otherwise, $Z^{\theta}(\cdot)$ is a strict local martingale and classical arbitrage is possible.
- From Itô's rule, we have

$$d\left(Z^{\theta}(t)S_i(t)\right) = Z^{\theta}(t)S_i(t)\sum_{k=1}^{K}\left(\sigma_{i,k}(t, S(t)) - \theta_k(t, S(t))\right)dW_k(t)$$

Everything an investor cares about: how and how much?

- We call *trading strategy* the number of shares held by an investor: $\eta(t) = (\eta_1(t), \dots, \eta_d(t))^\mathsf{T}$
- We assume that $\eta(\cdot)$ is progressively measurable with respect to $\mathbb{F}$ and self-financing.
- The corresponding wealth process $V^{v,\eta}(\cdot)$ for an investor with initial wealth $V^{v,\eta}(0) = v$ has dynamics

$$dV^{v,\eta}(t) = \sum_{i=1}^{d} \eta_i(t) dS_i(t).$$

- We restrict ourselves to trading strategies which satisfy $V^{1,\eta}(t) \geq 0$.

The terminal payoff

- Let $p : \mathbb{R}_+^d \to [0, \infty)$ denote a measurable function.
- The investor wants to have the payoff $p(S(T))$ at time T.
- For example,
  - market portfolio: $\tilde{p}(s) = \sum_{i=1}^d s_i$
  - money market: $p^0(s) = 1$
  - stock: $p^1(s) = s_1$
  - call: $p^C(s) = (s_1 - L)^+$ for some $L \in \mathbb{R}$.
- We define a candidate for the hedging price as

$$h^p(t, s) := \mathbb{E}^{t,s}\left[\tilde{Z}^\theta(T) p(S(T))\right],$$

where $\tilde{Z}^\theta(T) = Z^\theta(T)/Z^\theta(t)$ and $S(t) = s$ under the expectation operator $\mathbb{E}^{t,s}$.

Non path-dependent European claims

Assume that we have a contingent claim of the form $p(S(T)) \geq 0$ and that for all points of support (t,s) for $S(\cdot)$ with $t \in [0,T)$ we have $h^p \in C^{1,2}(\mathcal{U}_{t,s})$ for some neighborhood $\mathcal{U}_{t,s}$ of (t,s). Then, with $\eta_i^p(t,s) := D_i h^p(t,s)$ and $v^p := h^p(0, S(0))$, we get

$$V^{v^p,\eta^p}(t) = h^p(t, S(t)).$$

The strategy η^p is optimal in the sense that for any $\tilde{v} > 0$ and for any strategy $\tilde{\eta}$ whose associated wealth process is nonnegative and satisfies $V^{\tilde{v},\tilde{\eta}}(T) \geq p(S(T))$, we have $\tilde{v} \geq v^p$. Furthermore, h^p solves the PDE

$$\frac{\partial}{\partial t} h^p(t,s) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i s_j a_{i,j}(t,s) D^2_{i,j} h^p(t,s) = 0$$

at all points of support (t,s) for $S(\cdot)$ with $t \in [0,T)$.

The proof relies on Itô's formula.

- Define the martingale $N^p(\cdot)$ as

$$N^p(t) := \mathbb{E}[Z^\theta(T)p(S(T))|\mathcal{F}(t)] = Z^\theta(t)h^p(t, S(t)).$$

- Use a localized version of Itô's formula to get the dynamics of $N^p(\cdot)$. Since it is a martingale, its dt term must disappear which yields the PDE.
- Then, another application of Itô's formula yields

$$dh^p(t, S(t)) = \sum_{i=1}^{d} D_i h^p(t, S(t))dS_i(t) = dV^{\nu^p, \eta^p}(t).$$

- This yields directly $V^{\nu^p, \eta^p}(\cdot) \equiv h^p(\cdot, S(\cdot))$.

