Pricing American Step-Up and Step-Down Credit Default Swaps under Lévy Models

Kazutoshi Yamazaki

Center for the Study of Finance and Insurance, Osaka University

Joint Work with **Tim Siu-Tang Leung**, Johns Hopkins University

Modeling and Managing Financial Risks

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Outline

- American Step-up/down Callable/Putable CDSs.
  - Definition,
  - Relationship with Vanilla Credit Default Swaps and Swaptions.
- Solve analytically for a general exponential Lévy process with only negative jumps.
  - Spectrally Negative Lévy Processes and Scale Functions.
  - Solve optimal stopping problems.
- Numerical Example.

Model

- Let $X = \{X_t;\ t \geq 0\}$ be a Lévy process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- The value of the reference entity (a company stock or other assets) is assumed to evolve according to an exponential Lévy process

$$S_t = e^{X_t}, \quad t \geq 0.$$

- Following the Black-Cox structural approach, the default event is triggered by S crossing a lower level D.
- Without loss of generality, we can take $\log D = 0$ by shifting the initial value X_0. Henceforth, we shall work with the default time:

$$\theta := \inf\{\, t \geq 0 \,:\, X_t \,\leq\, 0\,\}.$$

Vanilla CDS

- From the buyer's perspective, the expected discounted payoff is given by

$$\bar{C}(x; p, \alpha, T) := \mathbb{E}^x \left[-\int_0^{\theta \wedge T} e^{-rt} p \, \mathrm{d}t + \alpha e^{-r\theta} 1_{\{\theta \leq T\}} \right],$$

where T is the maturity and $r > 0$ is the positive constant risk-free interest rate.

- The fair premium is

$$\bar{p}(x; \alpha, T) = \frac{\alpha \, r \, \zeta_T(x)}{1 - \zeta_T(x) - e^{-rT} \mathbb{P}^x\{\theta > T\}}$$

where

$$\zeta_T(x) := \mathbb{E}^x \left[e^{-r\theta} 1_{\{\theta \leq T\}} \right].$$

Vanilla CDS (Perpetual)

- When $T = +\infty$, the buyer's CDS price is

$$C(x; p, \alpha) := \mathbb{E}^x \left[-\int_0^\theta e^{-rt} p \,\mathrm{d}t + \alpha\, e^{-r\theta} \right]$$
$$= \left(\frac{p}{r} + \alpha \right) \zeta(x) - \frac{p}{r},$$

where

$$\zeta(x) := \mathbb{E}^x \left[e^{-r\theta} \right].$$

- The seller's CDS price is $-C(x; p, \alpha) = C(x; -p, -\alpha)$.
- Solving $C(x; p, \alpha) = 0$ yields the credit spread:

$$p(x; \alpha) = \frac{\alpha\, r\, \zeta(x)}{1 - \zeta(x)}.$$

American Perpetual Swaptions

The prices of (generalized American perpetual) payer and receiver swaptions are

$$v(x;\kappa,a,K) := \sup_{\tau\in\mathcal{T}} \mathbb{E}^x \left[e^{-r\tau} \left(C(X_\tau;\kappa,a) - K \right)^+ 1_{\{\theta>\tau\}} \right],$$
$$u(x;\kappa,a,K) := \sup_{\tau\in\mathcal{T}} \mathbb{E}^x \left[e^{-r\tau} \left(-C(X_\tau;\kappa,a) - K \right)^+ 1_{\{\theta>\tau\}} \right]$$

with

- default payment a;
- pre-specified spread κ;
- strike price K upon exercise.

Callable Step-Up/Down CDS

We consider a CDS contract with an embedded option that permits the protection buyer to change

- premium from p to $\hat{p}$,
- default payment from α to $\hat{\alpha}$,

any time before θ once for a fee γ. The value for the buyer is

$$
\begin{aligned}
&V(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) \\
&:= \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \Bigg[-\int_0^{\tau} e^{-rt} p \, \mathrm{d}t + 1_{\{\tau < \infty\}} \\
&\qquad \left(-\int_{\tau}^{\theta} e^{-rt} \hat{p} \, \mathrm{d}t - e^{-r\tau} \gamma 1_{\{\tau < \theta\}} + e^{-r\theta} (\hat{\alpha} 1_{\{\tau < \theta\}} + \alpha 1_{\{\tau = \theta\}}) \right) \Bigg],
\end{aligned}
$$

where $\mathcal{S} := \{ \nu \in \mathbb{F} : \tau \leq \theta \text{ a.s. } \}$.

