Riding on the Smiles

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Objectives:

- Studying the calibration properties of several stochastic volatility models
- Provide some price approximations allowing a better understanding of the models

On the calibration of the Heston (1993) model

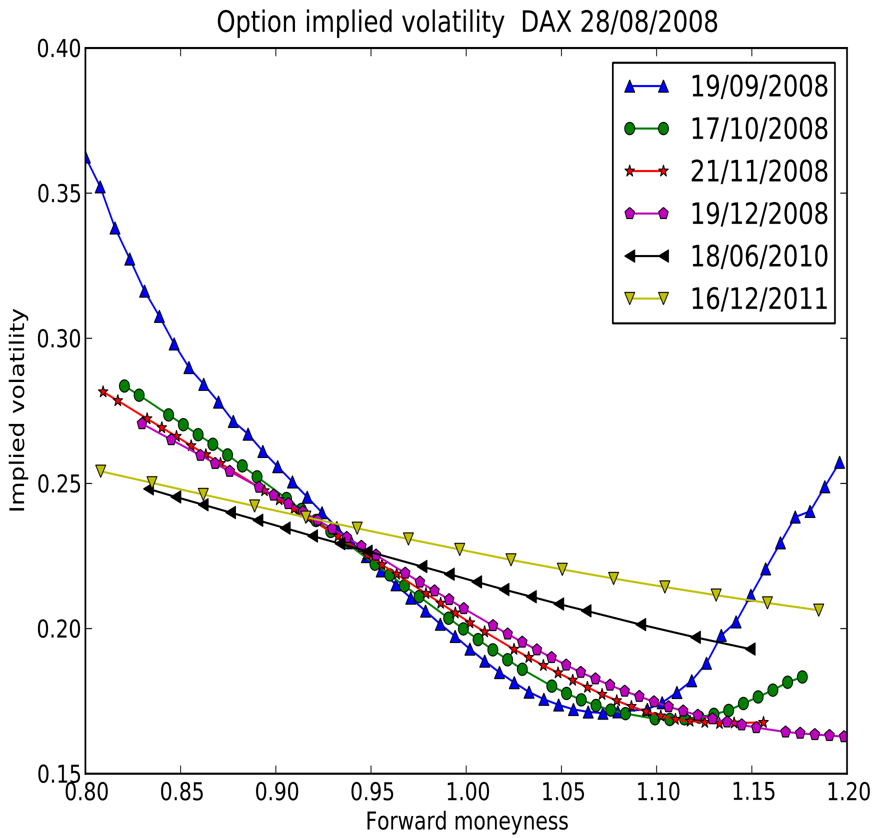
$$\begin{aligned} \frac{dF_t}{F_t} &= \sqrt{v_t}\, dW_t^1 \\ dv_t &= \kappa(\theta - v_t)\,dt + \sigma\sqrt{v_t}\, dW_t^2 \\ dW_t^1 dW_t^2 &= \rho\, dt \end{aligned}$$

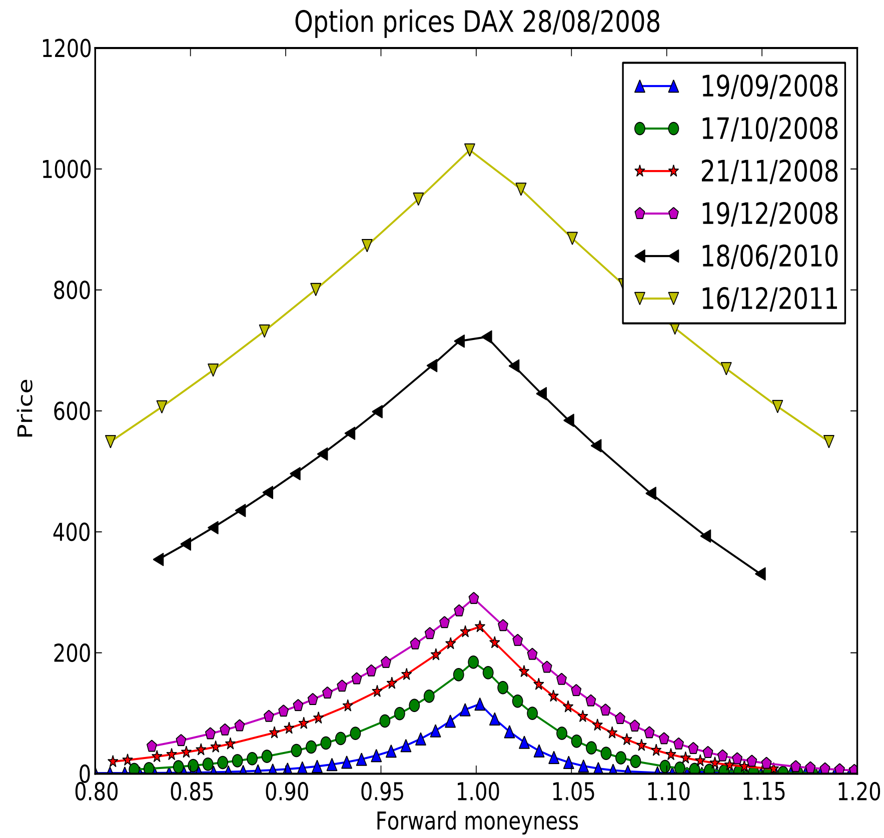
ρ controls the link between vol and asset returns

