Price modelling with microstructure via point processes.

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Content

Price representation

- Price processes behave differently at different scales:
  - Coarse scales (daily data): diffusions
  - Fine scales (tick data): marked point processes
- Breakdown of the diffusive behaviour in small scales
  - In dimension 1 : microstructure noise (variance instability).
  - In dimension 2 : Epps effect (covariance instability).

Microstructure noise and signature plot

- Itô semimartingale price model consensus (indifferently mid-price/traded-price)

$$S_t = \text{drift}_t + \int_0^t \sigma_s dB_s + (\text{jump process}_t)$$

- If S_t is observed over $[0, t]$ at times $0, \Delta, 2\Delta, \ldots$, convergence of the realized volatility

$$\boxed{V_\Delta\{S\}_t := \sum_{i\Delta \le t} \left(S_{i\Delta} - S_{(i-1)\Delta}\right)^2 \xrightarrow{\mathbb{P}} \int_0^t \sigma_s^2 ds}$$

as $\Delta \to 0$ with accuracy $\sqrt{\Delta}$.
- This suggests to pick Δ as small as possible... but

Signature plot

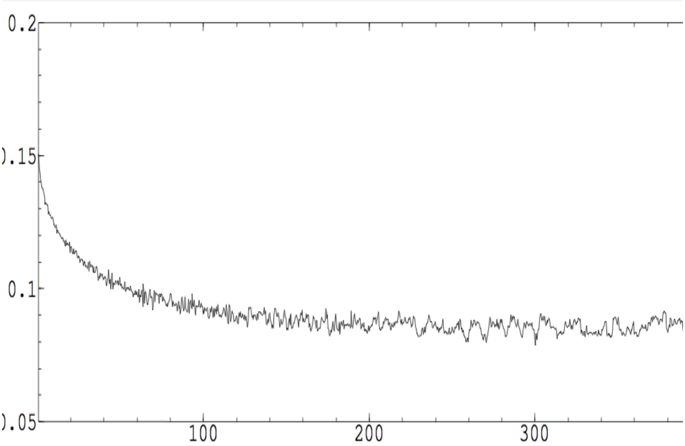


Figure: $\Delta \rightsquigarrow V_\Delta$ for FGBL (43 days, 9-11 AM) on Last Traded Ask.

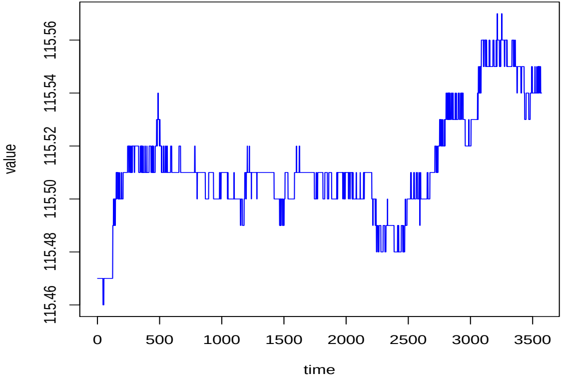


Figure: FGBL, 06 Feb 2007, 09:00–10:00 (UTC) 1 data per second.

In dimension 2: Epps effect

- In the same Itô semimartingale setting, we have convergence of the quadratic covariation

$$CV_\Delta\left\{S^{(1)},S^{(2)}\right\}_t := \sum_{i\Delta\le t}\left(S^{(1)}_{i\Delta}-S^{(1)}_{(i-1)\Delta}\right)\left(S^{(2)}_{i\Delta}-S^{(2)}_{(i-1)\Delta}\right) \xrightarrow{\mathbb{P}} \langle S^{(1)},S^{(2)}\rangle_t$$

- Same prescription as for the realized volatility: pick Δ as small as possible... but

Epps effect

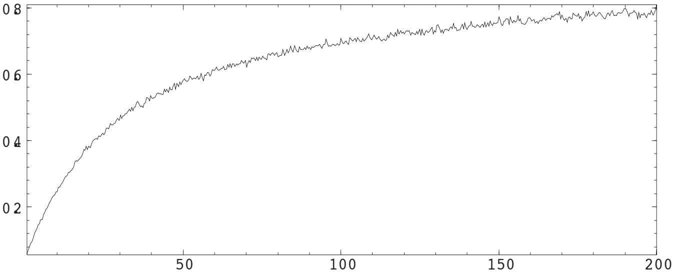


Figure: $\Delta \rightsquigarrow CV_\Delta\left\{S^{(1)}, S^{(2)}\right\}$ (normalized) with $S^{(1)} = FGBL$, $S^{(2)} = FGBM$, 40 days, 9-11AM.

In dimension 2

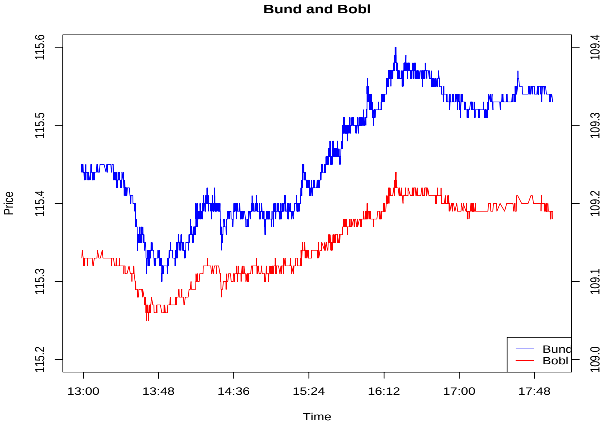


Figure: FGBL/FGBM

Motivation

We look for a "simple" multivariate price model with the following properties:

- Be defined in continuous time with discrete values in a microscopic scale.
- Incorporate microstructure noise and the Epps effect with "few" parameters.
- Diffuse in a macroscopic scale.

