Pricing American Step-Up and Step-Down Credit Default Swaps under Lévy Models

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Modeling and Managing Financial Risks

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Outline

- **American Step-up/down Callable/Putable CDSs.**
  - Definition,
  - Relationship with *Vanilla Credit Default Swaps and Swaptions*.
- Solve analytically for a general exponential Lévy process with only negative jumps.
  - *Spectrally Negative Lévy Processes and Scale Functions*.
  - Solve optimal stopping problems.
- Numerical Example.
Model

- Let $X = \{X_t; \ t \geq 0\}$ be a Lévy process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- The value of the reference entity (a company stock or other assets) is assumed to evolve according to an exponential Lévy process

$$S_t = e^{X_t}, \quad t \geq 0.$$ 

- Following the Black-Cox structural approach, the default event is triggered by $S$ crossing a lower level $D$.

- Without loss of generality, we can take $\log D = 0$ by shifting the initial value $X_0$. Henceforth, we shall work with the default time:

$$\theta := \inf\{t \geq 0 : X_t \leq 0\}.$$
Vanilla CDS

- From the buyer’s perspective, the expected discounted payoff is given by

\[
\tilde{C}(x; p, \alpha, T) := \mathbb{E}^x \left[ - \int_0^{\theta \wedge T} e^{-rt} p \, dt + \alpha e^{-r\theta} 1_{\{\theta \leq T\}} \right],
\]

where \( T \) is the maturity and \( r > 0 \) is the positive constant risk-free interest rate.

- The fair premium is

\[
\bar{p}(x; \alpha, T) = \frac{\alpha r \zeta_T(x)}{1 - \zeta_T(x) - e^{-rT} \mathbb{P}^x \{ \theta > T \}}
\]

where

\[
\zeta_T(x) := \mathbb{E}^x \left[ e^{-r\theta} 1_{\{\theta \leq T\}} \right].
\]
Vanilla CDS (Perpetual)

- When $T = +\infty$, the buyer’s CDS price is

$$C(x; p, \alpha) := \mathbb{E}^x \left[ -\int_0^\theta e^{-rt} \, p \, dt + \alpha e^{-r\theta} \right]$$

$$= \left( \frac{p}{r} + \alpha \right) \zeta(x) - \frac{p}{r},$$

where

$$\zeta(x) := \mathbb{E}^x \left[ e^{-r\theta} \right].$$

- The seller’s CDS price is $-C(x; p, \alpha) = C(x; -p, -\alpha)$.

- Solving $C(x; p, \alpha) = 0$ yields the credit spread:

$$p(x; \alpha) = \frac{\alpha r \zeta(x)}{1 - \zeta(x)}.$$
American Perpetual Swaptions

The prices of (generalized American perpetual) payer and receiver swaptions are

$$v(x; \kappa, a, K) := \sup_{\tau \in T} \mathbb{E}^x \left[ e^{-r\tau} \left( C(X_{\tau}; \kappa, a) - K \right)^+ 1_{\{\theta > \tau\}} \right],$$

$$u(x; \kappa, a, K) := \sup_{\tau \in T} \mathbb{E}^x \left[ e^{-r\tau} \left( -C(X_{\tau}; \kappa, a) - K \right)^+ 1_{\{\theta > \tau\}} \right]$$

with

- default payment $a$;
- pre-specified spread $\kappa$;
- strike price $K$ upon exercise.
Callable Step-Up/Down CDS

We consider a CDS contract with an embedded option that permits the protection buyer to change

- premium from $p$ to $\hat{p}$,
- default payment from $\alpha$ to $\hat{\alpha}$,

any time before $\theta$ once for a fee $\gamma$. The value for the buyer is

$$V(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) := \sup_{\tau \in S} \mathbb{E}^x \left[ - \int_0^\tau e^{-rt} p \, dt + 1_{\{\tau<\infty\}} \right. $$

$$\left. \left( - \int_\tau^\theta e^{-rt} \hat{p} \, dt - e^{-r\tau} \gamma 1_{\{\tau<\theta\}} + e^{-r\theta} (\hat{\alpha} 1_{\{\tau<\theta\}} + \alpha 1_{\{\tau=\theta\}}) \right) \right], $$

where $S := \{ \nu \in \mathbb{F} : \tau \leq \theta \ \text{a.s.} \}$. 
Step-Up/Down and Cancellation

