Correlations in Asynchronous Markets

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Motivation

Estimating correlations and volatilities in asynchronous markets

Historical correlations: Stoxx50 – S&P500 – Nikkei

Comparison with other heuristic estimators – options and correlation swaps

Correlations larger than 1

Conclusion
Motivation

- Equity derivatives generally involve baskets of stocks/indices traded in different geographical areas
- Operating hours of: Asian and European exchanges, Asian and American exchanges, usually have no overlap

- Standard methodology on equity derivatives desks:
  - Use standard multi-asset model based on assumption of continuously traded securities
  - Compute/trade deltas at the close of each market, using stale values for securities not trading at that time
  - Likewise, valuation is done using stale values for securities whose markets are closed

▶ How should we estimate volatility and correlation parameters?
Consider following situation:

- Valuation of the option is done at the close of the Stoxx50
- Deltas are computed and traded on the market close of each security

Daily P&L:

\[
P&L = - \left[ f \left( t_{i+1}, S_{1,i+1}, S_{2,i+1} \right) - f \left( t_i, S_{1,i}, S_{2,i} \right) \right] \\
+ \frac{df}{dS_1} \left( t_i, S_{1,i}, S_{2,i} \right) (S_{1,i+1} - S_{1,i}) \\
+ \frac{df}{dS_2} \left( t_i - \delta, S_{1,i-1}, S_{2,i} \right) (S_{2,i+1} - S_{2,i})
\]

Note that arguments of \( \frac{df}{dS_2} \) are different
• Rewrite delta on $S_2$ so that arguments are same as $f$ and $\frac{df}{dS_1}$

$$\frac{df}{dS_2}(t_i - \delta, S_{1,i-1}, S_{2,i}) = \frac{df}{dS_2}(t_i, S_{1,i}, S_{2,i}) - \frac{d^2f}{dS_1dS_2}(t_i, S_{1,i}, S_{2,i})(S_{1,i} - S_{1,i-1})$$

• Other correction terms contribute at higher order in $\Delta$

• P&L now reads:

$$P&L = - [f(t_{i+1}, S_{1,i+1}, S_{2,i+1}) - f(t_i, S_{1,i}, S_{2,i})]$$
$$+ \frac{df}{dS_1}(S_{1,i+1} - S_{1,i}) + \left[ \frac{df}{dS_2} - \frac{d^2f}{dS_1dS_2}(S_{1,i} - S_{1,i-1}) \right](S_{2,i+1} - S_{2,i})$$

• Expanding at 2nd order in $\delta S_1, \delta S_2$:

$$P&L = - \frac{df}{dt}\Delta - \left[ \frac{1}{2} \frac{d^2f}{dS_1^2} \delta S_{1+}^2 + \frac{1}{2} \frac{d^2f}{dS_1^2} \delta S_{2+}^2 + \frac{d^2f}{dS_1dS_2} \delta S_{1+} \delta S_{2+} \right] - \frac{d^2f}{dS_1dS_2} \delta S_{1-} \delta S_{2+}$$
- Assume $f$ is given by a Black-Scholes equation:

$$\frac{df}{dt} + \frac{\sigma_1^2}{2} S_1^2 \frac{d^2 f}{dS_1^2} + \frac{\sigma_2^2}{2} S_2^2 \frac{d^2 f}{dS_2^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{d^2 f}{dS_1 dS_2} = 0$$

- P&L now reads:

$$P&L = -\frac{1}{2} S_1^2 \frac{d^2 f}{dS_1^2} \left[ \left( \frac{\delta S_1^+}{S_1} \right)^2 - \sigma_1^2 \Delta \right] - \frac{1}{2} S_2^2 \frac{d^2 f}{dS_2^2} \left[ \left( \frac{\delta S_2^+}{S_2} \right)^2 - \sigma_2^2 \Delta \right]$$

$$- S_1 S_2 \frac{d^2 f}{dS_1 dS_2} \left[ \left( \frac{\delta S_1^-}{S_1} + \frac{\delta S_1^+}{S_1} \right) \frac{\delta S_2^+}{S_2} - \rho \sigma_1 \sigma_2 \Delta \right]$$