Point process approach

1. Price process = marked point process.
   - Marks : jumps up/down by 1 tick,
   - Jump times: time stamps of price changes.
2. The price process is the result (sum) of a "upward change or price" and a "downward change of price". Coupling random intensities (Hawkes process) $\rightarrow$ microstructure noise.
3. The price of two assets is obtained by coupling further (Hawkes process) the respective intensities of the "upward change of price" and "downward change of price" processes $\rightarrow$ dependence structure.

Compound Poisson process

- Let $N_t^{\mu_+}$ and $N_t^{\mu_-}$ be two independent Poisson processes with intensity $\mu_\pm$.
- Then

$$\boxed{M_t^{\mu_+,\mu_-} := N_t^{\mu_+} - N_t^{\mu_-} = \sum_{n=0}^{\infty} \varepsilon_n 1_{T_n \le t}}$$

is a compound Poisson process with

- $(T_n - T_{n-1})_{n\ge 1}$ i.i.d. exponential with parameters $\mu_+ + \mu_-$.
- Law of the jumps:

$$\mathbb{P}\big[\varepsilon_n = +1\big] = 1 - \mathbb{P}\big[\varepsilon_n = -1\big] = \frac{\mu_+}{\mu_+ + \mu_-}.$$

Scaling limit

- Macroscopic limit take $\mu_+ = \mu_- = \mu$ with $\delta \to 0$ (continuous limit in space).

$$\boxed{M_t^{\mu\delta^{-1}} = \sum_{0}^{N^\mu_{t\delta^{-1}} + N^\mu_{t\delta^{-1}}} \varepsilon_i}$$

with ε_i i.i.d. standard Bernoulli ± 1.

- Spatial renormalization

$$\sqrt{\delta} M^\mu_{t\delta^{-1}} \approx \sqrt{\delta} \sum_{1}^{\delta^{-1} 2\mu t} \varepsilon_i \approx B_{2\mu t},$$

where (B_t) is a standard Brownian motion. By scaling

$$B_{2\mu t} \stackrel{(d)}{=} \sqrt{2\mu} B_t$$

and $\sqrt{2\mu}$ is the macroscopic volatility$^{1/2}$.

Hawkes processes: the 1 dimensional case

- Start with a counting process N_t constructed via its stochastic intensity

$$\boxed{\lambda(t) = \mu + \int_0^t \phi(t-s)dN_s}$$

where $\phi(\cdot)$ is a *coupling* function. Standard $\phi(x) = \alpha e^{-\beta x}$. Interpretation of the parameters:

  - μ: exogenous intensity
  - α: (rather α/β): local self-exciting intensity.
  - β: temporal delay.

- (One has $\int_0^t \phi(t-s)dN_s = \sum_{T_n<t} \phi(t-T_n)$.) Essential constraint: $\int_0^{+\infty} \phi < 1$.

Remark on parameter inference

- The likelihood is explicit, given a continuous trajectory over $[0, T]$. If $\vartheta = (\mu, \alpha, \beta)$

$$\boxed{\log \ell(\vartheta) = \int_0^T \log(\lambda_\vartheta(s))dN_s - \int_0^T \lambda_\vartheta(s)ds.}$$

- But: maximization of the log-likelihood is computationnally intensive.

Price model in dimension 1

Let $S_t = N_t^+ - N_t^-$, with $N_t^\pm$ Hawkes processes with respective random intensities $\lambda_t^\pm$ given by

$$\boxed{\lambda^\pm(t) := \mu_\pm + \alpha \int_{[0,t)} e^{-\beta(t-s)} dN_s^\mp}$$

- $\mu_\pm$: exogeneous intensity.
- α et β: mutually exciting intensities generating a "mean-reverting effect" for S_t.
- $\alpha e^{-\beta x} \rightsquigarrow \Phi(x)$ with $\|\Phi\|_{L^1} < 1$ in the sequel.

Price in dimension 2

1. Start from two processes X and Y constructed as before.
2. Introduce a supplementary coupling on the intensities of the two processes and create a dependence structure Upward_X-Upward_Y and Downward_X-Downward_Y.
3. (We ignore further possible coupling Upward_X-Downward_Y and Downward_X-Upward_Y between X and Y.)

Representation of X and Y

Set

$$\boxed{X(t) = N_X^+(t) - N_X^-(t) \text{ and } Y(t) = N_Y^+(t) - N_Y^-(t)}$$

with

$$\lambda_X^{\pm}(t) = \mu_X^{\pm} + \int_{[0,t)} \Phi_{X,X}(t-s)dN_X^{\mp}(s) + \int_{[0,t)} \Phi_{X,Y}(t-s)dN_Y^{\pm}(s)$$

and

$$\lambda_Y^{\pm}(t) = \mu_Y^{\pm} + \int_{[0,t)} \Phi_{Y,Y}(t-s)dN_X^{\pm}(s) + \int_{[0,t)} \Phi_{Y,X}(t-s)dN_X^{\pm}(s)$$