This formulation covers default swaps with the following provisions:

1. **Step-up Option**: if \( \hat{p} > p \) and \( \hat{\alpha} > \alpha \), then the buyer is allowed to increase the coverage once from \( \alpha \) to \( \hat{\alpha} \) by paying the fee \( \gamma \) and a higher premium \( \hat{p} \) thereafter.

2. **Step-down Option**: when \( \hat{p} < p \) and \( \hat{\alpha} < \alpha \), then the buyer can reduce the coverage once from \( \alpha \) to \( \hat{\alpha} \) by paying the fee \( \gamma \) and a reduced premium \( \hat{p} \) thereafter.

3. **Cancellation Right**: as a special case of the step-down option with \( \hat{\alpha} = \hat{p} = 0 \), the resulting contract allows the buyer to terminate the CDS at time \( \tau \). This special case corresponds to an American callable step-down CDS.
Putable Step-Up/Down CDS

We consider a CDS contract with an embedded option that permits the protection seller to change

- premium from $p$ to $\hat{p}$,
- default payment from $\alpha$ to $\hat{\alpha}$,

any time before $\theta$ once for a fee $\gamma$. The value for the seller is

$$U(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) := \sup_{\tau \in S} \mathbb{E}^x \left[ \int_0^\tau e^{-rt} p \, dt + 1_{\{\tau < \infty\}} \right]$$

$$\left( \int_0^\theta e^{-rt} \hat{p} \, dt - e^{-r\theta} \gamma 1_{\{\tau < \theta\}} - e^{-r\theta} (\hat{\alpha} 1_{\{\tau < \theta\}} + \alpha 1_{\{\tau = \theta\}}) \right).$$
Decomposition

Let

$$\tilde{\alpha} := \alpha - \hat{\alpha}, \quad \text{and} \quad \tilde{p} := p - \hat{p}. $$

We can decompose callable and putable CDSs, respectively, by

$$ V(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) = C(x; p, \alpha) + v(x; -\tilde{p}, -\tilde{\alpha}, \gamma), $$
$$ U(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) = -C(x; p, \alpha) + u(x; -\tilde{p}, -\tilde{\alpha}, \gamma) $$

where

$$ C(x; p, \alpha) := \mathbb{E}^x \left[ - \int_0^\theta e^{-r\tau} p \, d\tau + \alpha e^{-r\theta} \right], $$
$$ v(x; \kappa, a, K) := \sup_{\tau \in T} \mathbb{E}^x \left[ e^{-r\tau} (C(X_\tau; \kappa, a) - K)^+ 1_{\{\theta > \tau\}} \right], $$
$$ u(x; \kappa, a, K) := \sup_{\tau \in T} \mathbb{E}^x \left[ e^{-r\tau} (-C(X_\tau; \kappa, a) - K)^+ 1_{\{\theta > \tau\}} \right]. $$
# Hedging

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<th>Protection Buyer’s Position</th>
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<td><strong>American Callable</strong></td>
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We also find the “put-call parity” and symmetry identities:

\[
V(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) - U(x; p, 2p - \hat{p}, \alpha, 2\alpha - \hat{\alpha}, \gamma) = 2 C(x; p, \alpha),
\]

\[
V(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) + U(x; p, 2p - \hat{p}, \alpha, 2\alpha - \hat{\alpha}, \gamma) = 2 v(x; -\hat{p}, -\hat{\alpha}, \gamma).
\]
What We Need to Solve

Because \( v(x; -\tilde{p}, -\tilde{\alpha}, \gamma) = u(x; \tilde{p}, \tilde{\alpha}, \gamma) \), we only need to solve for

1. the callable step-down CDS with \( \tilde{p} > 0 \) and \( \tilde{\alpha} > 0 \), and
2. the putable step-down CDS with \( \tilde{p} > 0 \) and \( \tilde{\alpha} > 0 \).

