Hedging under Arbitrage

Johannes Ruf

Columbia University, Department of Statistics

Modeling and Managing Financial Risks
January 12, 2011
Motivation

- Given: a frictionless market of stocks with continuous Markovian dynamics.
- **If there does not exist an equivalent local martingale measure can we have the concept of hedging?**
- **Answer:** Yes, if a square-integrable "market price of risk" exists.
- If there exists an equivalent local martingale measure and a stock price process is a "strict local martingale" what is the cheapest way to hold this stock at time $T$?
- **Answer:** Delta-hedging.
- **How can we compute hedging prices?**
- **Answer:** PDE techniques, (non-)equivalent changes of measures
- **Techniques:** Itô’s formula, PDE techniques to prove smoothness of hedging prices, Föllmer measure
Motivation

• Given: a frictionless market of stocks with continuous Markovian dynamics.

• If there does not exist an equivalent local martingale measure can we have the concept of hedging?

• Answer: Yes, if a square-integrable “market price of risk” exists.

• If there exists an equivalent local martingale measure and a stock price process is a “strict local martingale” what is the cheapest way to hold this stock at time $T$?

• Answer: Delta-hedging.

• How can we compute hedging prices?

• Answer: PDE techniques, (non-)equivalent changes of measures

• Techniques: Itô’s formula, PDE techniques to prove smoothness of hedging prices, Föllmer measure
Motivation

• Given: a frictionless market of stocks with continuous Markovian dynamics.

• If there does not exist an equivalent local martingale measure can we have the concept of hedging?

• Answer: Yes, if a square-integrable “market price of risk” exists.

• If there exists an equivalent local martingale measure and a stock price process is a “strict local martingale” what is the cheapest way to hold this stock at time $T$?

• Answer: Delta-hedging.

• How can we compute hedging prices?

• Answer: PDE techniques, (non-)equivalent changes of measures

• Techniques: Itô’s formula, PDE techniques to prove smoothness of hedging prices, Föllmer measure
Two generic examples

- Reciprocal of the three-dimensional Bessel process (NFLVR):

  \[ d\tilde{S}(t) = -\tilde{S}^2(t)dW(t) \]

- Three-dimensional Bessel process:

  \[ dS(t) = \frac{1}{S(t)} dt + dW(t) \]
Strict local martingales

- A stochastic process $X(\cdot)$ is a *local martingale* if there exists a sequence of stopping times $(\tau_n)$ with $\lim_{n \to \infty} \tau_n = \infty$ such that $X^{\tau_n}(\cdot)$ is a martingale.

- Here, in our context, a local martingale is a nonnegative stochastic process $X(\cdot)$ which does not have a drift:

$$dX(t) = X(t)\text{something}dW(t).$$

- Strict local martingales (local martingales, which are not martingales) do only appear in continuous time.

- Nonnegative local martingales are supermartingales.
We assume a Markovian market model.

- Our time is finite: \( T < \infty \). Interest rates are zero.
- The stocks \( S(\cdot) = (S_1(\cdot), \ldots, S_d(\cdot))^T \) follow

\[
dS_i(t) = S_i(t) \left( \mu_i(t, S(t)) dt + \sum_{k=1}^{K} \sigma_{i,k}(t, S(t)) dW_k(t) \right)
\]

with some measurability and integrability conditions.

- \( \to \) Markovian
- but not necessarily complete (\( K > d \) allowed).
- The covariance process is defined as

\[
a_{i,j}(t, S(t)) := \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))\sigma_{j,k}(t, S(t)).
\]

- The underlying filtration is denoted by \( \mathbb{F} = \{ \mathcal{F}(t) \}_{0 \leq t \leq T} \).
An important guy: the market price of risk.

- A *market price of risk* is an $\mathbb{R}^K$-valued process $\theta(\cdot)$ satisfying
  \[ \mu(t, S(t)) = \sigma(t, S(t))\theta(t). \]

- We assume it exists and
  \[ \int_0^T \|\theta(t)\|^2 dt < \infty. \]

- The market price of risk is not necessarily unique.
- We will always use a Markovian version of the form $\theta(t, S(t))$. (needs argument!)
Related is the stochastic discount factor.