Simulation over 1000 seconds

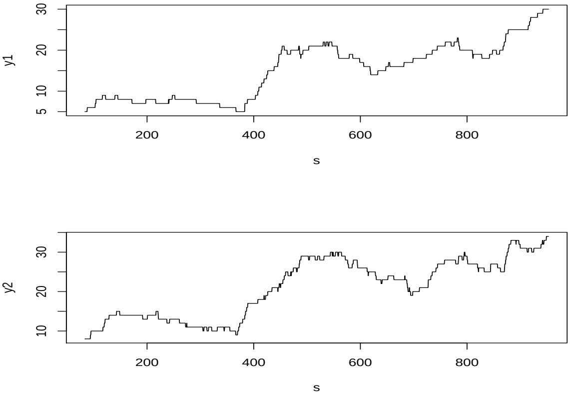


Figure: Sample simulation in dimension 2

Simulation over 1000 secondes

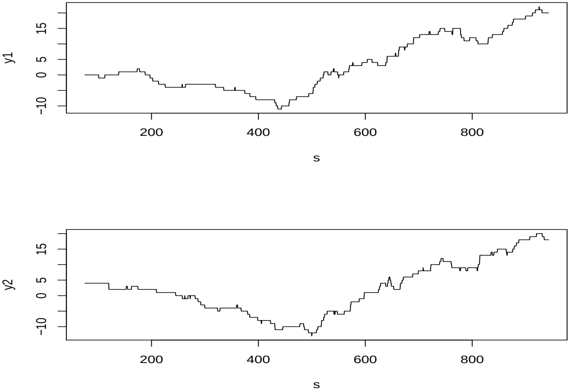


Figure: Another sample...

Scaling limits

- First step:
  1. closed-form formulas for the mean "signature plot" when $\Phi(x) = \alpha e^{-\beta x}$ (through the explicit computation of the Bartlett spectrum, case with stationary increments) in dimension 1 and 2.
  2. Statistical fits and discussion of further data filtering.
- Second step: diffusive limit (after spatial renormalization) for arbitrary Φ in dimension 1 (and arbitrary $\Phi_{X,Y}$, $\Phi_{Y,X}$ and $\Phi_{X,X} = \Phi_{Y,Y}$ in dimension 2).
- More scaling limits...
- Comparison with other models

Mean "signature plot" and scaling limits

- Time-space renormalization

$$\boxed{X^{(\delta)}(t) := \sqrt{\delta} X(\delta^{-1} t), \quad t \in [0,1]}$$

- Realized volatility

$$V_\Delta\{X^{(\delta)}\} := \sum_{i=1}^{\Delta^{-1}} \left(X^{(\delta)}(i\Delta) - X^{(\delta)}((i-1)\Delta) \right)^2$$
$$\approx \frac{1}{\Delta\delta^{-1}} \mathbb{E}\left[\left(X(\Delta\delta^{-1}) - X(0) \right)^2 \right]$$

- Mean signature plot

$$\boxed{\mathcal{V}(t) := \frac{1}{t} \mathbb{E}\left[\left(X(t) - X(0) \right)^2 \right]}$$

- Interpretation

$$\mathcal{V}(\Delta\delta^{-1}) \approx V_\Delta\{X^{(\delta)}\}.$$

Mean "Signature plot"

If $\Phi_{X,X}(x) = \Phi_{Y,Y}(x) = \alpha e^{-\beta x}$, $\Phi_{X,Y} = \Phi_{Y,X} = 0$ and $\mu^+ = \mu^- = \mu$ we have (via Bartlett spectrum) for X (or Y)

$$\mathcal{V}(t) = \frac{2\mu}{1-\alpha/\beta}\Big[\frac{1}{(1+\alpha/\beta)^2} + \\ +(1-\frac{1}{(1+\alpha/\beta)^2})\frac{1-\exp\big(-(\alpha+\beta)t\big)}{(\alpha+\beta)t}\Big]$$

Scaling limit in dimension 1, $\mu^{\pm} = \mu$

- Step 1 : price decomposition introducing a martingale

$$X^{(\delta)}(t) = \delta^{1/2}\big(N^{+}_{\delta^{-1}t} - N^{-}_{\delta^{-1}t}\big)$$
$$= M^{(\delta)}_t + B^{(\delta)}(t),$$

with

$$\boxed{M^{(\delta)}_t = \delta^{1/2}\big(N^{+}_{\delta^{-1}t} - N^{-}_{\delta^{-1}t}\big) - B^{(\delta)}_t \quad \text{martingale}}$$

and

$$\boxed{B^{(\delta)}_t = \delta^{1/2}\int_0^{\delta^{-1}t}\big(\lambda^{+}(s) - \lambda^{-}(s)\big)\,ds \quad \text{predictable}}$$

Scaling limit in dimension 1 (cont.)

- Step 2: Convergence of the compensator

$$
\begin{aligned}
B^{(\delta)}(t) &= \delta^{1/2} \int_0^{\delta^{-1}t} ds \int_0^s \Phi(s-u) d(N_u^- - N_u^+) \\
&= -\int_{[0,t)} dX^{(\delta)}(u) \int_0^{t-u} \delta^{-1}\Phi(\delta^{-1}s) ds \\
&= -\int_{[0,t)} X^{(\delta)}(u) \underbrace{\Phi_\delta(t-u)}_{\text{Dirac mass } \times \|\Phi\|_{L^1}} du \\
&\approx -\|\Phi\|_{L^1} X^{(0)}(t).
\end{aligned}
$$

- In the limit

$$X^{(0)}(t) = -\|\Phi\|_{L^1} X^{(0)}(t) + M_t^{(0)}$$

Scaling limit in dimension 1 (cont.)

