Price modelling with microstructure via point processes.

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Price representation

- Price processes behave differently at different scales:
  - Coarse scales (daily data): **diffusions**
  - Fine scales (tick data): **marked point processes**
- Breakdown of the diffusive behaviour in small scales
  - In dimension 1: **microstructure noise** (variance instability).
  - In dimension 2: **Epps effect** (covariance instability).
Itô semimartingale price model consensus (indifferently mid-price/traded-price)

\[ S_t = \text{drift}_t + \int_0^t \sigma_s dB_s + (\text{jump process}_t) \]

If \( S_t \) is observed over \([0, t]\) at times \(0, \Delta, 2\Delta, \ldots\), convergence of the realized volatility

\[
V_{\Delta}\{S\}_t := \sum_{i\Delta \leq t} (S_{i\Delta} - S_{(i-1)\Delta})^2 \xrightarrow{\mathbb{P}} \int_0^t \sigma_s^2 ds
\]

as \( \Delta \to 0 \) with accuracy \( \sqrt{\Delta} \).

This suggests to pick \( \Delta \) as small as possible... but
Figure: $\Delta \rightsquigarrow V_\Delta$ for FGBL (43 days, 9-11 AM) on Last Traded Ask.
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Construction of price models based on Hawkes processes

Scaling limit in dimension 1 and 2

More scaling limits

Comparing with the additive microstructure noise approach

Figure: FGBL, 06 Feb 2007, 09:00–10:00 (UTC) 1 data per second.
In dimension 2: Epps effect

- In the same Itô semimartingale setting, we have convergence of the quadratic covariation

\[
CV_\Delta \{S^{(1)}, S^{(2)}\}_t := \sum_{i\Delta \leq t} (S_{i\Delta}^{(1)} - S_{(i-1)\Delta}^{(1)})(S_{i\Delta}^{(2)} - S_{(i-1)\Delta}^{(2)})
\]

\[
\mathbb{P} \rightarrow \langle S^{(1)}, S^{(2)} \rangle_t
\]

- Same prescription as for the realized volatility: pick \(\Delta\) as small as possible... but
Epps effect

Figure: \( \Delta \sim CV_\Delta \{ S^{(1)}, S^{(2)} \} \) (normalized) with \( S^{(1)} = \text{FGBL}, \ S^{(2)} = \text{FGBM} \), 40 days, 9-11AM.
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**Figure: FGBL/FGBM**
We look for a “simple” multivariate price model with the following properties:

- Be defined in continuous time with discrete values in a microscopic scale.
- Incorporate microstructure noise and the Epps effect with “few” parameters.
- Diffuse in a macroscopic scale.
Point process approach

1. Price process = marked point process.
   - Marks: jumps up/down by 1 tick,
   - Jump times: time stamps of price changes.

2. The price process is the result (sum) of a “upward change or price” and a “downward change of price”. Coupling random intensities (Hawkes process) → microstructure noise.

3. The price of two assets is obtained by coupling further (Hawkes process) the respective intensities of the “upward change of price” and “downward change of price” processes → dependence structure.
Compound Poisson process

- Let $N^{\mu+}_t$ and $N^{\mu-}_t$ be two independent Poisson processes with intensity $\mu_{\pm}$.

- Then

$$M^{\mu+,:\mu-}_t := N^{\mu+}_t - N^{\mu-}_t = \sum_{n=0}^{\infty} \varepsilon_n 1_{T_n \leq t}$$

is a **compound Poisson process** with

- $(T_n - T_{n-1})_{n \geq 1}$ i.i.d. exponential with parameters $\mu_+ + \mu_-$. 

- Law of the jumps:

$$P[\varepsilon_n = +1] = 1 - P[\varepsilon_n = -1] = \frac{\mu_+}{\mu_+ + \mu_-}.$$
Scaling limit

- **Macroscopic limit** take $\mu_+ = \mu_- = \mu$ with $\delta \to 0$ (continuous limit in space).

\[
M_t^{\mu\delta^{-1}} = \sum_{0}^{N_t^{\mu}} + N_t^{\mu} - 1
\]

with $\varepsilon_i$ i.i.d. standard Bernoulli $\pm 1$.