$\Downarrow$

The Skew or Leverage

The smile


Option implied volatility DAX 28/08/2008
Implied volatility
Forward moneyness
19/09/2008
17/10/2008
21/11/2008
19/12/2008
18/06/2010
16/12/2011
0.40
0.35
0.30
0.25
0.20
0.15
0.80
0.85
0.90
0.95
1.00
1.05
1.10
1.15
1.20


Option prices DAX 28/08/2008
Price
Forward moneyness
19/09/2008
17/10/2008
21/11/2008
19/12/2008
18/06/2010
16/12/2011
1200
1000
800
600
400
200
0
0.80
0.85
0.90
0.95
1.00
1.05
1.10
1.15
1.20

the skew is controlled by ρ

$\Downarrow$

we have a term structure of skews

$\Downarrow$

we should have different values for ρ

$$\min \frac{1}{N} \sum_{i=1}^{N} (\sigma_{imp}^{model}(t, T_i, K_i) - \sigma_{imp}^{mkt}(t, T_i, K_i))^2$$

(norm in price won't work for multidimensional models).

Why extending the Heston model?

- The dynamics of the implied volatility surface (vanilla options) and the Variance Swap curve are driven by several factors

- On the FX market the skew is stochastic

- We have a term structure of skew: short term skew $\neq$ long term skew

Double-Heston model

(Christoffensen, Heston, Jacobs 2007)

$$\begin{aligned}
\frac{dF_t}{F_t} &= \sqrt{v_t^1}\,dZ_t^1 + \sqrt{v_t^2}\,dZ_t^2 \\
dv_t^1 &= \kappa^1(\theta^1 - v_t^1)dt + \sigma^1\sqrt{v_t^1}\,dW_t^1 \\
dv_t^2 &= \kappa^2(\theta^2 - v_t^2)dt + \sigma^2\sqrt{v_t^2}\,dW_t^2 \\
dZ_t^1 dW_t^1 &= \rho^1 dt \\
dZ_t^2 dW_t^2 &= \rho^2 dt
\end{aligned}$$

but

$$\underbrace{dZ_t^1 dZ_t^2 = dW_t^1 dW_t^2 = dZ_t^1 dW_t^2 = dZ_t^2 dW_t^1 = 0}_{AFFINITY\,Duffie-Filipovic-Schachermayer(2003)'s condition}$$

Wishart multi-dim Stochastic Vol

- $\frac{dF_t}{F_t} = Tr\left[\sqrt{\Sigma_t} dZ_t\right]$
- $d\Sigma_t = (\beta Q^\top Q + M\Sigma_t + \Sigma_t M^\top)dt + \sqrt{\Sigma_t} dW_t Q + Q^\top dW_t^\top \sqrt{\Sigma_t}$
- Z_t = Matrix Brownian Motion correlated with W_t (Matrix Brownian Motion)
- $Vol(F_t) = Tr\left[\Sigma_t\right]$ linear combination of the Wishart elements
- $Z_t = W_t R^\top + B_t \sqrt{\mathbb{I} - RR^\top}$ with B_t matrix BM.

- The Wishart Affine model is solvable. That is, the conditional characteristic function can be written as:

$$\mathbb{E}_t e^{i\omega \log(F_{t+\tau})} = e^{Tr[A(\tau)\Sigma_t]+B(\tau)\log(F_t)+C(\tau)}$$

- $A(\tau)$ solves a Riccati ODE.

$$\begin{aligned}
\partial_\tau A &= A\left(M + i\omega Q^\top R^\top\right) + \left(M + i\omega Q^\top R^\top\right)^\top A + 2AQ^\top QA - \frac{\omega^2 + i\omega}{2}\mathbb{I} \\
\partial_\tau c &= \mathrm{Tr}[\Omega\Omega^\top A(\tau)]
\end{aligned}$$

$$A(\tau) = A_{22}(\tau)^{-1} A_{21}(\tau),$$

with

$$\left(\begin{array}{cc} A_{11}(\tau) & A_{12}(\tau) \\ A_{21}(\tau) & A_{22}(\tau) \end{array} \right) = \exp \tau \left(\begin{array}{cc} M + i\omega Q^\top R^\top & -2Q^\top Q \\ \frac{i\omega(i\omega-1)}{2}\mathbb{I}_n & -\left(M + i\omega Q^\top R^\top\right)^\top \end{array} \right),$$

Stochastic correlation between stock returns and vol

$$Corr_t\left(dln(F), dVol\left(ln(F)\right)\right) = \quad \rho_t = \quad \frac{2Tr\left[\Sigma_t RQ\right]}{\sqrt{Tr\left[\Sigma_t\right]}\sqrt{Tr\left[\Sigma_t Q^\top Q\right]}}$$

- Stochastic correlation between the stock and its volatility
- Multi-dimensional correlation/volatility SHOULD allow for more complex skew effects