In particular, we need to solve the following optimal stopping problems:

\[
\begin{align*}
\nu(x) &:= \nu(x; -\tilde{p}, -\tilde{\alpha}, \gamma) = \sup_{\tau \in S} \mathbb{E}^x \left[ e^{-r \tau} h(X_\tau) 1_{\{\tau < \infty\}} \right], \\
\upsilon(x) &:= \upsilon(x; -\tilde{p}, -\tilde{\alpha}, \gamma) = \sup_{\tau \in S} \mathbb{E}^x \left[ e^{-r \tau} g(X_\tau) 1_{\{\tau < \infty\}} \right],
\end{align*}
\]

where

\[
\begin{align*}
h(x) &:= \left( \left( \frac{\tilde{p}}{r} - \gamma \right) - \left( \frac{\tilde{p}}{r} + \tilde{\alpha} \right) \zeta(x) \right) 1_{\{x > 0\}}, \\
g(x) &:= \left( \left( -\frac{\tilde{p}}{r} - \gamma \right) + \left( \frac{\tilde{p}}{r} + \tilde{\alpha} \right) \zeta(x) \right) 1_{\{x > 0\}}.
\end{align*}
\]
Spectrally Negative Lévy Processes

Defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), let \(X = \{X_t; t \geq 0\}\) be a spectrally negative Lévy process, i.e.

1. The paths are almost surely right continuous with left limits.
2. For \(0 \leq s \leq t\), \(X_t - X_s\) is equal in distribution to \(X_{t-s}\).
3. For \(0 \leq s \leq t\), \(X_t - X_s\) is independent of \(\{X_u: u \leq s\}\).
4. Jumps are almost surely negative (spectrally negative).

Laplace exponent is the logarithm of its Laplace transform:

\[
\psi(s) := \log \mathbb{E}^0 \left[ e^{sX_1} \right], \quad s \in \mathbb{C}.
\]

and we can write

\[
\psi(s) := cs + \frac{1}{2} \sigma^2 s^2 + \int_{(0,\infty)} (e^{-sx} - 1 + sx1_{0<x<1}) \Pi(dx), \quad s \in \mathbb{C}.
\]
Scale Functions

Associated with every spectrally negative Lévy process, there exists a \((r-)\)scale function

\[ W^{(r)} : \mathbb{R} \rightarrow \mathbb{R}, \]

whose Laplace transform is given by

\[
\int_{0}^{\infty} e^{-\beta x} W^{(r)}(x) dx = \frac{1}{\psi(\beta) - r}, \quad \beta > \Phi(r)
\]

where \(\Phi(r)\) is the (largest) positive root of

\[ \Phi(r) := \sup\{s \geq 0 : \psi(s) = r\}. \]

We assume \(W^{(r)}(x) = 0\) on \((-\infty, 0)\).
Scale Functions (Cont’d)

We define the first down- and up-crossing times, respectively, by

\[
\tau_a^- := \inf \{ t \geq 0 : X_t < a \}, \\
\tau_b^+ := \inf \{ t \geq 0 : X_t > b \}
\]

for any \(0 \leq a < x < b\). Then we have, for every \(0 \leq x < b\),

\[
\mathbb{E}^x \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < \tau_0^-\}} \right] = \frac{W^{(r)}(x)}{W^{(r)}(b)},
\]

\[
\mathbb{E}^x \left[ e^{-q\tau_0^-} 1_{\{\tau_b^+ > \tau_0^-\}} \right] = Z^{(r)}(x) - Z^{(r)}(b) \frac{W^{(r)}(x)}{W^{(r)}(b)}
\]

where \(Z^{(r)}(x) = 1 + q \int_0^x W^{(r)}(y) dy\) for every \(x \in \mathbb{R}\).
Callable Step-down CDS

