Running for the Exit: Distressed Selling and Endogenous Correlation

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The instability of realized correlation

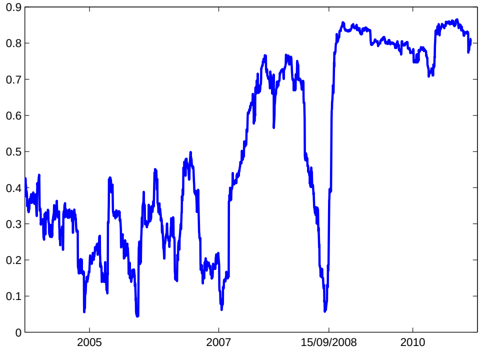

Figure: One year EWMA correlation between two ETF of S&P 500: SPDR XLE (energy) and SPDR XLK (technology)

Unexpected spikes of correlation

We witness **unexpected** 'correlation spikes' have been associated with the liquidation of large funds:

- LTCM: in Aug 1998, correlations of losses in various–previously uncorrelated– trades run by the hedge fund LTCM suddenly increased simultaneously, causing it to collapse after a few days.
- Brazil & Asia: the Asian market crisis in 2000 led to a collapse of the.. Brazilian equity index, BOVESPA.
- Subprime crisis: market losses in 'subprime MBS', largely seen as being uncorrelated to equity markets, led to huge falls in equity markets.
- August 2007: all long-short equity market neutral hedge funds lost $\sim 20\%$ per day between Aug 7-Aug 9, 2007 (Khandani & Lo, 2008), whereas major equity indices hardly moved!

Estimation of correlation: an exogenous parameter?

- Correlations and covariances between returns of assets, indices and funds are routinely estimated from historical data and used by market participants as inputs for trading, portfolio optimization and risk management.
- Correlation between two underlyings is often considered as a constant which supposedly reflects a structural correlation between 'fundamentals', therefore stable in time.
- Sophisticated models have been proposed for the distributional features of univariate returns but correlations are typically assumed to be constant or driven by exogenous factors.

The economic origin of correlations in returns

Such exogenous representation of correlation cannot explain spikes in realized correlations following the liquidation of a large fund. This leads us to distinguish two different origins for 'correlations' in returns:

- Correlation in fundamentals: common factors in returns (usual explanation)$\rightarrow$ correlation in "fundamentals", should not vary strongly in time
- Correlation from trading: generated by systematic supply/demand from investors

Objective: present a (tractable) framework for modeling endogenous correlation and its relation with liquidity

Framework

- Trading takes place at discrete dates: $t_k = k\tau$ ($\tau = \frac{T}{M}$)
- n assets; vector of price at t_k: $S_k = (S_k^1, ..., S_k^n)$
- At each period, the value of the assets moves due to exogenous economic factors. In absence of other effects, the return of asset i at period k would be $\sqrt{\tau}\xi_{k+1}^i$.
- $\xi_k = (\xi_k^1, ..., \xi_k^n))_{1\leq k\leq M}$ is a sequence of iid n-dimensional centered random variables, with covariance matrix Σ
- $(S_{k+1}^i)^* = S_k^i(1 + \sqrt{\tau}\xi_{k+1}^i)$

The leveraged fund

- Consider a leveraged fund holding α_i (≥ 0) units of asset i between dates $t = 0$ and T
- Between t_k and t_{k+1}, price moves due to exogenous economic factors move the value of the fund from $V_k = \sum_{i=1}^{n} \alpha_i S_k^i$ to

$$V_{k+1}^* = \sum_{i=1}^{n} \alpha_i (S_{k+1}^i)^* = V_k + \sum_{i=1}^{n} \alpha_i S_k^i \sqrt{\tau} \xi_{k+1}^i.$$

- Investors in the fund adopt a passive, buy and hold behavior as long as the fund is performing well. However, when the fund value drops below a threshold $\beta_0 V_0 < V_0$, investors progressively exit their positions

