Riding on the Smiles

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Objectives:

- Studying the calibration properties of several stochastic volatility models

- Provide some price approximations allowing a better understanding of the models
On the calibration of the Heston (1993) model

\[
\begin{align*}
\frac{dF_t}{F_t} &= \sqrt{v_t}dW_t^1 \\
\, dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^2 \\
\, dW_t^1 dW_t^2 &= \rho dt
\end{align*}
\]

\(\rho\) controls the link between vol and asset returns

\[\downarrow\]

The Skew or Leverage
The smile

Option implied volatility DAX 28/08/2008

Option prices DAX 28/08/2008
the skew is controlled by $\rho$

$\Downarrow$

we have a term structure of skews

$\Downarrow$

we should have different values for $\rho$

$$\min \frac{1}{N} \sum_{i=1}^{N} (\sigma_{imp}^{model}(t, T_i, K_i) - \sigma_{imp}^{mkt}(t, T_i, K_i))^2$$

(norm in price won’t work for multidimensional models).
Why extending the Heston model?

- The dynamics of the implied volatility surface (vanilla options) and the Variance Swap curve are driven by several factors.
- On the FX market the skew is stochastic.
- We have a term structure of skew: short term skew $\neq$ long term skew.
Double-Heston model

(Christoffensen, Heston, Jacobs 2007)

\[
\begin{align*}
\frac{dF_t}{F_t} &= \sqrt{v^1_t} dZ^1_t + \sqrt{v^2_t} dZ^2_t \\
 dv^1_t &= \kappa^1 (\theta^1 - v^1_t) dt + \sigma^1 \sqrt{v^1_t} dW^1_t \\
 dv^2_t &= \kappa^2 (\theta^2 - v^2_t) dt + \sigma^2 \sqrt{v^2_t} dW^2_t \\
 dZ^1_t dW^1_t &= \rho^1 dt \\
 dZ^2_t dW^2_t &= \rho^2 dt
\end{align*}
\]

but

\[
\underbrace{dZ^1_t dZ^2_t = dW^1_t dW^2_t = dZ^1_t dW^2_t = dZ^2_t dW^1_t = 0}_{AFFINITY \text{ } \text{Duffie–Filipovic–Schachermayer(2003)'}s \text{condition}}
\]
Wishart multi-dim Stochastic Vol

• \( \frac{dF_t}{F_t} = Tr \left[ \sqrt{\Sigma_t} dZ_t \right] \)

• \( d\Sigma_t = (\beta Q^T Q + M \Sigma_t + \Sigma_t M^\top) dt + \sqrt{\Sigma_t} dW_t Q + Q^T dW_t^\top \sqrt{\Sigma_t} \)

• \( Z_t = \) Matrix Brownian Motion correlated with \( W_t \) (Matrix Brownian Motion)

• \( Vol(F_t) = Tr [\Sigma_t] \) linear combination of the Wishart elements

• \( Z_t = W_t R^\top + B_t \sqrt{I - RR^\top} \) with \( B_t \) matrix BM.
• The **Wishart Affine model** is solvable. That is, the conditional characteristic function can be written as:

\[ \mathbb{E}_t e^{i\omega \log(F_{t+\tau})} = e^{Tr[A(\tau)\Sigma_t] + B(\tau) \log(F_t) + C(\tau)} \]

• **$A(\tau)$** solves a **Riccati ODE**.

\[
\begin{align*}
\partial_\tau A &= A(M + i\omega Q^T R^T) + (M + i\omega Q^T R^T)^T A + 2AQ^TQA - \frac{\omega^2 + i\omega}{2} \\
\partial_\tau c &= Tr[\Omega\Omega^T A(\tau)]
\end{align*}
\]

\[ A(\tau) = A_{22}(\tau)^{-1} A_{21}(\tau), \]
with
\[
\begin{pmatrix}
A_{11}(\tau) & A_{12}(\tau) \\
A_{21}(\tau) & A_{22}(\tau)
\end{pmatrix}
= \exp \tau \begin{pmatrix}
M + i\omega Q^\top R^\top & -2Q^\top Q \\
\frac{i\omega(i\omega - 1)}{2}I_n & -\left( M + i\omega Q^\top R^\top \right)^\top
\end{pmatrix},
\]
Stochastic correlation between stock returns and vol

\[
Corr_t (d\ln(F), d\text{Vol} (\ln(F))) = \rho_t = \frac{2Tr [\Sigma_t RQ]}{\sqrt{Tr [\Sigma_t]} \sqrt{Tr [\Sigma_t Q^\top Q]}}
\]

- Stochastic correlation between the stock and its volatility
- Multi-dimensional correlation/volatility **SHOULD** allow for more complex skew effects
The Multi-asset model

How to build a multi asset framework:

- Consistent with the smile in vanilla options
- With a general correlation structure
- Analytic as much as possible