- Prescription for estimating volatilities & correlations so that P&L vanishes on average:

$$\sigma_1^{*2} = \frac{1}{\Delta} \left\langle \left( \frac{\delta S_1^+}{S_1} \right)^2 \right\rangle \quad \sigma_2^{*2} = \frac{1}{\Delta} \left\langle \left( \frac{\delta S_2^+}{S_2} \right)^2 \right\rangle \quad \rho^* \sigma_1^* \sigma_2^* = \frac{1}{\Delta} \left\langle \left( \frac{\delta S_1^-}{S_1} + \frac{\delta S_1^+}{S_1} \right) \frac{\delta S_2^+}{S_2} \right\rangle$$

- Volatility estimators are the usual ones, involving daily returns

- The correlation estimator involves daily returns as well
Define \( r_i = \frac{\delta S_i^+}{S_i} \). At lowest order in \( \Delta \), \( \frac{\delta S_i^-}{S_i} \simeq \frac{\delta S_i^-}{S_{i-1}} \)

\[
\sigma_{1}^{*2} = \frac{1}{\Delta} \langle r_{1i}^2 \rangle \\
\sigma_{2}^{*2} = \frac{1}{\Delta} \langle r_{2i}^2 \rangle \\
\rho^* = \frac{\langle (r_{1i-1} + r_{1i}) r_{2i} \rangle}{\sqrt{\langle r_{1i}^2 \rangle \langle r_{2i}^2 \rangle}}
\]

- Had we chosen the close of the Nikkei for valuing the option: symmetrical estimator:

\[
\sigma_{1}^{*2} = \frac{1}{\Delta} \langle r_{1i}^2 \rangle \\
\sigma_{2}^{*2} = \frac{1}{\Delta} \langle r_{2i}^2 \rangle \\
\rho^* = \frac{\langle r_{1i} (r_{2i} + r_{2i+1}) \rangle}{\sqrt{\langle r_{1i}^2 \rangle \langle r_{2i}^2 \rangle}}
\]

- If returns are time-homogeneous \( \langle r_{1i-1} r_{2i} \rangle = \langle r_{1i} r_{2i+1} \rangle \)

In practice \( \frac{1}{N} \sum_1^N (r_{1i-1} + r_{1i}) r_{2i} - r_{1i} (r_{2i} + r_{2i+1}) = \frac{1}{N} (r_{10} r_{21} - r_{1N} r_{2N+1}) \)

- Difference between two estimators of \( \rho^* \): finite size effect of order \( \frac{1}{N} \)
Motivation

Correlation estimation in asynchronous markets

Historical correlations

Correlations larger than 1

In conclusion, in asynchronous markets: 2 correlations $\rho_S$, $\rho_A$:

$$\rho_S = \frac{\langle r_{1i} r_{2i} \rangle}{\sqrt{\langle r_{1i}^2 \rangle \langle r_{2i}^2 \rangle}} \quad \rho_A = \frac{\langle r_{1i} r_{2i+1} \rangle}{\sqrt{\langle r_{1i}^2 \rangle \langle r_{2i}^2 \rangle}}$$

and derivatives should be priced with $\rho^*$:

$$\rho^* = \rho_S + \rho_A$$

- Does $\rho^*$ depend on the particular delta strategy used in derivation?
- Is $\rho^*$ in $[-1, 1]$?
- How does $\rho^*$ compare to standard correlations estimators evaluated with 3-day, 5-day, $n$-day returns?
What if had computed deltas differently – for example "predicting" the value of the stock not trading at the time of computation?

