Non parametric test for a semi-martingale : Itō against multifractal

Laurent Duvernet[1], Christian Y. Robert[2], Mathieu Rosenbaum[3]

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[1]Université Paris 7
[2]Université Lyon 1
[3]CMAP-École polytechnique

Say we have a set of 1d, middle- or low-frequency financial data. What kind of model should we use?

Here we consider the following cases:

Itō semi-martingales vs. a class of (true) multifractal martingales.

- Does the data-generating process likely belong to one of the two classes?
- Is the number of available data large enough to answer the previous question?

Statistics for multifractal processes are a new topic! this an exploratory work.

Itō semi-martingales

- A semi-martingale is the sum of a finite variation process, a continuous local martingale, and a compensated jump process.
- It is in the Itō class if the finite variation process, the quadratic variation of the continuous local martingale and the compensator of the jump process are all absolutely continuous w.r.t. the Lebesgue measure.
- Very large and very natural family, especially for financial models.

Non-parametric tests for Itō semi-martingales

- Aït-Sahalia et Jacod (2009) test whether an Itō semi-martingale X is continuous or not, based on the observation of X at times $i/n\ i = 0, \ldots, n$.
- They base themselves on the following behavior of the p-variations of the process, $p > 0$:

$$n^{\tau(p)} \sum_{i=0}^{n-1} |X_{(k+1)/n} - X_{k/n}|^p \xrightarrow{\mathbb{P}} I > 0 \quad \text{pour } n \to +\infty$$

with :

- $\tau(p) = p/2 - 1$ if X has no jumps on $[0, 1]$,
- $\tau(p) = (p/2 - 1)_-$ if X jumps on $[0, 1]$.

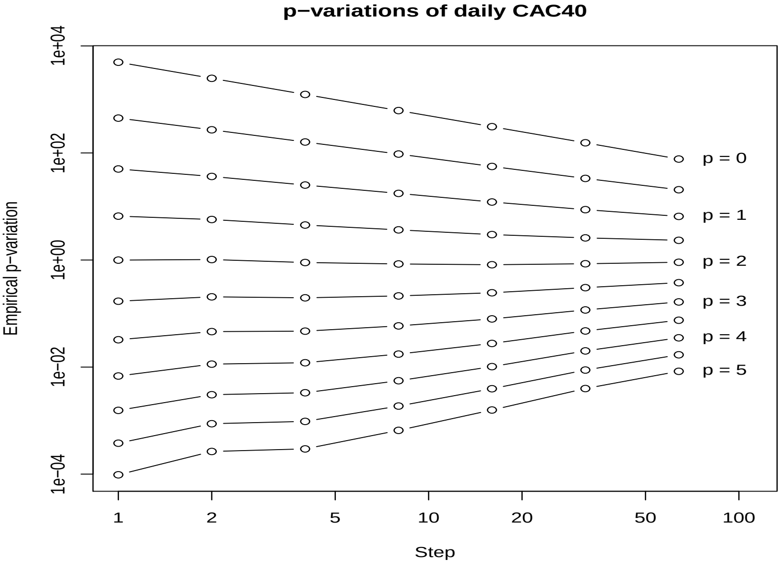
Multifractal models in finance

- Mandelbrot (1997), Calvet et Fisher (2001), Bacry, Muzy et Delour (2001): continuous, *multifractal* martingales as a model for financial assets prices.
- Multifractal processes:

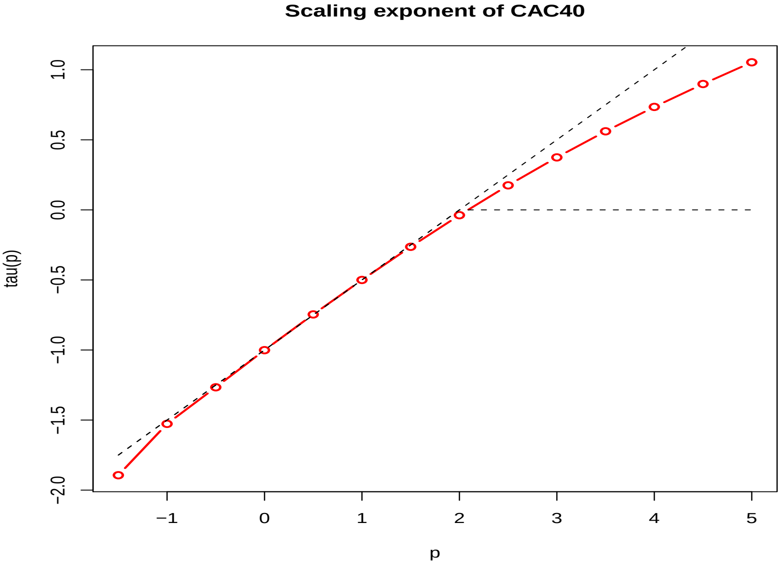
$$\sum_{i=0}^{n-1} |X_{(k+1)/n} - X_{k/n}|^p \approx c_p n^{-\tau(p)}, \quad n \to +\infty$$

  with $p \mapsto \tau(p)$ strictly concave (and not piecewise linear).
- This relation *seems* to be seen on the data.
- Also: the models reproduce the statistical regularities observed in practice while using *only a small number of scalar parameters*. Good results for risk prediction (see e.g. Duchon, Robert et Vargas 2008).