The Heston model leads to constraints on the correlation structure between assets (affinity)
The Wishart Affine Stochastic Correlation model

The model: \( F_t = (F_t^1, F_t^2)^\top \) and \( \Sigma_t \in M_{(2,2)} \)

\[
dF_t = \text{diag}(F_t) \left( \sqrt{\Sigma_t} dZ_t \right)
\]

\[
d\Sigma_t = \left( \Omega \Omega^\top + M \Sigma_t + \Sigma_t M^\top \right) dt + \sqrt{\Sigma_t} dW_t Q + Q^\top (dW_t)^\top \sqrt{\Sigma_t}
\]

\( dZ_t \) is a vector BM (n,1) and \( dW_t \) is a matrix BM (n,n):

\[
\frac{dF^1}{F^1} \frac{dF^2}{F^2} = \Sigma^{12} dt
\]

\[
dZ_t = dW_t \rho + \sqrt{1 - \rho^\top \rho} dB_t
\]

where \( \rho \) is a vector (n,1) and \( dB \) is a vector BM(n,1). We know how to compute the characteristic function as in the previous model.
Using perturbation method as in Benabid, Bensusan, El Karoui (2009) but for the vol of vol: \( Q \rightarrow \alpha Q \)

\[
\begin{align*}
\partial_\tau A &= A \left( M + i\omega Q^\top R^\top \right) + \left( M + i\omega Q^\top R^\top \right)^\top A + 2AQ^\top QA - \frac{\omega^2 + i\omega}{2} I \\
\partial_\tau c &= \text{Tr}[\Omega \Omega^\top A(\tau)] \\
A &= A^0 + \alpha A^1 + \alpha^2 A^2 \text{ in the above equation and collecting the terms:}
\end{align*}
\]

\[
\begin{align*}
\partial_\tau A^0 &= A^0 M + M^\top A^0 - \frac{\left( \omega^2 + i\omega \right)}{2} I \\
\partial_\tau A^1 &= A^1 M + M^\top A^1 + A^0 \left( i\omega Q^\top R^\top \right) + \left( i\omega Q^\top R^\top \right)^\top A^0 \\
\partial_\tau A^2 &= A^2 M + M^\top A^2 + A^1 \left( i\omega Q^\top R^\top \right) + \left( i\omega Q^\top R^\top \right)^\top A^1 + 2A^0 Q^\top QA^0.
\end{align*}
\]

\[e^{i\omega x-\frac{(\omega^2+i\omega)}{2}\sigma^2_x} \quad \text{BS}\]
**σ_{imp} for short time expansion**

Using perturbation method as in Benabid, Bensusan, El Karoui (2009) but for the vol of vol: (forward moneyness $m_f$)

\[
\sigma_{imp}^2 \sim v_1 + \frac{\rho_1 \sigma_1}{2} m_f \\
\sigma_{imp}^2 \sim v_1 + v_2 + \left(\frac{v_1 \rho_1 \sigma_1 + v_2 \rho_2 \sigma_2}{v_1 + v_2}\right) \frac{m_f}{2} \\
\sigma_{imp}^2 \sim \text{Tr}[\Sigma_t] + \frac{\text{Tr}[RQ\Sigma_t]}{\text{Tr}[\Sigma_t]} m_f \\
\sigma_{imp}^2 \sim \Sigma_{11}^{t} + \alpha m_f (\rho_1 Q_{11} + \rho_2 Q_{21})
\]

- calibration: Heston < Double-Heston and WMSV and WASC
- calibration: Double-Heston and WMSV: 2 regimes of VolOfVol (long/short)
- calibration: when aggregated the parameters give similar expansion (consistency)
- $\Sigma_{12}$ controls the slope of the skew and $\Sigma_{11} + \Sigma_{22}$ controls the level of the smile (if $RQ$ is non diagonal). In the Double-Heston the factors impact both level and skew!
\( \sigma_{imp} \) for short time expansion

\[
\sigma_{imp}^2 \sim v_1 + \frac{\rho_1 \sigma_1}{2} m_f \quad \text{Heston}
\]

\[
\sigma_{imp}^2 \sim v_1 + v_2 + \left( \frac{v_1 \rho_1 \sigma_1 + v_2 \rho_2 \sigma_2}{v_1 + v_2} \right) \frac{m_f}{2} \quad \text{Double-Heston}
\]

\[
\sigma_{imp}^2 \sim \text{Tr}[\Sigma_t] + \frac{\text{Tr}[RQ\Sigma_t]}{\text{Tr}[\Sigma_t]} m_f \quad \text{WMSV}
\]

\[
\sigma_{imp}^2 \sim \Sigma_t^{11} + \alpha m_f (\rho_1 Q_{11} + \rho_2 Q_{21}) \quad \text{Wasc}
\]

- calibration: WASC on DAX/FTSE or DAX/EuroStoxx same information on DAX smile! Vanilla products are basket product!
Thanks for your attention!