Proof (continued)

- Next, we prove optimality.
- Assume we have some initial wealth $\tilde{v} > 0$ and some strategy $\tilde{\eta}$ with nonnegative associated wealth process such that $V^{\tilde{v},\tilde{\eta}}(T) \geq p(S(T))$ is satisfied.
- Then, $Z^{\theta}(\cdot)V^{\tilde{v},\tilde{\eta}}(\cdot)$ is a supermartingale.
- This implies

$$\begin{aligned}\tilde{v} &\geq \mathbb{E}[Z^{\theta}(T)V^{\tilde{v},\tilde{\eta}}(T)] \geq \mathbb{E}[Z^{\theta}(T)p(S(T))] \\ &= \mathbb{E}[Z^{\theta}(T)V^{v^p,\eta^p}(T)] = v^p\end{aligned}$$

Non-uniqueness of PDE

- Usually,

$$\frac{\partial}{\partial t}v(t,s) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} s_i s_j a_{i,j}(t,s) D^2_{i,j} v(t,s) = 0$$

  does not have a unique solution.
- However, if h^p is sufficiently differentiable, it can be characterized as the minimal nonnegative solution of the PDE.
- This follows as in the proof of optimality. If $\tilde{h}$ is another nonnegative solution of the PDE with $\tilde{h}(T,s) = p(s)$, then $Z^{\theta}(\cdot)\tilde{h}(\cdot, S(\cdot))$ is a supermartingale.

Corollary: Modified put-call parity

For any $L \in \mathbb{R}$ we have the modified put-call parity for the call- and put-options $(S_1(T) - L)^+$ and $(L - S_1(T))^+$, respectively, with strike price L:

$$\mathbb{E}^{t,s}\left[\tilde{Z}^\theta(T)(L - S_1(T))^+\right] + h^{p^1}(t,s) = \mathbb{E}^{t,s}\left[\tilde{Z}^\theta(T)(S_1(T) - L)^+\right] + Lh^{p^0}(t,s),$$

where $p^0(\cdot) \equiv 1$ denotes the payoff of one monetary unit and $p^1(s) = s_1$ the price of the first stock for all $s \in \mathbb{R}_+^d$.

Role of Markovian market price of risk

Let $M \geq 0$ be a random variable measurable with respect to $\mathcal{F}^S(T)$. Let $\nu(\cdot)$ denote any MPR and $\theta(\cdot,\cdot)$ a Markovian MPR. Then, with

$$M^\nu(t) := \mathbb{E}\left[\frac{Z^\nu(T)}{Z^\nu(t)} M \,\middle|\, \mathcal{F}_t\right] \text{ and } M^\theta(t) := \mathbb{E}\left[\frac{Z^\theta(T)}{Z^\theta(t)} M \,\middle|\, \mathcal{F}_t\right]$$

for $t \in [0, T]$, we have $M^\nu(\cdot) \leq M^\theta(\cdot)$ almost surely.

Proof

- We define $c(\cdot) := \nu(\cdot) - \theta(\cdot, S(\cdot))$ and $c^n(\cdot) := c(\cdot)\mathbf{1}_{\{\tau_n \geq \cdot\}}$ for some localization sequence τ_n.
- Then,

$$\frac{Z^\nu(T)}{Z^\nu(t)} = \lim_{n\to\infty} \frac{Z^{c^n}(T)}{Z^{c^n}(t)}$$
$$\cdot \exp\left(-\int_t^T \theta^\mathsf{T}(dW(u) + c^n(u)du) - \frac{1}{2}\int_t^T \|\theta\|^2 du\right).$$

- Since $\int_0^T c^n(t)dt$ is bounded, $Z^{c^n}(\cdot)$ is a martingale.
- Fatou's lemma, Girsanov's theorem and Bayes' rule yield

$$M^\nu(t) \leq \liminf_{n\to\infty} \mathbb{E}^{\mathbb{Q}^n}\left[\exp\left(-\int_t^T \theta^\mathsf{T} dW^n(u) - \frac{1}{2}\int_t^T \| \|^2 du\right) M \middle| \mathcal{F}_t\right.$$

- Since $\sigma(\cdot, S(\cdot))c^n(\cdot) \equiv 0$ the process $S(\cdot)$ has the same dynamics under $\mathbb{Q}^n$ as under $\mathbb{P}$.