- **Spatial renormalization**

\[
\sqrt{\delta}M_t^{\mu\delta^{-1}} \approx \sqrt{\delta} \sum_{1}^{\delta^{-1}2\mu t}\varepsilon_i \approx B_{2\mu t},
\]

where $(B_t)$ is a standard Brownian motion. By scaling

\[
B_{2\mu t} \overset{(d)}{=} \sqrt{2\mu}B_t
\]

and $\sqrt{2\mu}$ is the macroscopic volatility $1/2$. 
Hawkes processes: the 1 dimensional case

- Start with a **counting** process $N_t$ constructed via its stochastic intensity

$$
\lambda(t) = \mu + \int_0^t \phi(t - s) dN_s
$$

where $\phi(\cdot)$ is a **coupling** function. Standard $\phi(x) = \alpha e^{-\beta x}$. Interpretation of the parameters:

- $\mu$: exogenous intensity
- $\alpha$: (rather $\alpha/\beta$): local self-exciting intensity.
- $\beta$: temporal delay.

(One has $\int_0^t \phi(t - s) dN_s = \sum_{T_n < t} \phi(t - T_n)$.) **Essential constraint**: $\int_0^{+\infty} \phi < 1$. 
Remark on parameter inference

- The likelihood is explicit, given a continuous trajectory over $[0, T]$. If $\vartheta = (\mu, \alpha, \beta)$

$$\log \ell(\vartheta) = \int_0^T \log(\lambda_\vartheta(s))dN_s - \int_0^T \lambda_\vartheta(s)ds.$$  

- **But:** maximization of the log-likelihood is computationally intensive.
Price model in dimension 1

Let $S_t = N_t^+ - N_t^-$, with $N_t^\pm$ Hawkes processes with respective random intensities $\lambda_t^\pm$ given by

$$\lambda^\pm(t) := \mu^\pm + \alpha \int_{[0,t]} e^{-\beta(t-s)} dN_s^\pm$$

- $\mu^\pm$: exogeneous intensity.
- $\alpha$ et $\beta$: mutually exciting intensities generating a “mean-reverting effect” for $S_t$.
- $\alpha e^{-\beta x} \xrightarrow{\text{as}} \Phi(x)$ with $\|\Phi\|_{L^1} < 1$ in the sequel.
Price in dimension 2

1. Start from two processes $X$ and $Y$ constructed as before.
2. Introduce a supplementary coupling on the intensities of the two processes and create a dependence structure $\text{Upward}_X - \text{Upward}_Y$ and $\text{Downward}_X - \text{Downward}_Y$.
3. (We ignore further possible coupling $\text{Upward}_X - \text{Downward}_Y$ and $\text{Downward}_X - \text{Upward}_Y$ between $X$ and $Y$.)
Representation of $X$ and $Y$

Set

$$X(t) = N^+_X(t) - N^-_X(t) \text{ and } Y(t) = N^+_Y(t) - N^-_Y(t)$$

with

$$\lambda^\pm_X(t) = \mu^\pm_X + \int_{[0,t]} \Phi_{X,X}(t-s) dN^\pm_X(s) + \int_{[0,t]} \Phi_{X,Y}(t-s) dN^\pm_Y(s)$$

and

$$\lambda^\pm_Y(t) = \mu^\pm_Y + \int_{[0,t]} \Phi_{Y,X}(t-s) dN^\pm_X(s) + \int_{[0,t]} \Phi_{Y,Y}(t-s) dN^\pm_Y(s)$$
Simulation over 1000 seconds

Figure: Sample simulation in dimension 2
Simulation over 1000 secondes

Figure: Another sample...
Scaling limits

- **First step:**
  1. Closed-form formulas for the mean “signature plot” when \( \Phi(x) = \alpha e^{-\beta x} \) (through the explicit computation of the Bartlett spectrum, case with stationary increments) in dimension 1 and 2.
  2. Statistical fits and discussion of further data filtering.

- **Second step:** diffusive limit (after spatial renormalization) for arbitrary \( \Phi \) in dimension 1 (and arbitrary \( \Phi_{X,Y}, \Phi_{Y,X} \) and \( \Phi_{X,X} = \Phi_{Y,Y} \) in dimension 2).

- More scaling limits...