- Step 3: Convergence of the martingale part

$$
\begin{aligned}
\left\langle M^{(\delta)} \right\rangle_t &= \delta \int_0^{\delta^{-1}t} \left(\lambda^+(s) + \lambda^-(s)\right) ds \\
&= 2\mu t + \delta \int_0^{\delta^{-1}t} ds \int_0^s \phi(s-u) d\left(N^+(u) + N^-(u)\right) \\
&= 2\mu t + \int_0^t \left[M^{(\delta)}\right]_u \phi_\delta(t-u) du \\
&\approx 2\mu t + \int_0^t \langle M^{(\delta)} \rangle_u \phi_\delta(t-u) du \\
&\approx 2\mu t + \|\Phi\|_{L^1} \langle M^{(\delta)} \rangle_t
\end{aligned}
$$

- Conclusion

$$
\left\langle M^{(\delta)} \right\rangle_t \xrightarrow{\mathbb{P}} \frac{2\mu}{1 - \|\Phi\|_{L^1}} t
$$

Scaling limit in dimension 1 (cont.)

- We obtain a)

$$M^{(\delta)}(t) \xrightarrow{d} \sqrt{\frac{2\mu}{1-\|\Phi\|_{L^1}}}W_t,$$

  where W is a Wiener process
- and b) the representation

$$X^{(0)}(t) = -\|\Phi\|_{L^1}X^{(0)}(t) + M_t^{(0)}$$

- a) + b) yield the final result:

$$\boxed{X^{(\delta)}(t) \xrightarrow{d} \frac{1}{1+\|\Phi\|_{L^1}}\sqrt{\frac{2\mu}{1-\|\Phi\|_{L_1}}}W_t}$$

Discussion

- Microscopic variance

$$\boxed{\mathbb{E}[\lambda^+ + \lambda^-] = \frac{2\mu}{1 - ||\Phi||_1}}$$

- Macroscopic variance

$$\boxed{\sigma^2 = \frac{2\mu}{1 - ||\Phi||_1} \frac{1}{(1 + ||\Phi||_1)^2}}$$

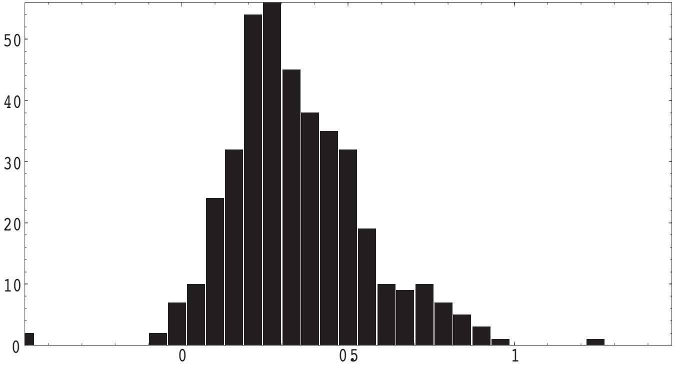
- **However the influence of Φ does not disappear at large scale**
- This influence can be quantified by looking at the function

$$||\phi||_1 = x \in [0, 1) \rightsquigarrow f(x) = \frac{1}{1 - x} \frac{1}{(1 + x)^2}$$

$f(x) \leq f(0) \Longrightarrow x \approx 0.61$ and f minimum at $x = \frac{1}{3}$.

Influence of $||\Phi||_1$ on the macroscopic variance

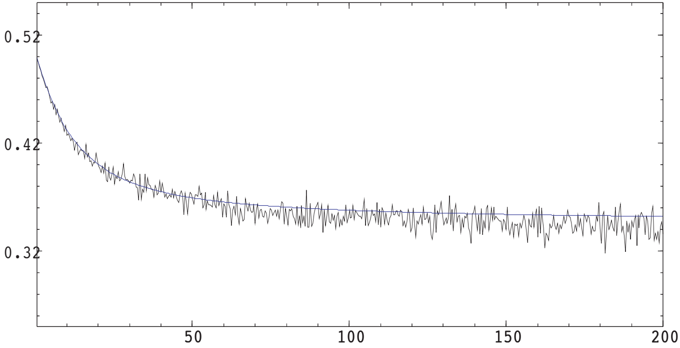
Histogram of $||\Phi||_1$ fitted (mean square) on the signature plot of
Bund 10Y 140 days - 9 : 11am - 12am: 2pm - 2 : 4pm



Mean ≈ 0.34

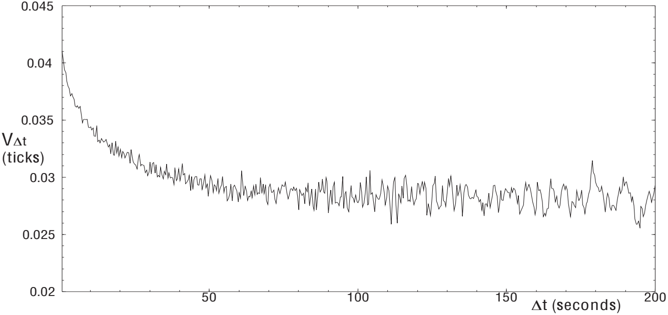
Mean signature plot on simulated data

Signature plot on 11 hours simulated data



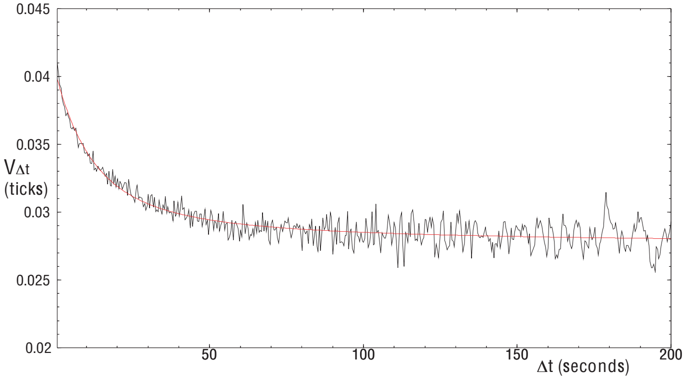
Mean signature plot on real data

- Bund 10Y : 21 days, 9-11 AM - Last Traded Ask (7000 points)