Step-Up/Down and Cancellation

This formulation covers default swaps with the following provisions:

1. Step-up Option: if $\hat{p} > p$ and $\hat{\alpha} > \alpha$, then the buyer is allowed to increase the coverage once from α to $\hat{\alpha}$ by paying the fee γ and a higher premium $\hat{p}$ thereafter.
2. Step-down Option: when $\hat{p} < p$ and $\hat{\alpha} < \alpha$, then the buyer can reduce the coverage once from α to $\hat{\alpha}$ by paying the fee γ and a reduced premium $\hat{p}$ thereafter.
3. Cancellation Right: as a special case of the step-down option with $\hat{p} = \hat{\alpha} = 0$, the resulting contract allows the buyer to terminate the CDS at time τ. This special case corresponds to an American callable step-down CDS.

Putable Step-Up/Down CDS

We consider a CDS contract with an embedded option that permits the protection seller to change

- premium from p to $\hat{p}$,
- default payment from α to $\hat{\alpha}$,

any time before θ once for a fee γ. The value for the seller is

$$U(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) := \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \Bigg[\int_0^\tau e^{-rt} p \, \mathrm{d}t + 1_{\{\tau < \infty\}} \left(\int_\tau^\theta e^{-rt} \hat{p} \, \mathrm{d}t - e^{-r\tau} \gamma 1_{\{\tau < \theta\}} - e^{-r\theta} (\hat{\alpha} 1_{\{\tau < \theta\}} + \alpha 1_{\{\tau = \theta\}}) \right) \Bigg].$$

Decomposition

Let

$$\tilde{\alpha} := \alpha - \hat{\alpha}, \quad \text{and} \quad \tilde{p} := p - \hat{p}.$$

We can decompose callable and putable CDSs, respectively, by

$$V(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) = C(x; p, \alpha) + v(x; -\tilde{p}, -\tilde{\alpha}, \gamma),$$
$$U(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) = -C(x; p, \alpha) + u(x; -\tilde{p}, -\tilde{\alpha}, \gamma)$$

where

$$C(x; p, \alpha) := \mathbb{E}^x \left[-\int_0^\theta e^{-rt} p \, \mathrm{d}t + \alpha \, e^{-r\theta} \right],$$
$$v(x; \kappa, a, K) := \sup_{\tau \in \mathcal{T}} \mathbb{E}^x \left[e^{-r\tau} \left(C(X_\tau; \kappa, a) - K \right)^+ 1_{\{\theta > \tau\}} \right],$$
$$u(x; \kappa, a, K) := \sup_{\tau \in \mathcal{T}} \mathbb{E}^x \left[e^{-r\tau} \left(-C(X_\tau; \kappa, a) - K \right)^+ 1_{\{\theta > \tau\}} \right].$$

Hedging

	Protection Buyer's Position	*Protection Seller's Position*
American Callable Step-Up CDS	long a vanilla CDS + long American payer default swaption	short a vanilla CDS + short American payer default swaption
American Callable Step-Down CDS	long a vanilla CDS + long American receiver default swaption	short a vanilla CDS + short American receiver default swaption

	Protection Buyer's Position	*Protection Seller's Position*
American Putable Step-Up CDS	long a vanilla CDS + short American receiver default swaption	short a vanilla CDS + long American receiver default swaption
American Putable Step-Down CDS	long a vanilla CDS + short American payer default swaption	short a vanilla CDS + long American payer default swaption

We also find the "put-call parity" and symmetry identities:

$$V(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) - U(x; p, 2p - \hat{p}, \alpha, 2\alpha - \hat{\alpha}, \gamma) = 2\, C(x; p, \alpha),$$
$$V(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) + U(x; p, 2p - \hat{p}, \alpha, 2\alpha - \hat{\alpha}, \gamma) = 2\, v(x; -\tilde{p}, -\tilde{\alpha}, \gamma).$$

What We Need to Solve

Because $v(x; -\tilde{p}, -\tilde{\alpha}, \gamma) = u(x; \tilde{p}, \tilde{\alpha}, \gamma)$, we only need to solve for

1. the callable step-down CDS with $\tilde{p} > 0$ and $\tilde{\alpha} > 0$, and
2. the putable step-down CDS with $\tilde{p} > 0$ and $\tilde{\alpha} > 0$.