We need to solve

\[ v(x) := \sup_{\tau \in S} \mathbb{E}^x \left[ e^{-r\tau} h(X_\tau) 1\{\tau < \infty\} \right], \]

where

\[ h(x) := \left( \left( \frac{\tilde{p}}{r} - \gamma \right) - \left( \frac{\tilde{p}}{r} + \tilde{\alpha} \right) \zeta(x) \right) 1\{x \geq 0\} \]

and

\[ \zeta(x) = Z^{(r)}(x) - \frac{r}{\Phi(r)} W^{(r)}(x), \quad x \in \mathbb{R}. \]

Using the scale functions, the stopping value becomes

\[ h(x) = \left[ \tilde{p} \left( \frac{1}{r} \left( 1 - Z^{(r)}(x) \right) + \frac{1}{\Phi(r)} W^{(r)}(x) \right) \right. \]

\[ \left. - \tilde{\alpha} \left( Z^{(r)}(x) - \frac{r}{\Phi(r)} W^{(r)}(x) \right) - \gamma \right] 1\{x > 0\}, \quad x \in \mathbb{R}. \]
**Optimal Stopping Time**

We can guess that the optimal stopping time is in the form

$$
\tau^+_B := \inf \{ t \geq 0 : X_t \notin (0, B) \}.
$$

Define

$$
\varphi(B) := (\tilde{p} + \tilde{\alpha}r) W^{(r)}(B) - \frac{W^{(r)'}(B)}{W^{(r)}(B)} G^{(r)}(B)
$$

where

$$
G^{(r)}(B) := \frac{\tilde{p}}{r} \left( Z^{(r)}(B) - 1 \right) + \tilde{\alpha} Z^{(r)}(B) + \gamma, \quad B \geq 0.
$$

Suppose that $B^*$ is such threshold level that attains

$$
\varphi(B^*) = 0.
$$

When there does not exist such threshold, we set $B^* = \infty$ (or $\tau^+_B = \theta$) if $\varphi(0+) < 0$ and $B^* = 0$ (or $\tau^+_B = 0$) if $\varphi(0+) \geq 0$. 
Optimal Stopping Time (Cont’d)

Proposition

The optimal stopping time is given by

\[ \tau_{B^*}^+ := \inf \{ t \geq 0 : X_t \notin (0, B^*) \} . \]

Remark

We have \( B^* = 0 \) if and only if

1. \( \sigma = 0 \),
2. \( \Pi(0, \infty) < \infty \), and
3. \( \tilde{p} - r\gamma - (\tilde{a} + \gamma)\Pi(0, \infty) \geq 0. \)
Value function

1. when $B^* = 0$, 
   \[ v(x) = h(x), \quad x \in \mathbb{R}, \]

2. when $0 < B^* < \infty$, 
   \[ v(x) = \begin{cases} 
   W^{(r)}(x) \left( \frac{\bar{p} + \alpha r}{\Phi(r)} - \frac{G^{(r)}(B^*)}{W^{(r)}(B^*)} \right) 1\{x \neq 0\}, & -\infty < x < B^*, \\
   h(x), & x \geq B^*, 
   \end{cases} \]

3. when $B^* = \infty$, 
   \[ v(x) = 0, \quad x \in \mathbb{R}. \]

The values of callable step-down and putable step-up CDSs are

\[ V(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) = C(x; p, \alpha) + v(x), \]
\[ U(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) = -C(x; p, \alpha) + v(x). \]
Figure: Illustrating the continuous and smooth fits of the value function $v_{B^*}(\cdot)$ (solid curve) and stopping value $h(\cdot)$ (dashed curve). Here $v_{B^*}(0+) = 0$ for the unbounded variation case while $v_{B^*}(0+) > 0$ for the bounded variation case.
Putable Step-down CDS

We can guess that the optimal stopping time is in the form

\[ \tau_A^- := \inf \{ t \geq 0 : X_t \leq A \} . \]