- Comparison with other models
Mean “signature plot” and scaling limits

- Time-space renormalization
  \[ X^{(\delta)}(t) := \sqrt{\delta} X(\delta^{-1}t), \quad t \in [0, 1] \]

- Realized volatility
  \[ V_\Delta \{ X^{(\delta)} \} := \sum_{i=1}^{\Delta^{-1}} \left( X^{(\delta)}(i\Delta) - X^{(\delta)}((i - 1)\Delta) \right)^2 \]
  \[ \approx \frac{1}{\Delta \delta^{-1}} \mathbb{E} \left[ \left( X(\Delta \delta^{-1}) - X(0) \right)^2 \right] \]

- Mean signature plot
  \[ \mathcal{V}(t) := \frac{1}{t} \mathbb{E} \left[ (X(t) - X(0))^2 \right] \]

- Interpretation
  \[ \mathcal{V}(\Delta \delta^{-1}) \approx V_\Delta \{ X^{(\delta)} \}. \]
Mean “Signature plot”

If \( \Phi_{X,X}(x) = \Phi_{Y,Y}(x) = \alpha e^{-\beta x} \), \( \Phi_{X,Y} = \Phi_{Y,X} = 0 \) and \( \mu^+ = \mu^- = \mu \) we have (via Bartlett spectrum) for \( X \) (or \( Y \))

\[
\mathcal{N}(t) = \frac{2\mu}{1 - \alpha/\beta} \left[ \frac{1}{(1 + \alpha/\beta)^2} + \right.
\]

\[
+ \left( 1 - \frac{1}{(1 + \alpha/\beta)^2} \right) \frac{1 - \exp \left( - \left( \alpha + \beta \right) t \right)}{(\alpha + \beta) t} \]
\]
Scaling limit in dimension 1, $\mu^{\pm} = \mu$

- **Step 1**: price decomposition introducing a martingale

\[ X^{(\delta)}(t) = \delta^{1/2} \left( N_{\delta^{-1}t}^{+} - N_{\delta^{-1}t}^{-} \right) \]
\[ = M_{t}^{(\delta)} + B^{(\delta)}(t), \]

with

\[ M_{t}^{(\delta)} = \delta^{1/2} \left( N_{\delta^{-1}t}^{+} - N_{\delta^{-1}t}^{-} \right) - B_{t}^{(\delta)}, \text{ martingale} \]

and

\[ B_{t}^{(\delta)} = \delta^{1/2} \int_{0}^{\delta^{-1}t} \left( \lambda^{+}(s) - \lambda^{-}(s) \right) ds, \text{ predictable} \]
Scaling limit in dimension 1 (cont.)

- **Step 2: Convergence of the compensator**

\[ B^{(\delta)}(t) = \delta^{1/2} \int_0^{\delta^{-1}t} ds \int_0^s \Phi(s - u) d(N_u^- - N_u^+) \]

\[ = - \int_{[0,t)} dX^{(\delta)}(u) \int_0^{t-u} \delta^{-1} \Phi(\delta^{-1}s) ds \]

\[ = - \int_{[0,t)} X^{(\delta)}(u) \left[ \Phi(\delta(t - u)) d \left( \delta^{-1} \int_s^t \Phi(\delta^{-1}s) ds \right) \right] du \]

\[ \approx - \| \Phi \|_{L^1} X^{(0)}(0) + M_t^{(0)} \]

- **In the limit**
Scaling limit in dimension 1 (cont.)

- **Step 3: Convergence of the martingale part**

\[
\langle M^{(\delta)} \rangle_t = \delta \int_0^{\delta^{-1}t} (\lambda^+(s) + \lambda^-(s)) \, ds
\]

\[
= 2\mu t + \delta \int_0^{\delta^{-1}t} ds \int_0^s \phi(s-u) d(N^+(u) + N^-(u))
\]

\[
= 2\mu t + \int_0^t [M^{(\delta)}]_u \phi_\delta(t-u) \, du
\]

\[
\approx 2\mu t + \int_0^t \langle M^{(\delta)} \rangle_u \phi_\delta(t-u) \, du
\]

\[
\approx 2\mu t + \|\Phi\|_{L^1} \langle M^{(\delta)} \rangle_t
\]

- **Conclusion**

\[
\langle M^{(\delta)} \rangle_t \xrightarrow{\mathbb{P}} \frac{2\mu}{1 - \|\Phi\|_{L^1}} t
\]
Scaling limit in dimension 1 (cont.)