Multifractal scaling of the French stock index



Multifractal scaling of the French stock index



MRW model of Bacry and Muzy (2003)

$$X_t = B_{\theta_t},\ t \geq 0,$$

with B a standard Brownian motion and θ an increasing process indep. of B :

$$\theta_t = \sigma^2 \lim_{l \to 0} \int_0^t \mathrm{e}^{w_l(u)} \mathrm{d}u.$$

- log-normal case : $\big(w_l(t),\ t \geq 0\big)$ is a stationary Gaussian process such that

$$\mathbb{C}\mathrm{ov}\big[w_l(0), w_l(t)\big] \uparrow \lambda^2 \log_+(T/t) \quad \text{for } l \to 0$$
$$\text{and } \mathbb{E}\big[\mathrm{e}^{w_l(t)}\big] = 1 \quad \text{for all } l \text{ and } t.$$

Properties of the model

- Multifractal scaling: $\tau(p) = p/2 - 1 - \lambda^2 p(p-2)/8$, $\lambda^2 \approx 0.1$ in finance.
- Note that as $l \to 0$, we have $\mathrm{e}^{w_l(u)} \to 0$ a.s. and $\mathrm{e}^{w_l(u)} \to +\infty$ in L^p, p>1. The process

$$\theta_t = \lim_{l \to 0} \int_0^t \mathrm{e}^{w_l(u)} \mathrm{d}u$$

  is non degenerate but also not Lebesgue-absolutely continuous. Hence $X = \left(B_{\theta_t},\ t \geq 0\right)$ is a continuous martingale that is not Itō.
- Also: construction of a more general MRW class with $\left(w_l(t), t \geq 0\right)$ that has an *infinitely divisible* distribution.

Theoretical problem

Consider a process $X = \big(X_t, 0 \le t \le 1\big)$ observed at dyadic times, and define

$$S(p, 2^{-N}) = \sum_{k=0}^{2^N-1} |X_{(k+1)2^{-N}} - X_{k2^{-N}}|^p.$$

Based on $S(p, 2^{-N})$, find a statistic that converges to a known distribution under

$$H_0 \ : \ X \text{ is an Itō semi-martingale}$$

and becomes degenerate under

$$H_1 \ : \ X \text{ is an MRW process.}$$

Also: same question when you exchange H_0 and H_1.

Case H_0: X is Itō

Proposition 1 (Aït-Sahalia and Jacod)

If X is Itō with no jumps, then for $p > 2$

$$c(p)\frac{S(p,2^{-N}))}{(S(2p,2^{-N}))^{1/2}}\left(\frac{S(p,2^{-(N-1)})}{S(p,2^{-N})} - 2^{p/2-1}\right) \xrightarrow{\mathcal{L}} N(0,1).$$

The strict concavity of $p \mapsto \tau(p)$ shows that this goes to $+\infty$ if X is an MRW.

However, if X is Itō with jumps, this goes to an unknown r.v.

Test for H_0: X is Itō

Choose (k_N) a sequence such that $k_N \leq 1$, $k_N \to 1$ and $(1 - k_N)N \to +\infty$.

Theorem 1

Consider

$$T_N^{Ito} = c(p) 2^{(p/2-1)(\lfloor k_N N \rfloor - N)} \frac{S(p, 2^{-\lfloor k_N N \rfloor}))}{(S(2p, 2^{-N}))^{1/2}} \left(\frac{S(p, 2^{-(N-1)})}{S(p, 2^{-N})} - 2^{p/2-1} \right).$$

Then if X is Itō with jumps, $T_N^{Ito} \to 0$ in probability. If X is Itō with no jumps, $T_N^{Ito} \xrightarrow{\mathcal{L}} N(0,1)$. If X is an MRW, $T_N^{Ito} \to +\infty$ in probability.

Case H_0: X is an MRW

Proposition 2

If X is an MRW, alors

$$\frac{\sqrt{3}}{\sqrt{2(2^{\tau(4)-1})}}\frac{S(2,2^{-N})-S(2,2^{-N+1})}{\sqrt{S(4,2^{-N})}} \xrightarrow{\mathcal{L}} N(0,1).$$

If X is Itō with jumps, this goes to 0.
However, if X is Itō with no jumps, this is of order 1.