In particular, we need to solve the following optimal stopping problems:

$$v(x) := v(x; -\tilde{p}, -\tilde{\alpha}, \gamma) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} h(X_\tau) 1_{\{\tau < \infty\}} \right],$$

$$u(x) := u(x; -\tilde{p}, -\tilde{\alpha}, \gamma) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} g(X_\tau) 1_{\{\tau < \infty\}} \right],$$

where

$$h(x) := \left(\left(\frac{\tilde{p}}{r} - \gamma \right) - \left(\frac{\tilde{p}}{r} + \tilde{\alpha} \right) \zeta(x) \right) 1_{\{x>0\}},$$

$$g(x) := \left(\left(-\frac{\tilde{p}}{r} - \gamma \right) + \left(\frac{\tilde{p}}{r} + \tilde{\alpha} \right) \zeta(x) \right) 1_{\{x>0\}}.$$

Spectrally Negative Lévy Processes

Defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $X = \{X_t; t \geq 0\}$ be a spectrally negative Lévy process, i.e.

1. The paths are almost surely right continuous with left limits.
2. For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.
3. For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.
4. Jumps are almost surely negative (spectrally negative).

Laplace exponent is the logarithm of its Laplace transform:

$$\psi(s) := \log \mathbb{E}^0 \left[e^{sX_1} \right], \quad s \in \mathbb{C}.$$

and we can write

$$\psi(s) := cs + \frac{1}{2}\sigma^2 s^2 + \int_{(0,\infty)} (e^{-sx} - 1 + sx 1_{\{0<x<1\}}) \Pi(\mathrm{d}x), \quad s \in \mathbb{C}.$$

Scale Functions

Associated with every spectrally negative Lévy process, there exists a *(r-)scale function*

$$W^{(r)} : \mathbb{R} \mapsto \mathbb{R},$$

whose Laplace transform is given by

$$\int_0^\infty e^{-\beta x} W^{(r)}(x)\mathrm{d}x = \frac{1}{\psi(\beta) - r}, \qquad \beta > \Phi(r)$$

where $\Phi(r)$ is the (largest) *positive root* of

$$\Phi(r) := \sup\{s \geq 0 : \psi(s) = r\}.$$

We assume $W^{(r)}(x) = 0$ on $(-\infty, 0)$.

Scale Functions (Cont'd)

We define the *first down-* and *up-crossing times*, respectively, by

$$\tau_a^- := \inf\{t \geq 0 : X_t < a\},$$
$$\tau_b^+ := \inf\{t \geq 0 : X_t > b\}$$

for any $0 \leq a < x < b$. Then we have, for every $0 \leq x < b$,

$$\mathbb{E}^x\left[e^{-q\tau_b^+}1_{\{\tau_b^+<\tau_0^-\}}\right] = \frac{W^{(r)}(x)}{W^{(r)}(b)},$$
$$\mathbb{E}^x\left[e^{-q\tau_0^-}1_{\{\tau_b^+>\tau_0^-\}}\right] = Z^{(r)}(x) - Z^{(r)}(b)\frac{W^{(r)}(x)}{W^{(r)}(b)}$$

where $Z^{(r)}(x) = 1 + q\int_0^x W^{(r)}(y)\mathrm{d}y$ for every $x \in \mathbb{R}$.