Mean signature plot on real data - Mean square regression

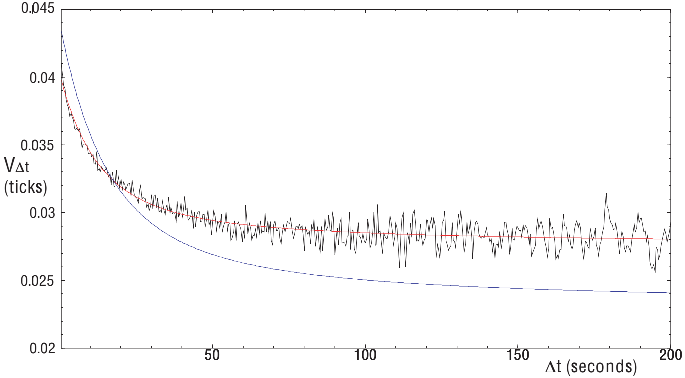
- Bund 10Y : 21 days, 9-11 AM - Last Traded Ask
  Mean square regression fit



$\Longrightarrow$ Fairly good modelling of the 1d microstructure noise.

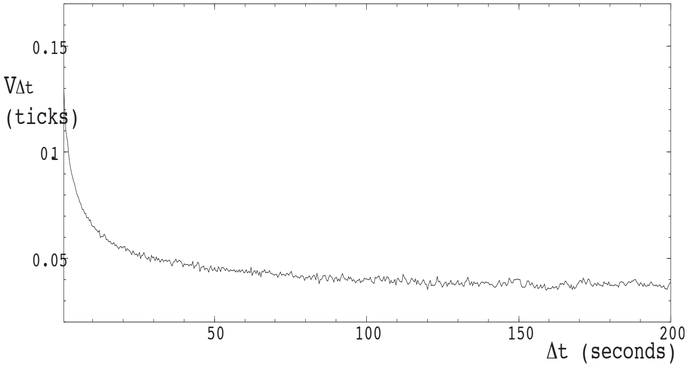
Mean signature plot on real data - MLE

- Bund 10Y : 21 days, 9-11 AM - Last Traded Ask
  Maximum likelihood fit



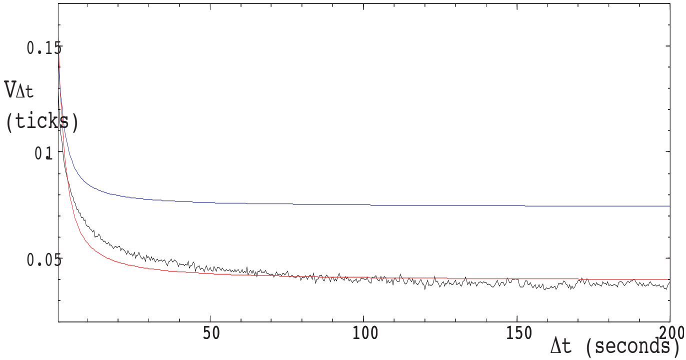
Mean signature plot on real data

- Bund 10Y : 26 days, 9-11 AM - Last Traded Price (29000 points)



Mean signature plot on real data - MLE

- 10Y Bund data : 26 days, 9-11 AM - Last Traded Price
  Mean square regression fit
  Maximum likelihood fit



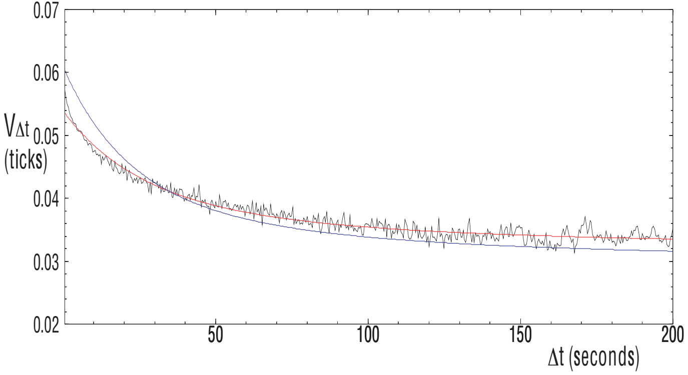
Instabilities of the MLE fit

The 1d model is a very good model for 1d microstructure noise but it remains a "first-brick" model for tick-by-tick time-series themselves :

- "Naive" model
  - Arbitrary parametric shape $\phi(t) = \alpha e^{-\beta t}$
  - Fully symmetric constant parameters : $\mu^+ = \mu^-$, $\alpha^+ = \alpha^-$, $\beta^+ = \beta^-$
  - No volume in the model !
- tick-by-tick time-series : Arbitrary projection of a very complex phenomenon (orderbook dynamics)