- Option delta-hedged one way minus option delta-hedged the other way. Final P&L is:
  \[ \sum (\Delta_t^a - \Delta_t^b)(S_{t+\Delta} - S_t) \]
- Price of pure delta strategy is zero: **correlation estimator is independent on delta strategy used in derivation**

Imagine processes are continuous yet observations are asynchronous: assume that \( \rho \sigma_1 \sigma_2, \sigma_1^2, \sigma_2^2 \) are periodic functions with period \( \Delta = 1 \) day:

\[
\rho_S = \frac{\frac{1}{\Delta} \int_t^{t+\Delta-\delta} \rho \sigma_1 \sigma_2 \, ds}{\sqrt{\frac{1}{\Delta} \int_t^{t+\Delta} \sigma_1^2 \, ds} \sqrt{\frac{1}{\Delta} \int_{t-\delta}^{t+\Delta-\delta} \sigma_2^2 \, ds}}
\]

\[
\rho_A = \frac{\frac{1}{\Delta} \int_t^{t+\Delta} \rho \sigma_1 \sigma_2 \, ds}{\sqrt{\frac{1}{\Delta} \int_t^{t+\Delta} \sigma_1^2 \, ds} \sqrt{\frac{1}{\Delta} \int_{t-\delta}^{t+\Delta-\delta} \sigma_2^2 \, ds}}
\]

\[ \rho^* = \rho_S + \rho_A \]

- Recovers value of "synchronous correlation": **no bias**
Motivation
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**Historical correlations**

- $\rho_S$ (blue), $\rho_A$ (pink), $\rho^* = \rho_S + \rho_A$ (green) – 6-month EWMA

![Graphs showing historical correlations between Stoxx50/SP500, Nikkei/Stoxx50, and Nikkei/SP500 with different correlation indices over time from 4-Dec-99 to 1-Dec-09.](image)
- Is the signal for $\rho_A$ in the Stoxx50/S&P500 case real?
- Switch time series of Stoxx50 and S&P500 and redo computation:

▷ In the reversed situation, $\rho_A$ hovers around 0.
• \( \rho_S, \rho_A \) seem to move antithetically
  
  • Imagine \( \sigma_1(s) = \sigma_1 \lambda(s), \sigma_2(s) = \sigma_2 \lambda(s), \rho \) constant, with \( \lambda(s) \) such that 
  \[
  \frac{1}{\Delta} \int_0^\Delta \lambda^2(s) ds = 1.
  \]
  Then:
  
  \[
  \rho_S = \rho \frac{1}{\Delta} \int_0^{\Delta-\delta} \lambda^2(s) ds
  \]
  \[
  \rho_A = \rho \frac{1}{\Delta} \int_{\Delta-\delta}^\Delta \lambda^2(s) ds
  \]
  
  and \( \rho^* \) is given by:
  
  \[
  \rho^* = \rho_S + \rho_A = \rho
  \]
  
  • By changing \( \lambda(s) \) we can change \( \rho_S, \rho_A \), while \( \rho^* \) stays fixed.

▷ The relative sizes of \( \rho_S, \rho_A \) are given by the intra-day distribution of the realized covariance.
Comparison with heuristic estimators

- Trading desks have long ago realized that merely using $\rho_S$ is inadequate
  - Standard fix: compute standard correlation using 3-day, 5-day, you-name-it, rather than daily returns
  - How do these estimators differ from $\rho^*$?

- Connected issue: how do we price an $n$-day correlation swap?