Net supply due to distressed selling

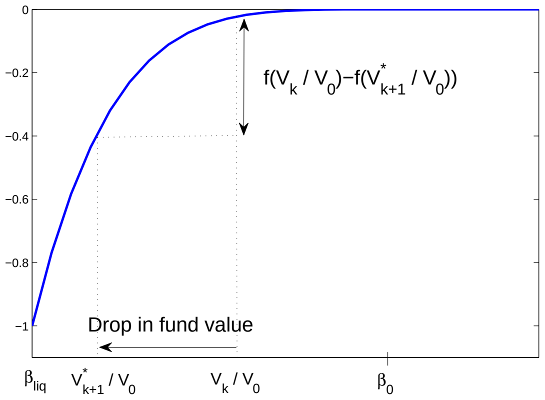

Figure: Net supply due to distressed selling is equal to $-\alpha_i(f(\frac{V^*_{k+1}}{V_0}) - f(\frac{V_k}{V_0}))$

Net supply due to distressed selling

- $f : \mathbb{R} \to \mathbb{R}$ is increasing, constant on $[\beta_0, +\infty[$
- The fund is liquidated when the value reaches $\beta_{liq} V_0$ where $\beta_{liq} < \beta_0$
- As the fund loses value and approaches liquidation, distressed selling becomes more intense: this feature is captured by choosing f to be concave
- The supply/demand pattern generated by these distressed sellers exiting the fund may be amplified by short sellers or predatory trading

Price impact of distressed selling

- This distressed selling activity impacts prices in a non-random manner. Empirical studies (Obizhaeva 2008; Cont Kukanov Stoikov 2010) provide evidence for the linearity of this price impact at daily and intraday frequencies.
- Market impact on asset i's return is equal to $\frac{\alpha_i}{\lambda_i}(f(\frac{V^*_{k+1}}{V_0}) - f(\frac{V_k}{V_0}))$
- λ_i represent the *depth* of the market in asset i: a net demand of $\frac{\lambda_i}{100}$ shares for security i moves i's price by one percent.

Price dynamics

$$\begin{tikzcd}
t_k \arrow[d] & & t_{k+1} \arrow[d] \\
\boxed{S_k, V_k} \arrow[r, "\text{exogenous}", "\text{economic factors}"'] & \boxed{S^*_{k+1}, V^*_{k+1}} \arrow[r, "\text{distressed selling}", "\text{short selling}"'] & \boxed{S_{k+1}, V_{k+1}}
\end{tikzcd}$$

$$S^i_{k+1} = S^i_k \left(1 + \sqrt{\tau}\xi^i_{k+1} + \frac{\alpha_i}{\lambda_i}\left(f\left(\frac{V_k}{V_0} + \sum_{i=1}^{n} \frac{\alpha_i S^i_k}{V_0}\sqrt{\tau}\xi^i_{k+1}\right) - f\left(\frac{V_k}{V_0}\right)\right)\right) \tag{1}$$

where

$$V_k = \sum_{i=1}^{n} \alpha_i S^i_k \tag{2}$$

Correlation between assets

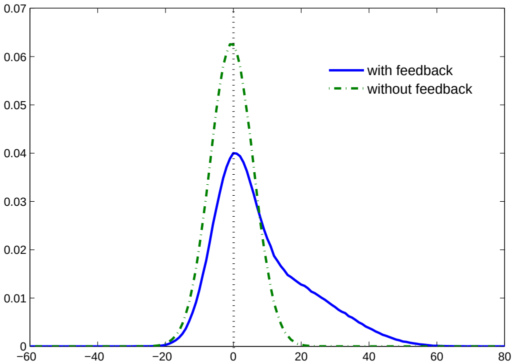

Figure: Distribution of realized correlation between the two securities (with $\rho = 0$) with and without feedback effects due to distressed selling

Correlation between assets

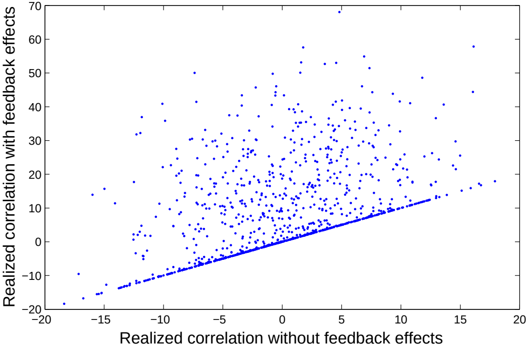

Figure: Scatter plot of realized correlation with and without feedback effects due to distressed selling (each data point represents one simulated scenario)