Callable Step-down CDS

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We need to solve

$$v(x) := \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} h(X_\tau) 1_{\{\tau < \infty\}} \right],$$

where

$$h(x) := \left(\left(\frac{\tilde{p}}{r} - \gamma \right) - \left(\frac{\tilde{p}}{r} + \tilde{\alpha} \right) \zeta(x) \right) 1_{\{x \geq 0\}}$$

and

$$\zeta(x) = Z^{(r)}(x) - \frac{r}{\Phi(r)} W^{(r)}(x), \quad x \in \mathbb{R}.$$

Using the scale functions, the stopping value becomes

$$h(x) = \left[\tilde{p} \left(\frac{1}{r} \left(1 - Z^{(r)}(x) \right) + \frac{1}{\Phi(r)} W^{(r)}(x) \right) - \tilde{\alpha} \left(Z^{(r)}(x) - \frac{r}{\Phi(r)} W^{(r)}(x) \right) - \gamma \right] 1_{\{x > 0\}}, \quad x \in \mathbb{R}.$$

Optimal Stopping Time

We can guess that the optimal stopping time is in the form

$$\tau_B^+ := \inf\left\{t \geq 0 : X_t \notin (0, B)\right\}.$$

Define

$$\varrho(B) := (\tilde{p} + \tilde{\alpha} r)\, W^{(r)}(B) - \frac{W^{(r)\prime}(B)}{W^{(r)}(B)} G^{(r)}(B)$$

where

$$G^{(r)}(B) := \frac{\tilde{p}}{r}\left(Z^{(r)}(B) - 1\right) + \tilde{\alpha} Z^{(r)}(B) + \gamma, \quad B \geq 0.$$

Suppose that B^* is such threshold level that attains

$$\varrho(B^*) = 0.$$

When there does not exist such threshold, we set $B^* = \infty$ (or $\tau_{B^*}^+ = \theta$) if $\varrho(0+) < 0$ and $B^* = 0$ (or $\tau_{B^*}^+ = 0$) if $\varrho(0+) \geq 0$.

Optimal Stopping Time (Cont'd)

Proposition

The optimal stopping time is given by

$$\tau_{B^*}^{+} := \inf\left\{t \geq 0 : X_t \notin (0, B^*)\right\}.$$

Remark

We have $B^* = 0$ *if and only if*

1. $\sigma = 0$,
2. $\Pi(0, \infty) < \infty$, *and*
3. $\tilde{p} - r\gamma - (\tilde{\alpha} + \gamma)\Pi(0, \infty) \geq 0$.

Value function

1. when $B^* = 0$,

$$v(x) = h(x), \quad x \in \mathbb{R},$$

2. when $0 < B^* < \infty$,

$$v(x) = \begin{cases} W^{(r)}(x) \left(\frac{\tilde{p}+\tilde{\alpha} r}{\Phi(r)} - \frac{G^{(r)}(B^*)}{W^{(r)}(B^*)} \right) 1_{\{x \neq 0\}}, & -\infty < x < B^*, \\ h(x), & x \geq B^*, \end{cases}$$

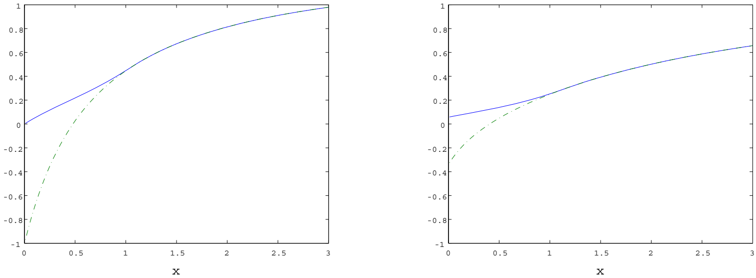
3. when $B^* = \infty$,

$$v(x) = 0, \quad x \in \mathbb{R}.$$

The values of callable step-down and putable step-up CDSs are

$$V(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) = C(x; p, \alpha) + v(x),$$
$$U(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) = -C(x; p, \alpha) + v(x).$$

Smooth Fit



unbounded variation case

bounded variation case

Figure: Illustrating the continuous and smooth fits of the value function $v_{B^*}(\cdot)$ (solid curve) and stopping value $h(\cdot)$ (dashed curve). Here $v_{B^*}(0+) = 0$ for the unbounded variation case while $v_{B^*}(0+) > 0$ for the bounded variation case.

Putable Step-down CDS

We can guess that the optimal stopping time is in the form

$$\tau_A^- := \inf\{t \geq 0 : X_t \leq A\}.$$

Define

$$\rho(A) := \int_A^\infty \Pi(\mathrm{d}u)\left(1 - e^{-\Phi(r)(u-A)}\right), \quad A > 0.$$

Here, ϱ_A monotonically decreases in A because

$$\frac{\partial}{\partial A}\rho(A) = -\int_A^\infty \Pi(\mathrm{d}u)\Phi(r)e^{-\Phi(r)(u-A)} < 0.$$

Let A^* be such that

$$(\tilde{\alpha} - \gamma)\rho(A) = (\gamma r + \tilde{p})$$

if it exists and $A^* = 0$ if it does not exist.