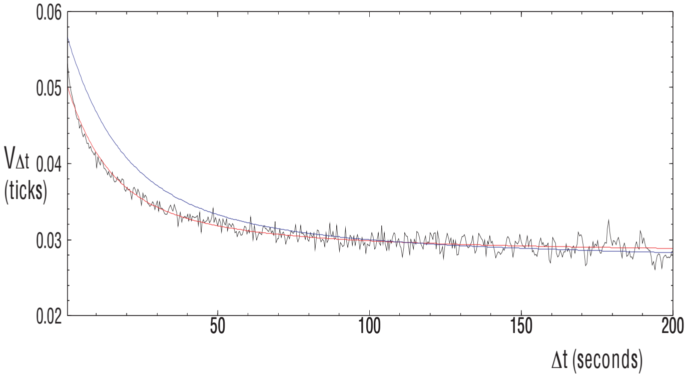
Mean signature plot on real data

- 10Y Bund data : 26 days, 9-11 AM - Last Traded Price Volume > 1 - 11000 points



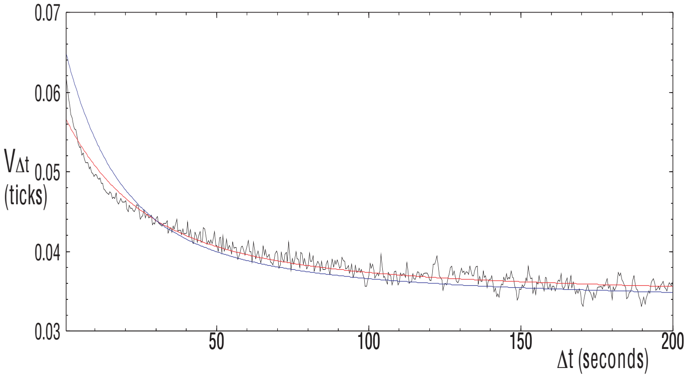
Mean signature plot on real data

- 10Y Bund data : 21 days, 9-11 AM - Last Traded Price Volume > 1 - 8600 points



Mean signature plot on real data

- 10Y Bund data : 41 days, 9-11 AM - Last Traded Price Volume $>$ 1 - 20000 points



Scaling limit in dimension 2

- For simplicity $\mu_+ = \mu_-$, $\Phi_{X,X} = \Phi_{Y,Y} = \Phi_{\text{self}}$
- In the same way, $X(t) = M_X(t) + B_X(t)$ with $B_X(t)$ given by

$$\int_0^t \left[\lambda_X^+(s) - \lambda_X^-(s)\right] ds$$
$$= \int_0^t \Big[\int_0^s \left(\Phi_{\text{self}}(s-u)dN_X^-(u) + \Phi_{XY}(s-u)dN_Y^+(u)\right)$$
$$- \int_0^s \left(\Phi_{\text{self}}(s-u)dN_X^+(u) + \Phi_{XY}(s-u)dN_Y^-(u)\right)\Big] ds$$

- After scaling+same kind of approximation as in the 1d case

$$X^{(\delta)}(t) \approx -\|\Phi_{\text{self}}\|_{L^1} X^{(\delta)}(t) + \|\Phi_{XY}\|_{L^1} Y^{(\delta)}(t) + M_X^{(\delta)}(t).$$

Scaling limit in dimension 2 (cont.)

- By symmetry, we obtain in the limit

$$X^{(0)}(t) = M_X^{(0)}(t) - \|\Phi_{\text{self}}\|_{L^1} X^{(0)}(t) + \|\Phi_{XY}\|_{L^1} Y^{(0)}(t)$$
$$Y^{(0)}(t) = M_Y^{(0)}(t) - \|\Phi_{\text{self}}\|_{L^1} Y^{(0)}(t) + \|\Phi_{YX}\|_{L^1} X^{(0)}(t)$$

- Convergence of the martingale part

$$\left(M_X^{(\delta)}, M_Y^{(\delta)}\right) \xrightarrow{d} \sigma_{\|\Phi_s\|,\|\Phi_{XY}\|,\|\Phi_{YX}\|}\left(W^{(1)}, W^{(2)}\right)$$

where $W^{(1)}$ and $W^{(2)}$ are two independent Brownian motions. (We need $t \rightsquigarrow t\Phi_{XY}(t)$ and $t\Phi_{YX}(t)$ in L^1).

Scaling limit in dimension 2 (cont.)

- We have in the limit $\delta \to 0$

$$X^{(\delta)} \xrightarrow{d} \frac{\sigma_{\|\Phi_s\|,\|\Phi_{XY}\|,\|\Phi_{YX}\|}}{(1+\|\Phi_s\|)^2-\|\Phi_{XY}\|\|\Phi_{YX}\|}\left[(1+\|\Phi_s\|)W^{(1)}+\|\Phi_{XY}\|W^{(2)}\right].$$

and (by symmetry)

$$Y^{(\delta)} \xrightarrow{d} \frac{\sigma_{\|\Phi_s\|,\|\Phi_{XY}\|,\|\Phi_{YX}\|}}{(1+\|\Phi_s\|)^2-\|\Phi_{XY}\|\|\Phi_{YX}\|}\left[\|\Phi_{YX}\|W^{(1)}+(1+\|\Phi_s\|)W^{(2)}\right].$$

- Macroscopic correlation formula

$$\boxed{C(X,Y) = \frac{(\|\Phi_{XY}\| + \|\Phi_{YX}\|)(1+\|\Phi_s\|)}{\|\Phi_{XY}\|\|\Phi_{YX}\| + (1+\|\Phi_s\|)^2}}$$

The mean Epps effect the dimension 2 model

- Daily "correlation" estimator : $C_{\Delta t} = \tilde{C}_{\Delta t}/\tilde{C}_0$

$$\tilde{C}_{\Delta t} = \sum_{n=0}^{1day/\Delta t} (X((n+1)\Delta t) - X(n\Delta t))(Y((n+1)\Delta t) - Y(n\Delta t))$$

- the mean Epps effect

$$MEpps_{\Delta t} = \frac{E(X(\Delta t)Y(\Delta t))}{\sqrt{E(X(\Delta t)^2)E(Y(\Delta t)^2)}} \qquad (1)$$

with initial condition : $X(0) = 0$

- **closed-form formula for the mean Epps effect** when $\Phi_{X,X}$, $\Phi_{Y,Y}$, $\Phi_{X,Y}$, $\Phi_{Y,X}$ are of the form $\alpha e^{-\beta x}$
$\rightarrow$ through the explicit computation of the **Bartlett spectrum (1963)**.