Correlation between assets

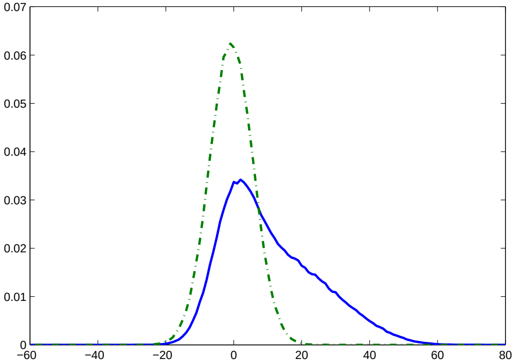

Figure: Distribution of realized correlation in scenarios where fund value reaches $\beta_0 V_0$ between 0 and T (plain line) and in scenarios where fund value remains above $\beta_0 V_0$ (dotted line)

Fund volatility

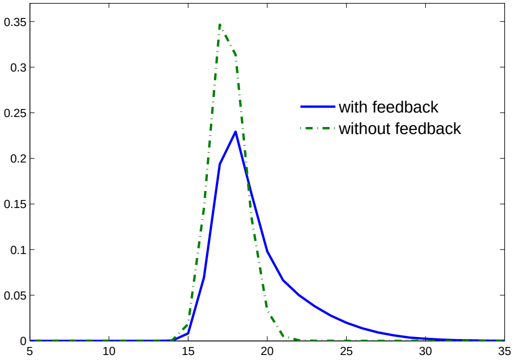

Figure: Distribution of realized volatility of the fund's portfolio with and without feedback effects

Fund volatility

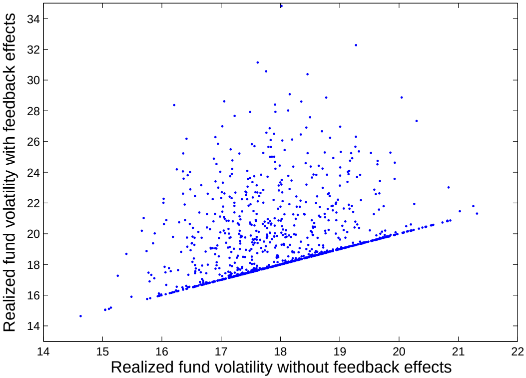

Figure: Scatter plot of realized volatility of the fund's portfolio with and without feedback effects

Fund volatility

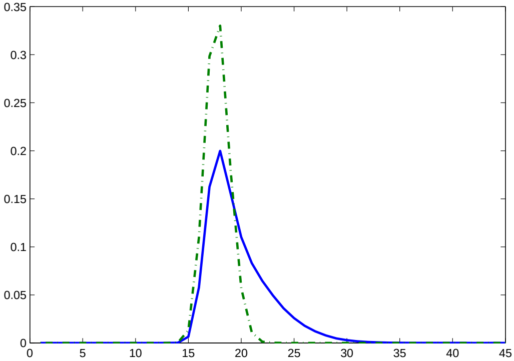

Figure: Distribution of realized fund volatility in scenarios where fund value reaches $\beta_0 V_0$ between 0 and T (plain line) and in scenarios where fund value remains above $\beta_0 V_0$ (dotted line)

- In presence of feedback effects, correlation and fund volatility are path-dependent: their distributions (compared to their distributions without feedback effects, reflecting statistical error) reflect the actions of distressed sellers and short sellers.
- Distressed selling by investors exiting the fund can generate significant realized correlation, even between assets with zero fundamental correlation, resulting in higher fund volatility.
- To confirm that the phenomena observed in the numerical experiments are not restricted to particular parameter choices or a particular choice of the function f, we will now analyze the continuous-time limit of our discrete-time model: the study of this limit allows to obtain analytical formulas for realized correlation which confirm quantitatively the effects observed in the numerical experiments.

Diffusion limit

Theorem

Under the assumption that $f \in C_b^3$ *such that* $\max \frac{\alpha_i}{\lambda_i} \| Id \times f' \|_\infty < 1$ *and that* $\mathbb{E}(|\xi|^4) < \infty$, $S^{(\tau)}_{\lfloor t\tau \rfloor}$ *converges weakly towards a diffusion* $P_t = (P_t^1, ...P_t^n)^t$ *when* τ *goes to* 0 *where*

$$\frac{dP_t^i}{P_t^i} = (\mu_t)_i dt + (\sigma_t dW_t)_i \qquad 1 \leq i \leq n$$

$$(\mu_t)_i = \frac{\alpha_i}{2\lambda_i} f''(\frac{V_t}{V_0}) \frac{< \pi_t, \Sigma \pi_t >}{V_0^2}; (\sigma_t)_{i,j} = A_{i,j} + \frac{\alpha_i}{\lambda_i} f'(\frac{V_t}{V_0}) \frac{(A^t \pi_t)_j}{V_0}$$