- The *stochastic discount factor* corresponding to $\theta$ is denoted by
  \[ Z^\theta(t) := \exp \left( -\int_0^t \theta^\top(u, S(u))dW(u) - \frac{1}{2} \int_0^t \|\theta(u, S(u))\|^2 du \right). \]

- It has dynamics
  \[ dZ^\theta(t) = -\theta^\top(t, S(t))Z^\theta(t)dW(t). \]

- If $Z^\theta(\cdot)$ is a martingale, that is, if $E[Z^\theta(T)] = 1$, then it defines a risk-neutral measure $\mathbb{Q}$ with $d\mathbb{Q} = Z^\theta(T)d\mathbb{P}$.

- Otherwise, $Z^\theta(\cdot)$ is a strict local martingale and classical arbitrage is possible.

- From Itô’s rule, we have
  \[ d \left( Z^\theta(t)S_i(t) \right) = Z^\theta(t)S_i(t) \sum_{k=1}^K (\sigma_i, k(t, S(t)) - \theta_k(t, S(t))) dW_k(t). \]
Everything an investor cares about: how and how much?

- We call *trading strategy* the number of shares held by an investor: \( \eta(t) = (\eta_1(t), \ldots, \eta_d(t))^T \)
- We assume that \( \eta(\cdot) \) is progressively measurable with respect to \( \mathbb{F} \) and self-financing.
- The corresponding wealth process \( V^{v,\eta}(\cdot) \) for an investor with initial wealth \( V^{v,\eta}(0) = v \) has dynamics

\[
dV^{v,\eta}(t) = \sum_{i=1}^{d} \eta_i(t) dS_i(t).
\]
- We restrict ourselves to trading strategies which satisfy \( V^{1,\eta}(t) \geq 0 \).
The terminal payoff

- Let $p : \mathbb{R}_+^{d} \rightarrow [0, \infty)$ denote a measurable function.
- The investor wants to have the payoff $p(S(T))$ at time $T$.
- For example,
  - market portfolio: $\tilde{p}(s) = \sum_{i=1}^{d} s_i$
  - money market: $p^0(s) = 1$
  - stock: $p^1(s) = s_1$
  - call: $p^c(s) = (s_1 - L)^+$ for some $L \in \mathbb{R}$.
- We define a candidate for the hedging price as
  \[
  h^p(t, s) := \mathbb{E}^{t, s} \left[ \tilde{Z}^\theta(T) p(S(T)) \right],
  \]
  where $\tilde{Z}^\theta(T) = Z^\theta(T)/Z^\theta(t)$ and $S(t) = s$ under the expectation operator $\mathbb{E}^{t, s}$.
Non path-dependent European claims

Assume that we have a contingent claim of the form $p(S(T)) \geq 0$ and that for all points of support $(t, s)$ for $S(\cdot)$ with $t \in [0, T)$ we have $h^p \in C^{1,2}(U_{t,s})$ for some neighborhood $U_{t,s}$ of $(t, s)$. Then, with $\eta^p_i(t, s) := D_i h^p(t, s)$ and $\nu^p := h^p(0, S(0))$, we get

$$V^\nu^p,\eta^p(t) = h^p(t, S(t)).$$

The strategy $\eta^p$ is optimal in the sense that for any $\tilde{\nu} > 0$ and for any strategy $\tilde{\eta}$ whose associated wealth process is nonnegative and satisfies $V^{\tilde{\nu},\tilde{\eta}}(T) \geq p(S(T))$, we have $\tilde{\nu} \geq \nu^p$. Furthermore, $h^p$ solves the PDE

$$\frac{\partial}{\partial t} h^p(t, s) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i s_j a_{i,j}(t, s) D_{i,j}^2 h^p(t, s) = 0$$

at all points of support $(t, s)$ for $S(\cdot)$ with $t \in [0, T)$. 
The proof relies on Itô’s formula.

• Define the martingale $N^p(\cdot)$ as

$$N^p(t) := \mathbb{E}[Z^\theta(T)p(S(T))|\mathcal{F}(t)] = Z^\theta(t)h^p(t, S(t)).$$

• Use a localized version of Itô’s formula to get the dynamics of $N^p(\cdot)$. Since it is a martingale, its $dt$ term must disappear which yields the PDE.

• Then, another application of Itô’s formula yields

$$dh^p(t, S(t)) = \sum_{i=1}^{d} D_i h^p(t, S(t))dS_i(t) = dV^{v^p, \eta^p}(t).$$

• This yields directly $V^{v^p, \eta^p}(\cdot) \equiv h^p(\cdot, S(\cdot))$. 
Proof (continued)

• Next, we prove optimality.