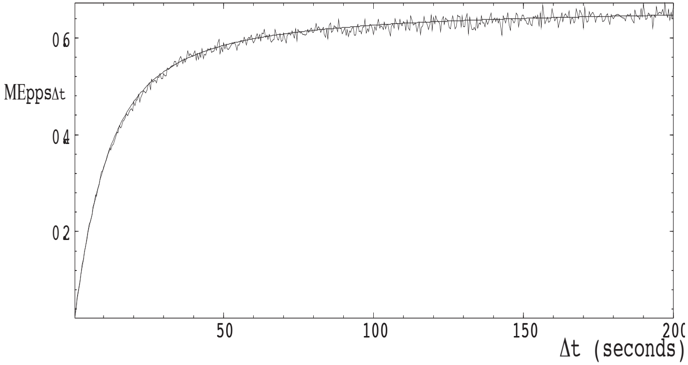
Closed form for the mean Epps effect in dimension 2

Closed form formula for the mean Epps effect in dimension 2

- General case $\rightarrow$ too many parameters...
- Reducing the parameters
  - μ_X, μ_Y
  - $\alpha_{same} = \alpha_{X,X} = \alpha_{X,Y}$,
  - $\alpha_{cross} = \alpha_{X,Y} = \alpha_{Y,X}$,
  - $\beta = \beta_{X,Y} = \beta_{Y,X} = \beta_{X,X} = \beta_{Y,Y}$

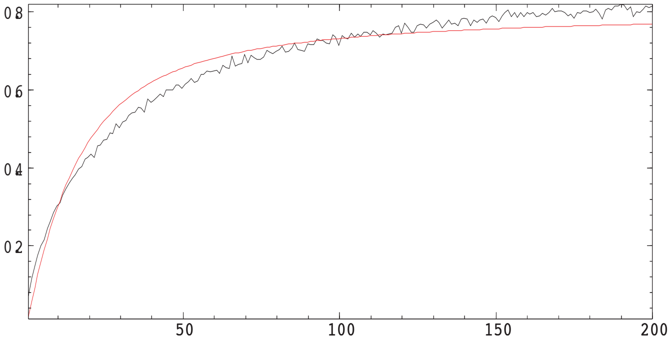
Mean Epps effect on simulated data

Mean Epps effect on 50 hours simulated data



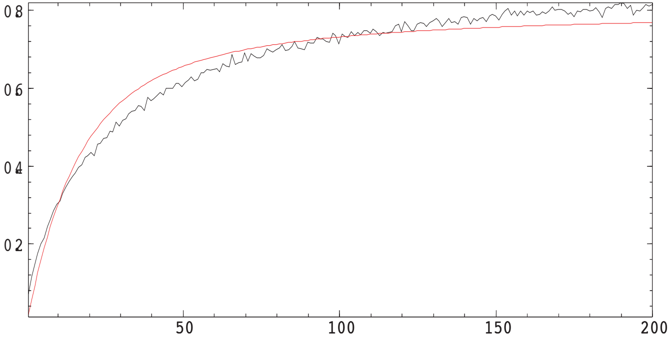
Mean Epps effect on real data

- Bund 10Y / Bobl 5Y : 41 days, 9-11 AM - Last Traded



Mean Epps effect on real data

- Bund 10Y / Bobl 5Y : 41 days, 9-11 AM - Last Traded



with $\alpha_{Bobl} = \alpha_{Bund}$ no way to perform good fits for the two individual signature plots **and** the Epps effect at the same time.

Instabilities

The 2d model accounts for 2d microstructure noise but it remains a "first-brick" model for tick-by-tick time-series themselves :

- "Naive" model
  - Arbitrary parametric shape $\phi(t) = \alpha e^{-\beta t}$
  - Fully symmetric constant parameters
    $\rightarrow$ clearly not the case at all in the real life!
- tick-by-tick time-series : Arbitrary projection of a complex phenomenon (orderbook dynamics)

Moreover

- "filtering" is even more arbitrary than in the 1d case
  $\rightarrow$ No reason to use the same filtering rule for each asset

More scaling limits! (in dimension 1)

- Bachelier (additive) limit with arbitrary $\mu_+ \neq \mu_-$, $\Phi_+ \neq \Phi_-$
- Black-Scholes (multiplicative) limit
- How to – simply– obtain a continuous diffusion process as macroscopic limit
- Toward macroscopic stochastic volatility diffusion via a Nelson type argument

Microstructure noise: the latent price approach

- The fair/efficient price (S_t) is a diffusion of the form

$$dS_t = b_t dt + \sigma_t dB_t, \quad t \in [0,1]$$

  but cannot be observed.
- What we can observe $= \big(Y_1, \dots, Y_{\Delta^{-1}}\big)$, where

$$\boxed{\mathrm{Law}\big(Y_k \,\big|\, (S_t)_t\big) = K_\Delta(S_{k\Delta}, dx)}$$