- $\pi_t = (\alpha_1 P_t^1, ..., \alpha_n P_t^n)^t$ *is the (dollar) allocation of the fund*
- $V_t = \sum_{1 \leq i \leq n} \alpha_i P_t^i$ *is the value of the fund*
- A *is a square-root of the fundamental covariance matrix:* $AA^t = \Sigma$

Realized covariance

Proposition

The realized covariance between securities i and j between 0 and t is equal to $\frac{1}{t}\int_0^t C_s^{i,j}\,\mathrm{d}s$, where $C_s^{i,j}$, the instantaneous covariance between i and j, is given by:

$$C_s^{i,j} = \Sigma_{i,j} + \frac{\alpha_j}{\lambda_j} f'\left(\frac{V_s}{V_0}\right)\frac{(\Sigma\pi_t)_i}{V_0} + \frac{\alpha_i}{\lambda_i} f'\left(\frac{V_s}{V_0}\right)\frac{(\Sigma\pi_t)_j}{V_0}$$

$$+ \frac{\alpha_i\alpha_j}{\lambda_i\lambda_j}(f')^2\left(\frac{V_s}{V_0}\right)\frac{< \pi_t, \Sigma\pi_t >}{V_0^2}$$

with $\pi_t = (\alpha_1 P_t^1, ..., \alpha_n P_t^n)^t$.

Realized covariance is *path-dependent*. It is the sum of a fundamental covariance and a *liquidity-dependent excess covariance term*. The impact of the liquidation of a fund is *computable* under our model assumptions.

Realized correlation is a deterministic function of π_t

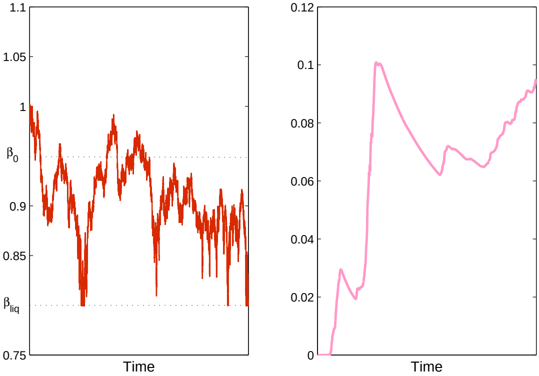

Figure: $\frac{V_t}{V_0}$ (left) and realized correlation on $[0, t]$ (right)

Excess correlation is exacerbated by illiquidity

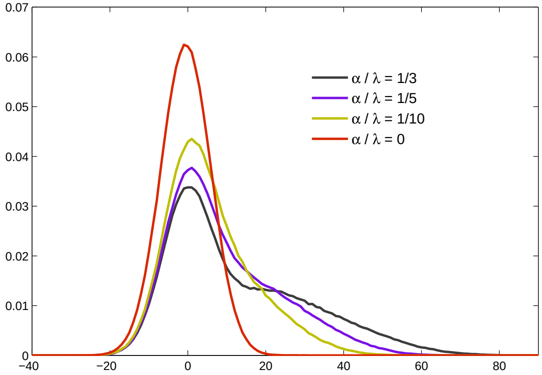

Figure: Distribution of realized correlation for different values of $\frac{\alpha}{\lambda}$

Realized covariance: case of null fundamental correlations

Corollary

If the fundamental covariance matrix Σ is diagonal, then, for all $1 \leq i, j \leq n$, the instantaneous covariance between i and j $(i \neq j)$ is given by:

$$C_t^{i,j} = \frac{\alpha_j}{\lambda_j} f^{'}(\frac{V_t}{V_0}) \frac{\alpha_i}{V_0} P_t^i \sigma_i^2 + \frac{\alpha_i}{\lambda_i} f^{'}(\frac{V_t}{V_0}) \frac{\alpha_j}{V_0} P_t^j \sigma_j^2$$
$$+ \frac{\alpha_i \alpha_j}{\lambda_i \lambda_j} (f^{'})^2 (\frac{V_t}{V_0}) \sum_{1 \leq l \leq n} (\frac{\alpha_l}{V_0} P_t^l \sigma_l)^2 > 0$$

In absence of fundamental correlation, distressed selling/short selling creates positive correlation between the fund's assets.