Value function

We have $\tau^-_{A^*}$ is an optimal stopping time and the value function is given by $u(x)$ where it equals for $x > 0$

$$(\tilde{\alpha} - \gamma)\left(\frac{1}{r}\int_{A^*}^{\infty} \Pi(\mathrm{d}u)\left[Z^{(r)}(x - A^*) - Z^{(r)}(x-u)\right]\right) - \left(\gamma + \frac{\tilde{p}}{r}\right) Z^{(r)}(x - A^*) + \left(\frac{\tilde{p}}{r} + \tilde{\alpha}\right)\zeta(x)$$

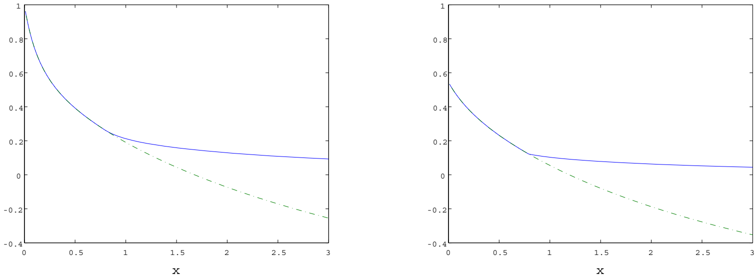
and

$$(\tilde{\alpha} - \gamma)(\zeta(x) - \Gamma(x, 0))$$

respectively when $A^* > 0$ and $A^* = 0$ ($u(x) = 0$ on $(-\infty, 0]$). Here

$$\Gamma(x, A) = \int_A^{\infty} \Pi(\mathrm{d}u)\left\{\frac{1}{\Phi(r)} W^{(r)}(x - A)\left(1 - e^{-\Phi(r)(u-A)}\right) - \int_0^{u-A} \mathrm{d}z W^{(r)}(x - A - z)\right\}.$$

Continuous and Smooth Fit



unbounded variation case   bounded variation case

Figure: Illustrating the continuous and smooth fits of the value function $u_{A^*}(\cdot)$ (solid curve) and stopping value $g(\cdot)$ (dashed curve). Here $u_{A^*}(\cdot)$ is C^1 at A^* for the unbounded variation case while it is C^0 for the bounded variation case.

Numerical Example

Let X be a spectrally negative Lévy process of the form

$$X_t - X_0 = \mu t + \sigma B_t - \sum_{n=1}^{N_t} Z_n, \quad 0 \le t < \infty$$

where

- $B = \{B_t; t \ge 0\}$ is a standard Brownian motion,
- $N = \{N_t; t \ge 0\}$ is a Poisson process with arrival rate λ,
- $Z = \{Z_n; n = 1, 2, \ldots\}$ is an i.i.d. sequence of Pareto random variables with $a = 1.2$ and $b = 5$.

The Pareto distribution with positive parameters a and b is given by

$$F(t) = 1 - (1 + bt)^{-a}, \quad t \ge 0.$$

Numerical Example

Let

$$q := \hat{p}/p = \hat{\alpha}/\alpha,$$

and consider, for both callable and putable CDSs, the following three cases:

(C) $q = 0$ (cancellable case),

(D) $q = 0.5$ (step-down),

(U) $q = 1.5$ (step-up).

We assume $r = 0.03$ and $\sigma = 0.2$, $x = 1.5$ and $\gamma = 50bps = 0.005$, and change the values of λ and μ so that it satisfies the risk-neutral condition $\psi(1) = r$.