The Multi-asset model

How to build a multi asset framework:

- Consistent with the smile in vanilla options

- With a general correlation structure

- Analytic as much as possible

The Heston model leads to constraints on the correlation structure between assets (affinity)

The Wishart Affine Stochastic Correlation model

<u>The model</u>: $F_t = (F_t^1, F_t^2)^\top$ and $\Sigma_t \in M_{(2,2)}$

$$dF_t = diag[F_t]\left(\sqrt{\Sigma_t}dZ_t\right)$$
$$d\Sigma_t = \left(\Omega\Omega^\top + M\Sigma_t + \Sigma_t M^\top\right)dt + \sqrt{\Sigma_t}dW_tQ + Q^\top(dW_t)^\top\sqrt{\Sigma_t}$$

dZ_t is a vector BM (n,1) and dW_t is a matrix BM (n,n):

$$\frac{dF^1}{F^1}\frac{dF^2}{F^2} = \Sigma^{12}dt$$

$$dZ_t = dW_t\rho + \sqrt{1-\rho^\top\rho}dB_t$$

where ρ is a vector (n,1) and dB is a vector BM(n,1). We know how to compute the characteristic function as in the previous model.

σ_{imp} short time expansion

Using perturbation method as in Benabid, Bensusan, El Karoui (2009) but for the vol of vol: $Q \to \alpha Q$

$$\begin{aligned}
\partial_\tau A &= A\left(M + i\omega Q^\top R^\top\right) + \left(M + i\omega Q^\top R^\top\right)^\top A + 2AQ^\top QA - \frac{\omega^2 + i\omega}{2}\mathbb{I} \\
\partial_\tau c &= \mathrm{Tr}[\Omega\Omega^\top A(\tau)]
\end{aligned}$$

$A = A^0 + \alpha A^1 + \alpha^2 A^2$ in the above equation and collecting the terms:

$$\begin{aligned}
\partial_\tau A^0 &= A^0 M + M^\top A^0 - \frac{(\omega^2 + i\omega)}{2}\mathbb{I} \\
\partial_\tau A^1 &= A^1 M + M^\top A^1 + A^0\left(i\omega Q^\top R^\top\right) + \left(i\omega Q^\top R^\top\right)^\top A^0 \\
\partial_\tau A^2 &= A^2 M + M^\top A^2 + A^1\left(i\omega Q^\top R^\top\right) + \left(i\omega Q^\top R^\top\right)^\top A^1 + 2A^0 Q^\top Q A^0.
\end{aligned}$$

$$e^{i\omega x - \frac{(\omega^2 + i\omega)}{2}\sigma^2\tau} \qquad \text{BS}$$

σ_{imp} for short time expansion

Using perturbation method as in Benabid, Bensusan, El Karoui (2009) but for the vol of vol: (forward moneyness m_f)

$$\sigma_{imp}^2 \sim v_1 + \frac{\rho_1 \sigma_1}{2} m_f \qquad \text{Heston}$$

$$\sigma_{imp}^2 \sim v_1 + v_2 + \left(\frac{v_1 \rho_1 \sigma_1 + v_2 \rho_2 \sigma_2}{v_1 + v_2} \right) \frac{m_f}{2} \qquad \text{Double-Heston}$$

$$\sigma_{imp}^2 \sim \mathrm{Tr}[\Sigma_t] + \frac{\mathrm{Tr}[RQ\Sigma_t]}{\mathrm{Tr}[\Sigma_t]} m_f \qquad \text{WMSV}$$

$$\sigma_{imp}^2 \sim \Sigma_t^{11} + \alpha m_f (\rho_1 Q_{11} + \rho_2 Q_{21}) \qquad \text{Wasc}$$

- calibration : Heston $<$ Double-Heston and WMSV and WASC
- calibration : Double-Heston and WMSV: 2 regimes of VolOfVol (long/short)
- calibration: when aggregated the parameters give similar expansion (consistency)
- Σ_{12} controls the slope of the skew and $\Sigma_{11} + \Sigma_{22}$ controls the level of the smile (if RQ is non diagonal). In the Double-Heston the factors impact both level and skew!

σ_{imp} for short time expansion

$$\sigma_{imp}^2 \sim v_1 + \frac{\rho_1\sigma_1}{2} m_f \qquad \text{Heston}$$

$$\sigma_{imp}^2 \sim v_1 + v_2 + \left(\frac{v_1\rho_1\sigma_1 + v_2\rho_2\sigma_2}{v_1 + v_2}\right)\frac{m_f}{2} \qquad \text{Double-Heston}$$

$$\sigma_{imp}^2 \sim \mathrm{Tr}[\Sigma_t] + \frac{\mathrm{Tr}[RQ\Sigma_t]}{\mathrm{Tr}[\Sigma_t]} m_f \qquad \text{WMSV}$$

$$\sigma_{imp}^2 \sim \Sigma_t^{11} + \alpha m_f(\rho_1 Q_{11} + \rho_2 Q_{21}) \qquad \text{Wasc}$$

- calibration : WASC on DAX/FTSE or DAX/EuroStoxx same information on DAX smile! Vanilla products are basket product!

Thanks for your attention!