- $K_\Delta(s, dx)$ Markov kernel.
- Conditional on the latent $(S_t)_t$, the Y_i are independent.
- Popular model: additive microstructure (white) noise

$$\boxed{Y_i = S_{i\Delta} + \xi_{i,\Delta},\ i = 1, \dots, \Delta^{-1},\ \mathbb{E}[\xi_{i,\Delta}] = 0}$$

Some references

- Latent price approach
  - In statistics: Gloter and Jacod (2001), Munk and Schmiedt-Hieber (2009), Reiß (2010)
  - In financial econometrics: Ait-Sahalia, Mykland and Zhang (2003 to 2006).
  - And many more... Podolkii, Vetter, Jacod, Mykland, Zhang, Bandi, Russell, Diebold, Strasser, Barndorff-Nielsen, Hansen, Lund, Shepard,
- Other approaches for modelling microstructure noise:
  - Engle Russell (2002), Robert and Rosenbaum (2009)
  - Econophysics literature Order book oriented modelling...

Comparison: additive microstructure noise vs. Hawkes

- Latent price approach. One observes

$$Y_{i,\Delta} = S_{i\Delta} + \xi_i^{\Delta},\;\; \mathbb{E}[\xi_{i,\Delta}] = 0,\; \mathbb{E}[\xi_{i,\Delta}^2] = \rho^2 > 0,$$

  with $dS_t = \sigma(t)dB_t$.
- Take $\sigma(t) \equiv \sigma$ for simplicity...
- This is not a microscopic model in our terminology!
- Indeed: the observation horizon $[0, 1]$ is fixed irrespectively of the sampling observation frequency Δ^{-1}.

Additive microstructure noise

Indeed

- As $\Delta \to 0$, one equivalently observes (in a distributional sense) (Reiß, 2010)

$$Y(dt) = X_t + \rho \Delta^{1/2} \dot{B}(dt), \quad t \in [0, 1]$$

- Hence infinite information over fixed time as $\Delta \to 0$.
- In our setting, we can observe continuously $(X(t), t \in [0, 1])$. This observation contains finite information only about μ and Φ. (Equivalently: one cannot recover μ nor Φ from $(X(t), t \in [0, 1])$.)

So?

- How to reconcile both approaches and compare them?
- Recast the additive microstructure noise model into microscopic time, over the horizon $[0, \delta^{-1}]$ with $\delta \approx 0$.
- In this setting, we have data at (microscopic) times

$$0, \Delta, 2\Delta, \ldots, n\Delta = \delta^{-1}$$

- We can compare now additive microstructure noise data $\{Y_{i\Delta}\}$ and Hawkes data $\{X^{(\delta)}(i\Delta)\}$, for $i = 1, \ldots, n$ (same sample size).

Mean "signature plot"

- Recall the time-space renormalization

$$\boxed{X^{(\delta)}(t) := \sqrt{\delta} X(\delta^{-1} t), \quad t \in [0,1]}$$

- Realized volatility

$$V_\Delta \{X^{(\delta)}\} := \sum_{i=1}^{\Delta^{-1}} \left(X^{(\delta)}(i\Delta) - X^{(\delta)}((i-1)\Delta) \right)^2$$

$$\approx \frac{1}{\Delta \delta^{-1}} \mathbb{E}\left[\left(X(\Delta \delta^{-1}) - X(0) \right)^2 \right]$$

- Mean signature plot

$$\boxed{\mathcal{V}(t) := \frac{1}{t} \mathbb{E}\left[\left(X(t) - X(0) \right)^2 \right]}$$

- Interpretation

$$\mathcal{V}(\Delta \delta^{-1}) \approx V_\Delta \{X^{(\delta)}\}.$$

Comparing signature plots

- Transform the additive microstructure noise model $Y_{i,\Delta} = \sigma B_{i\Delta} + \rho\,\xi_{i,\Delta}$ into

$$Y^{(\delta)}_{i\Delta} = \sqrt{\delta}\sigma B_{i\Delta\delta^{-1}} + \rho\sqrt{\delta}\xi_{i,\Delta}.$$

- Historic volatility approximation $V_\Delta\{Y^{(\delta)}\}$

$$\boxed{\sum_{i=1}^{\Delta^{-1}} \left(Y^{(\delta)}_{i\Delta} - Y^{(\delta)}_{(i-1)\Delta}\right)^2 \approx \sigma^2 + 2\rho^2\delta\Delta^{-1} =: \mathcal{V}_{\text{add micro}}(\delta^{-1}\Delta)}$$

Conclusion

- Additive microstructure signature plot ($t = \Delta\delta^{-1}$):

$$\boxed{\mathcal{V}_{\text{add micro}}(t) = \sigma^2 + \frac{2\rho^2}{t}}$$

- Hawkes signature plot:

$$\boxed{\mathcal{V}_{\text{Hawkes}}(t) = \sigma^2 + \sigma^2\left\{(1+\|\Phi\|)^2 - 1\right\}G(t)}$$

with $G(t) = \frac{1-e^{-(\alpha+\beta)t}}{(\alpha+\beta)t} \sim (\alpha+\beta)^{-1}/t$ (large t) and with the identification

$$\boxed{\sigma^2 = \frac{2\mu}{1-\|\Phi\|}\frac{1}{(1+\|\Phi\|)^2}}$$

- $\mathcal{V}_{\text{add micro}}(t)$ cannot be consistent with empirical data in the regime $t \approx 0$ unless $\rho = \rho(t)$.