Optimal threshold levels

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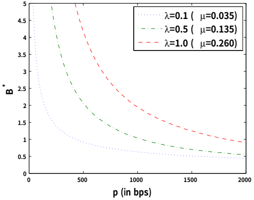
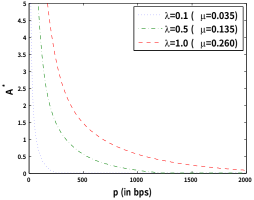
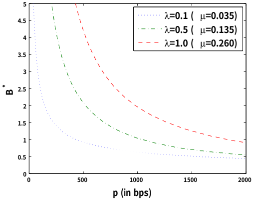


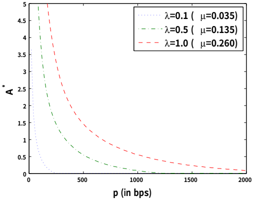
callable-(C)



putable-(C)

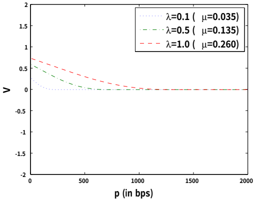


callable-(D)/putable-(U)

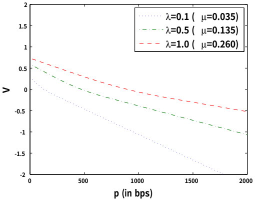


callable-(U)/putable-(D)

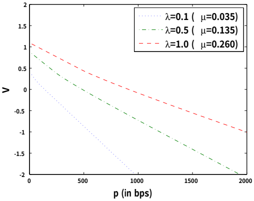
Numerical Example (Cont'd)



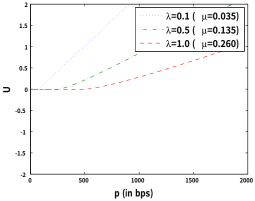
callable-(C)



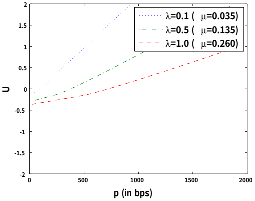
callable-(D)



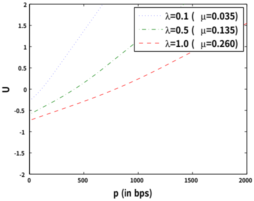
callable-(U)



putable-(C)

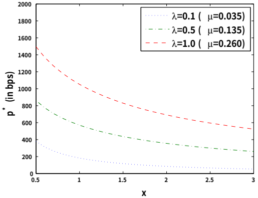


putable-(D)

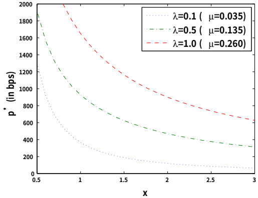


putable-(U)

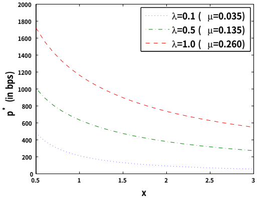
Premiums for callable CDSs



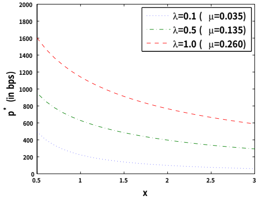
vanilla



callable-(C)

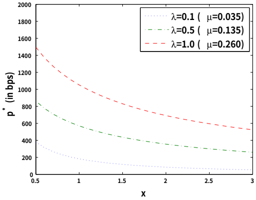


callable-(D)

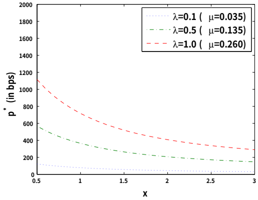


callable-(U)

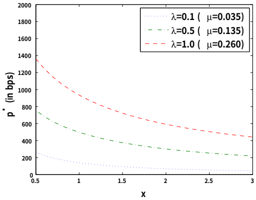
Premiums for putable CDSs



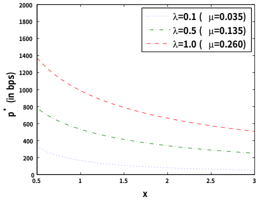
vanilla



putable-(C)



putable-(D)



putable-(U)

References

[1] T. S-T Leung and K. Yamazaki. *Pricing American Step-Up and Step-Down Credit Default Swaps under Lévy Models (2010)*. CSFI Discussion Paper, Osaka University.

[2] M. Egami and K. Yamazaki. *Solving Optimal Dividend Problems via Phase-Type Fitting Approximation of Scale Functions (2010)*. CSFI Discussion Paper, Osaka University.