Test for the case H_0: X is an MRW

Theorem 2

Fix $k \in (0,1)$ and define

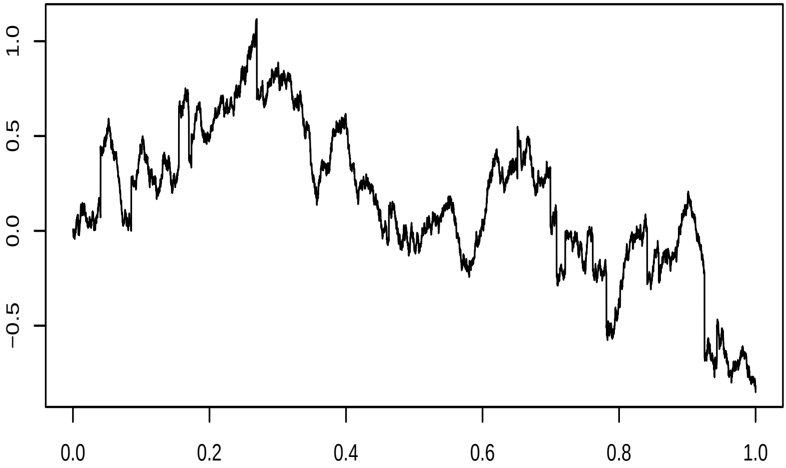
$$T_N^{MRW} = \frac{\sqrt{3}}{\sqrt{2(2^{\tau(4)-1})}} 2^{(N-\lfloor kN \rfloor)\tau(4)/2} \frac{S(2,2^{-N}) - S(2,2^{-N+1})}{\sqrt{S(4,2^{-\lfloor kN \rfloor})}}.$$

Then if X is Itō, T_N^{MRW} goes to 0 in probability. If X is an MRW, then $T_N^{MRW} \xrightarrow{\mathcal{L}} N(0,1)$.

NB: in practice $\tau(4)$ is unknown. We also have a theoretical result for the case where it is replaced by a consistent estimator.

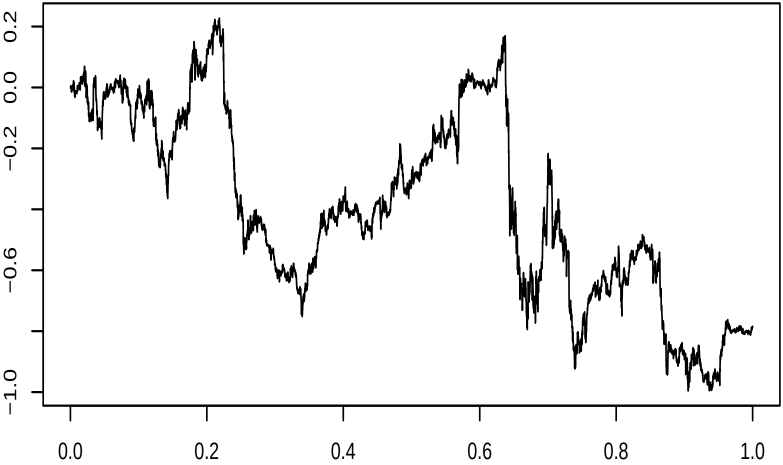
Simulations: Itō semi-martingales

Either BM, or BM with a few (≈ 30) large jumps $\mathcal{U}([-1/2, 1/2])$.



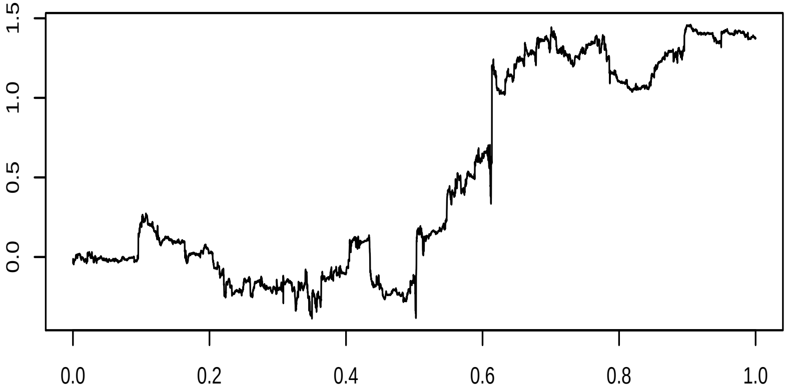
Simulations: MRW

Log-normal MRW with $\lambda^2 = 0.1$:
$\tau(p) = -0.0125p^2 + 0.525p - 1.$



Simulations: MRW

Log-normal MRW with $\lambda^2 = 0.7$:
$\tau(p) = -0.0875p^2 + 0.675p - 1.$



Simulation results

Simulated process	MRW, $\lambda^2 = 0.1$		MRW, $\lambda^2 = 0.7$	
Number 2^N of data	32 768	1 048 576	32 768	1 048 576
Level of the test				
10%	11	11	10	10
5%	8	5	4	4
1%	1	2	2	1

Simulated process	Brownian motion		Brownian motion + large jumps	
Level of the test				
10%	29	63	66	100
5%	12	21	31	82
1%	3	4	4	23

Table: *Number of rejects of H_0: $X = MRW$, $\tau(4) = 1 - \lambda^2$ known, for 100 simulations of the processes.*