▶ An $n$–day correlation swap should be priced with $\rho_n$ given by:

$$\rho_n = \rho_S + \frac{n-1}{n} \rho_A$$

- For $n = 3$, $\rho_3 = \rho_S + \frac{2}{3} \rho_A$

- If no serial correlation in historical sample, standard correlation estimator applied to $n$-day returns yields $\rho_n$
Historical $n$-day correlations

- $n$-day correlations evaluated on 2004-2009 with:
  - $n$-day returns (dark blue)
  - using $\rho_S + \frac{n-1}{n} \rho_A$ (light blue)
- compared to $\rho^*$ (purple line)

Common estimators $\rho_3, \rho_5$ underestimate $\rho^*$
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The S&P500 and Stoxx50 as synchronous securities

- European and American exchanges have some overlap. We can either:
  - delta-hedge asynchronously the S&P500 at 4pm New York time and the Stoxx50 at 5:30pm Paris time
  - delta-hedge simultaneously both futures at – say – 4pm Paris time

- 1st case: use $\rho^*$, 2nd case: use standard correlation for synchronous securities – are they different?

- $\rho^*$ (light blue), standard sync. correlation (dark blue) – 3-month EWMA

- Matches well, but not identical: difference stems from residual realized serial correlations.

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- Example of RBS/Citigroup correlations: $\rho_S$ (blue), $\rho_A$ (pink), $\rho^*$ (green) – 3-month EWMA

- Are instances when $\rho^* > 1$ an artifact? Do they have financial significance?
Consider a situation when no serial correlation is present. The global correlation matrix is positive, by construction. How large can $\rho_S + \rho_A$ be?

![Diagram showing correlation matrix with $\rho_S$ and $\rho_A$]

Compute eigenvalues of full correlation matrix:

- assume both ladder uprights consist of $N$ segments, with periodic boundary conditions
- assume eigenvalues have components $e^{ik\theta}$ on higher upright, $\alpha e^{ik\theta}$ on lower upright
- express that $\lambda$ is an eigenvalue:
  
  \[ \alpha \rho_S + 1 + \alpha e^{i\theta} = \lambda \]
  
  \[ \rho_S + \alpha + e^{-i\theta} \rho_A = \lambda \alpha \]

yields:

\[ \lambda = 1 \pm \sqrt{(\rho_S + \rho_A \cos \theta)^2 + \rho_A^2 \sin^2 \theta} \]
• Periodic boundary conditions impose $\theta = \frac{2n\pi}{N}$, where $n = 0 \ldots N - 1$
• $\lambda(\theta)$ extremal for $\theta = 0, \pi$. For these values $\lambda = 1 \pm |\rho_S \pm \rho_A|$
• $\lambda > 0$ implies:

\[
-1 \leq \rho_S + \rho_A \leq 1 \\
-1 \leq \rho_S - \rho_A \leq 1 
\]

▷ If no serial correlations $\rho^* \in [-1, 1]$

▷ Instances when $\rho^* > 1$: evidence of serial correlations

▷ Impact of $\rho^* > 1$ on trading desk: price with the right realized volatilities, 100% correlation $\rightarrow$ lose money !!
Example with basket option

- Sell 6-month basket option on basket of Japanese stock & French stock.
  Payoff is \( \left( \sqrt{\frac{S_1^T S_2^T}{S_1^0 S_2^0}} - 1 \right)^+ \)
- Basket is lognormal with volatility given by \( \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho \sigma_1 \sigma_2} \)
- Use following "historical" data:
  - Realized vols are 21.8% for \( S_1 \), 23.6% for \( S_2 \). Realized correlations are \( \rho_S = 63.3\% \), \( \rho_A = 57.6\% \): \( \rho^* = 121\% \).
• Backtest delta-hedging of option with:
  • implied vols = realized vols
  • different implied correlations

• Initial option price and final P&L:

![](image)

• Final P&L vanishes when one prices and risk-manages option with an implied correlation $\rho \approx 125\%$. 
It is possible to price and risk-manage options on asynchronous securities using the standard synchronous framework, provided special correlation estimator is used.

Correlation estimator quantifies correlation that is materialized as cross-Gamma P&L.

Correlation swaps and options have to be priced with different correlations.

Serial correlations may push realized value of $\rho^*$ above 1: a short correlation position will lose money, even though one uses the right vols and 100% correlation.