• Assume we have some initial wealth $\tilde{v} > 0$ and some strategy $\tilde{\eta}$ with nonnegative associated wealth process such that $V^{\tilde{v}, \tilde{\eta}}(T) \geq p(S(T))$ is satisfied.

• Then, $Z^\theta(\cdot) V^{\tilde{v}, \tilde{\eta}}(\cdot)$ is a supermartingale.

• This implies

$$
\tilde{v} \geq \mathbb{E}[Z^\theta(T) V^{\tilde{v}, \tilde{\eta}}(T)] \geq \mathbb{E}[Z^\theta(T) p(S(T))] \\
= \mathbb{E}[Z^\theta(T) V^{\nu_p, \eta_p}(T)] = \nu^p
$$
Non-uniqueness of PDE

• Usually,

\[
\frac{\partial}{\partial t} v(t, s) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i s_j a_{i,j}(t, s) D_{i,j}^2 v(t, s) = 0
\]

does not have a unique solution.

• However, if \( h^p \) is sufficiently differentiable, it can be characterized as the minimal nonnegative solution of the PDE.

• This follows as in the proof of optimality. If \( \tilde{h} \) is another nonnegative solution of the PDE with \( \tilde{h}(T, s) = p(s) \), then \( Z^\theta(\cdot)\tilde{h}(\cdot, S(\cdot)) \) is a supermartingale.
Corollary: Modified put-call parity

For any $L \in \mathbb{R}$ we have the modified put-call parity for the call- and put-options $(S_1(T) - L)^+$ and $(L - S_1(T))^+$, respectively, with strike price $L$:

$$
\mathbb{E}^{t,s} \left[ \tilde{Z}^\theta(T)(L - S_1(T))^+ \right] + h^p_1(t, s) = \mathbb{E}^{t,s} \left[ \tilde{Z}^\theta(T)(S_1(T) - L)^+ \right] + Lh^p_0(t, s),
$$

where $p^0(\cdot) \equiv 1$ denotes the payoff of one monetary unit and $p^1(s) = s_1$ the price of the first stock for all $s \in \mathbb{R}^d_+$. 
Role of Markovian market price of risk

Let $M \geq 0$ be a random variable measurable with respect to $\mathcal{F}^S(T)$. Let $\nu(\cdot)$ denote any MPR and $\theta(\cdot, \cdot)$ a Markovian MPR. Then, with

$$M^\nu(t) := \mathbb{E} \left[ \frac{Z^\nu(T)}{Z^\nu(t)} M \bigg| \mathcal{F}_t \right] \quad \text{and} \quad M^\theta(t) := \mathbb{E} \left[ \frac{Z^\theta(T)}{Z^\theta(t)} M \bigg| \mathcal{F}_t \right]$$

for $t \in [0, T]$, we have $M^\nu(\cdot) \leq M^\theta(\cdot)$ almost surely.
Proof

• We define $c(\cdot) := \nu(\cdot) - \theta(\cdot, S(\cdot))$ and $c^n(\cdot) := c(\cdot)1_{\{\tau_n \geq \cdot\}}$ for some localization sequence $\tau_n$.

• Then,

$$\frac{Z^\nu(T)}{Z^\nu(t)} = \lim_{n \to \infty} \frac{Z^{c^n}(T)}{Z^{c^n}(t)} \cdot \exp \left( - \int_t^T \theta^T (dW(u) + c^n(u)du) - \frac{1}{2} \int_t^T \|\theta\|^2 du \right).$$

• Since $\int_0^T c^n(t)dt$ is bounded, $Z^{c^n}(\cdot)$ is a martingale.

• Fatou’s lemma, Girsanov’s theorem and Bayes’ rule yield

$$M^\nu(t) \leq \liminf_{n \to \infty} \mathbb{E}^{Q^n} \left[ \exp \left( - \int_t^T \theta^T dW^n(u) - \frac{1}{2} \int_t^T \|\theta\|^2 du \right) M \bigg| \mathcal{F}_t \right].$$

• Since $\sigma(\cdot, S(\cdot))c^n(\cdot) \equiv 0$ the process $S(\cdot)$ has the same dynamics under $Q^n$ as under $P$. 
We can change the measure to compute $h^p$

- There exists not always an equivalent local martingale measure.
- However, after making some technical assumptions on the probability space and the filtration we can construct a new measure $\mathbb{Q}$ which corresponds to a “removal of the stock price drift”.
- Based on the work of Föllmer and Meyer and along the lines of Delbaen and Schachermayer.
Theorem: Under a new measure \( \mathbb{Q} \) the drifts disappear.