Define

\[ \rho(A) := \int_A^\infty \Pi(du) \left( 1 - e^{-\Phi(r)(u-A)} \right), \quad A > 0. \]

Here, \( \varrho_A \) monotonically decreases in \( A \) because

\[ \frac{\partial}{\partial A} \rho(A) = -\int_A^\infty \Pi(du) \Phi(r)e^{-\Phi(r)(u-A)} < 0. \]

Let \( A^* \) be such that

\[ (\tilde{\alpha} - \gamma)\rho(A) = (\gamma r + \tilde{p}) \]

if it exists and \( A^* = 0 \) if it does not exist.
Value function

We have $\tau_{A^*}^{-}$ is an optimal stopping time and the value function is given by $u(x)$ where it equals for $x > 0$

$$(\tilde{\alpha} - \gamma) \left( \frac{1}{r} \int_{A^*}^{\infty} \Pi(du) \left[ Z^{(r)}(x - A^*) - Z^{(r)}(x - u) \right] \right)$$

$$- \left( \gamma + \tilde{p} \right) Z^{(r)}(x - A^*) + \left( \frac{\tilde{p}}{r} + \tilde{\alpha} \right) \zeta(x)$$

and

$$(\tilde{\alpha} - \gamma)(\zeta(x) - \Gamma(x, 0))$$

respectively when $A^* > 0$ and $A^* = 0$ ($u(x) = 0$ on $(-\infty, 0]$). Here

$$\Gamma(x, A) = \int_{A}^{\infty} \Pi(du) \left\{ \frac{1}{\Phi(r)} W^{(r)}(x - A) \left( 1 - e^{-\Phi(r)(u-A)} \right) \right.$$

$$\left. - \int_{0}^{u-A} dz W^{(r)}(x - A - z) \right\}.$$
Continuous and Smooth Fit

**Figure:** Illustrating the continuous and smooth fits of the value function $u_{A^*}(\cdot)$ (solid curve) and stopping value $g(\cdot)$ (dashed curve). Here $u_{A^*}(\cdot)$ is $C^1$ at $A^*$ for the unbounded variation case while it is $C^0$ for the bounded variation case.
Numerical Example

Let \( X \) be a spectrally negative Lévy process of the form

\[
X_t - X_0 = \mu t + \sigma B_t - \sum_{n=1}^{N_t} Z_n, \quad 0 \leq t < \infty
\]

where

- \( B = \{ B_t; t \geq 0 \} \) is a standard Brownian motion,
- \( N = \{ N_t; t \geq 0 \} \) is a Poisson process with arrival rate \( \lambda \),
- \( Z = \{ Z_n; n = 1, 2, \ldots \} \) is an i.i.d. sequence of Pareto random variables with \( a = 1.2 \) and \( b = 5 \).

The Pareto distribution with positive parameters \( a \) and \( b \) is given by

\[
F(t) = 1 - (1 + bt)^{-a}, \quad t \geq 0.
\]
Numerical Example

Let

\[ q := \hat{p}/p = \hat{\alpha}/\alpha, \]

and consider, for both callable and putable CDSs, the following three cases:

(C) \( q = 0 \) (cancellable case),
(D) \( q = 0.5 \) (step-down),
(U) \( q = 1.5 \) (step-up).

We assume \( r = 0.03 \) and \( \sigma = 0.2 \), \( x = 1.5 \) and \( \gamma = 50 \text{bps} = 0.005 \), and change the values of \( \lambda \) and \( \mu \) so that it satisfies the risk-neutral condition \( \psi(1) = r \).
Optimal threshold levels

callable-(C)
callable-(D)/putable-(U)
callable-(U)/putable-(D)
putable-(C)
Numerical Example (Cont’d)

callable-(C)
callable-(D)
callable-(U)

putable-(C)
putable-(D)
putable-(U)
Premiums for callable CDSs

vanilla

callable-(C)

callable-(D)

callable-(U)
Premiums for putable CDSs

vanilla

putable-(C)

putable-(D)

putable-(U)
References