We can change the measure to compute h^p

- There exists not always an equivalent local martingale measure.
- However, after making some technical assumptions on the probability space and the filtration we can construct a new measure $\mathbb{Q}$ which corresponds to a "removal of the stock price drift".
- Based on the work of Föllmer and Meyer and along the lines of Delbaen and Schachermayer.

Theorem: Under a new measure $\mathbb{Q}$ the drifts disappear.

There exists a measure $\mathbb{Q}$ such that $\mathbb{P} \ll \mathbb{Q}$. More precisely, for all nonnegative $\mathcal{F}(T)$-measurable random variables Y we have

$$\mathbb{E}^{\mathbb{P}}[Z^\theta(T)Y] = \mathbb{E}^{\mathbb{Q}}\left[Y\mathbf{1}_{\left\{\frac{1}{Z^\theta(T)}>0\right\}}\right].$$

Under this measure $\mathbb{Q}$, the stock price processes follow

$$dS_i(t) = S_i(t)\sum_{k=1}^{K}\sigma_{i,k}(t,S(t))\,d\widetilde{W}_k(t)$$

up to time $\tau^\theta := \inf\{t \in [0,T] : 1/Z^\theta(t) = 0\}$. Here,

$$\widetilde{W}_k(t\wedge\tau^\theta) := W_k(t\wedge\tau^\theta) + \int_0^{t\wedge\tau^\theta}\theta_k(u,S(u))du$$

is a K-dimensional $\mathbb{Q}$-Brownian motion stopped at time τ^θ.

What happens in between time 0 and time T: Bayes' rule.

For all nonnegative $\mathcal{F}(T)$-measurable random variables Y the representation

$$\mathbb{E}^{\mathbb{Q}}\left[Y\mathbf{1}_{\{1/Z^{\theta}(T)>0\}} \middle| \mathcal{F}(t)\right] = \mathbb{E}^{\mathbb{P}}[Z^{\theta}(T)Y|\mathcal{F}(t)]\frac{1}{Z^{\theta}(t)}\mathbf{1}_{\{1/Z^{\theta}(t)>0\}}$$

holds $\mathbb{Q}$-almost surely (and thus $\mathbb{P}$-almost surely) for all $t \in [0, T]$.

After a change of measure, the Bessel process becomes Brownian motion.

- We look at a market with one stock:

$$dS(t) = \frac{1}{S(t)}dt + dW(t).$$

- We have $S(t) > 0$ for all $t \geq 0$.
- The market price of risk is $\theta(t, s) = 1/s$.
- The inverse stochastic discount factor $1/Z^{\theta}$ becomes zero exactly when $S(t)$ hits 0.
- Removing the drift with a change of measure as before makes $S(\cdot)$ a Brownian motion (up to the first hitting time of zero by $1/Z^{\theta}(\cdot)$) under $\mathbb{Q}$.

The optimal strategy for getting one dollar at time T can be explicitly computed.

- For $p(s) \equiv p^0(s) \equiv 1$ we get

$$h^{p^0}(t,s) = \mathbb{E}^{\mathbb{P}}\left[\frac{Z^\theta(T)}{Z^\theta(t)} \cdot 1 \middle| \mathcal{F}_t\right]\Big|_{S(t)=s} = \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{1/Z^\theta(T)>0\}}|\mathcal{F}_t]|_{S(t)=s}$$

$$= 2\Phi\left(\frac{s}{\sqrt{T-t}}\right) - 1.$$

- This yields the optimal strategy

$$\eta^0(t,s) = \frac{2}{\sqrt{T-t}}\phi\left(\frac{s}{\sqrt{T-t}}\right).$$

- The hedging price h^p satisfies on all points $\{s > 0\}$ the PDE

$$\frac{\partial}{\partial t}h^p(t,s) + \frac{1}{2}D^2 h^p(t,s) = 0.$$

Conclusion

- No equivalent local martingale measure needed to find an optimal hedging strategy based upon the familiar delta hedge.
- Sufficient conditions are derived for the necessary differentiability of expectations indexed over the initial market configuration.
- The dynamics of stochastic processes under a non-equivalent measure and a generalized Bayes' rule might be of interest themselves.
- We have computed some optimal trading strategies in standard examples for which so far only ad-hoc and not necessarily optimal strategies have been known.

Thank you!