There exists a measure \( \mathbb{Q} \) such that \( \mathbb{P} \ll \mathbb{Q} \). More precisely, for all nonnegative \( \mathcal{F}(T) \)-measurable random variables \( Y \) we have

\[
\mathbb{E}^{\mathbb{P}}[Z^{\theta}(T)Y] = \mathbb{E}^{\mathbb{Q}}\left[Y1\left\{ \frac{1}{Z^{\theta}(T)}>0 \right\} \right].
\]

Under this measure \( \mathbb{Q} \), the stock price processes follow

\[
dS_i(t) = S_i(t) \sum_{k=1}^{K} \sigma_{i,k}(t, S(t)) \, d\widehat{W}_k(t)
\]

up to time \( \tau^{\theta} := \inf\{t \in [0, T] : 1/Z^{\theta}(t) = 0\} \). Here,

\[
\widehat{W}_k(t \wedge \tau^{\theta}) := W_k(t \wedge \tau^{\theta}) + \int_{0}^{t \wedge \tau^{\theta}} \theta_k(u, S(u)) \, du
\]

is a \( K \)-dimensional \( \mathbb{Q} \)-Brownian motion stopped at time \( \tau^{\theta} \).
What happens in between time 0 and time $T$: Bayes’ rule.

For all nonnegative $\mathcal{F}(T)$-measurable random variables $Y$ the representation

$$
\mathbb{E}^Q \left[ Y 1\{1/Z^\theta(T)>0\} \bigg| \mathcal{F}(t) \right] = \mathbb{E}^P[Z^\theta(T)Y|\mathcal{F}(t)] \frac{1}{Z^\theta(t)} 1\{1/Z^\theta(t)>0\}
$$

holds $Q$-almost surely (and thus $P$-almost surely) for all $t \in [0, T]$. 

After a change of measure, the Bessel process becomes Brownian motion.

• We look at a market with one stock:

\[ dS(t) = \frac{1}{S(t)} dt + dW(t). \]

• We have \( S(t) > 0 \) for all \( t \geq 0 \).

• The market price of risk is \( \theta(t, s) = 1/s \).

• The inverse stochastic discount factor \( 1/Z^\theta \) becomes zero exactly when \( S(t) \) hits 0.

• Removing the drift with a change of measure as before makes \( S(\cdot) \) a Brownian motion (up to the first hitting time of zero by \( 1/Z^\theta(\cdot) \)) under \( \mathbb{Q} \).
The optimal strategy for getting one dollar at time $T$ can be explicitly computed.

- For $p(s) \equiv p^0(s) \equiv 1$ we get

$$h^{p^0}(t, s) = \mathbb{E}^P\left[\frac{Z^\theta(T)}{Z^\theta(t)} \cdot 1 \bigg| \mathcal{F}_t \right]_{S(t)=s} = \mathbb{E}^Q[\mathbf{1}_{\left\{1/Z^\theta(T)>0\right\}}|\mathcal{F}_t]|S(t)=s$$

$$= 2\Phi\left(\frac{s}{\sqrt{T-t}}\right) - 1.$$  

- This yields the optimal strategy

$$\eta^0(t, s) = \frac{2}{\sqrt{T-t}}\phi\left(\frac{s}{\sqrt{T-t}}\right).$$

- The hedging price $h^p$ satisfies on all points $\{s > 0\}$ the PDE

$$\frac{\partial}{\partial t} h^p(t, s) + \frac{1}{2} D^2 h^p(t, s) = 0.$$
Conclusion

- No equivalent local martingale measure needed to find an optimal hedging strategy based upon the familiar delta hedge.
- Sufficient conditions are derived for the necessary differentiability of expectations indexed over the initial market configuration.
- The dynamics of stochastic processes under a non-equivalent measure and a generalized Bayes' rule might be of interest themselves.
- We have computed some optimal trading strategies in standard examples for which so far only ad-hoc and not necessarily optimal strategies have been known.
Thank you!