Impact on fund variance

Proposition

The fund's realized variance between 0 and t is equal to $\frac{1}{t}\int_0^t \Gamma_s \,\mathrm{d}s$ where Γ_s, the instantaneous variance of the fund, is given by:

$$\Gamma_s V_s^2 = <\pi_s, \Sigma\pi_s> + \frac{2}{V_0} f^{'}(\frac{V_s}{V_0}) <\pi_s, \Sigma\pi_s><\Lambda, \pi_s>$$

$$+\frac{1}{V_0^2}(f^{'}(\frac{V_s}{V_0}))^2 <\pi_s, \Sigma\pi_s> (<\Lambda, \pi_s>)^2$$

with

- $\pi_t = (\alpha_1 P_t^1, ..., \alpha_n P_t^n)^t$ denotes the (dollar) holdings of the fund,
- $\Lambda = (\frac{\alpha_1}{\lambda_1}, ..., \frac{\alpha_n}{\lambda_n})^t$ represents the positions of the fund in each market as a fraction of the respective market depth.

Limits of diversification

- Distressed selling increases fund volatility, exactly in scenarios where the fund experiences difficulty, reducing the benefit of diversification
- Without liquidity drying up (λ constant), feedback effects can modify significantly fund volatility when investors exit large positions
- Spikes in correlation and fund volatility can be triggered by investors exiting their positions, even in absence of predatory trading by short sellers

Spillover effects

Consider now a small target fund with positions $(\mu_t^i, i = 1..n)$.

Proposition

Under the assumption that M has negligible impact on market prices:

$$d[M]_t =< \pi_t^\mu, \Sigma\pi_t^\mu > dt + \left(\frac{2}{V_0} f^{'}(\frac{V_t}{V_0}) < \pi_t^\mu, \Sigma\pi_t^\alpha >< \Lambda, \pi_t^\mu > \right) dt$$

$$+ \left(\frac{1}{V_0^2} (f^{'}(\frac{V_t}{V_0}))^2 < \pi_t^\alpha, \Sigma\pi_t^\alpha > (< \Lambda, \pi_t^\mu >)^2 \right) dt$$

where

- $M_t = \sum_{1 \le i \le n} \mu_t^i P_t^i$, $\pi_t^\alpha = (\alpha_1 P_t^1, ..., \alpha_n P_t^n)^t$, $\pi_t^\mu = (\mu_t^1 P_t^1, ..., \mu_t^n P_t^n)^t$
- $\Lambda = (\frac{\alpha_1}{\lambda_1}, ..., \frac{\alpha_n}{\lambda_n})^t$ represents the positions of the fund in each market as a fraction of the respective market depth.

Orthogonality condition

If the allocations of the two funds verify the 'orthogonality' condition:

$$< \Lambda, \pi_t^{\mu} >= \sum_{1 \le i \le n} \frac{\alpha_i}{\lambda_i} \mu_t^i P_t^i = 0$$

distressed selling and short selling on the reference fund *do not affect* the target fund's variance:

$$d[M]_t =< \pi_t^{\mu}, \Sigma \pi_t^{\mu} > dt$$

On the contrary, volatility is exacerbated for funds similar to α.

Strategy crowding: the example of August 2007

- Investors exiting a large market-neutral long short fund lead to high losses/excess volatility for *similar* long short funds.
- However, *index funds*, being *orthogonal* to the reference fund, were unaffected.
- This can happen *without liquidity drying up* ($\neq$ explanation of Khandani and Lo).
- *Crowding* was a major risk factor in this market.

Our framework allows us to quantify *strategy crowding* risk.

Some concluding remarks

- **Correlation risk** should be understood dynamically rather than in terms of static covariance matrices.
- Realized correlations among funds and indices can be highly variable and quite different from 'fundamental' correlations suggested by economic analysis.
- Realized correlations depend more on the type/market cap of various strategies used by market players than on 'fundamental' correlations.

Some concluding remarks

- 'Correlation risk' and liquidity risk cannot be separated in a realistic stress testing framework.
- When liquidity effects are accounted for, observed levels of realized correlation across asset classes are compatible with the null hypothesis of absence of correlation in fundamentals!
- The benefit of fund diversification is reduced and spillover effects can be observed.
- **Strategy crowding** should be considered as a risk factor. We provide a quantitative framework to evaluate such risk.