- We obtain a)

\[ M^{(\delta)}(t) \overset{d}{\to} \sqrt{\frac{2\mu}{1 - \|\Phi\|_{L^1}}} W_t, \]

where \( W \) is a Wiener process

- and b) the representation

\[ X^{(0)}(t) = -\|\Phi\|_{L^1} X^{(0)}(t) + M_t^{(0)} \]

- a) + b) yield the final result:

\[ X^{(\delta)}(t) \overset{d}{\to} \frac{1}{1 + \|\Phi\|_{L^1}} \sqrt{\frac{2\mu}{1 - \|\Phi\|_{L^1}}} W_t \]
Discussion

- **Microscopic variance**
  \[\mathbb{E}[\lambda^+ + \lambda^-] = \frac{2\mu}{1 - \|\Phi\|_1}\]

- **Macroscopic variance**
  \[\sigma^2 = \frac{2\mu}{1 - \|\Phi\|_1} \cdot \frac{1}{(1 + \|\Phi\|_1)^2}\]

- **However the influence of \(\Phi\) does not disappear at large scale**

- **This influence can be quantified by looking at the function**
  \[\|\phi\|_1 = x \in [0, 1) \leadsto f(x) = \frac{1}{1 - x} \cdot \frac{1}{(1 + x)^2}\]
  \[f(x) \leq f(0) \implies x \approx 0.61\text{ and } f\text{ minimum at } x = \frac{1}{3} \].
Influence of $||\Phi||_1$ on the macroscopic variance

Histogram of $||\Phi||_1$ fitted (mean square) on the signature plot of
Bund 10Y 140 days - 9:11am - 12am: 2pm - 2:4pm

Mean $\approx 0.34$
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Construction of price models based on Hawkes processes.

Scaling limit in dimension 1 and 2

More scaling limits

Comparing with the additive microstructure noise approach

Mean signature plot on simulated data

Signature plot on 11 hours simulated data
Bund 10Y : 21 days, 9-11 AM - Last Traded Ask (7000 points)
Bund 10Y: 21 days, 9-11 AM - Last Traded Ask

Mean square regression fit

⇒ Fairly good modelling of the 1d microstructure noise.
Bund 10Y : 21 days, 9-11 AM - Last Traded Ask

Maximum likelihood fit
Bund 10Y : 26 days, 9-11 AM - Last Traded Price (29000 points)
10Y Bund data: 26 days, 9-11 AM - Last Traded Price

Mean square regression fit

Maximum likelihood fit

Mean signature plot on real data - MLE
Instabilities of the MLE fit

The 1d model is a very good model for 1d microstructure noise but it remains a ”first-brick” model for tick-by-tick time-series themselves :

- ”Naive” model
  - Arbitrary parametric shape $\phi(t) = \alpha e^{-\beta t}$
  - Fully symmetric constant parameters :
    $\mu^+ = \mu^-$, $\alpha^+ = \alpha^-$, $\beta^+ = \beta^-$
  - No volume in the model !

- tick-by-tick time-series : Arbitrary projection of a very complex phenomenon (orderbook dynamics)
Mean signature plot on real data

- 10Y Bund data: 26 days, 9-11 AM - Last Traded Price
  Volume > 1 - 11000 points

\[
V_{\Delta t}(\text{ticks}) \quad \Delta t (\text{seconds})
\]
Mean signature plot on real data

- 10Y Bund data: 21 days, 9-11 AM - Last Traded Price
  Volume > 1 - 8600 points
Mean signature plot on real data

- 10Y Bund data: 41 days, 9-11 AM - Last Traded Price
  Volume > 1 - 20000 points
For simplicity \( \mu_+ = \mu_- \), \( \Phi_X, X = \Phi_Y, Y = \Phi_{\text{self}} \).

In the same way, \( X(t) = M_X(t) + B_X(t) \) with \( B_X(t) \) given by

\[
\int_0^t \left[ \lambda_X^+(s) - \lambda_X^-(s) \right] ds
\]

\[
= \int_0^t \left[ \int_0^s (\Phi_{\text{self}}(s - u)dN_X^-(u) + \Phi_{XY}(s - u)dN_Y^+(u))
- \int_0^s (\Phi_{\text{self}}(s - u)dN_X^+(u) + \Phi_{XY}(s - u)dN_Y^-(u)) \right] ds
\]

After scaling, the same kind of approximation as in the 1d case

\[
X^{(\delta)}(t) \approx -\|\Phi_{\text{self}}\|_{L^1} X^{(\delta)}(t) + \|\Phi_{XY}\|_{L^1} Y^{(\delta)}(t) + M_X^{(\delta)}(t).
\]
Scaling limit in dimension 2 (cont.)

- By symmetry, we obtain in the limit

\[ X^{(0)}(t) = M^{(0)}_X(t) - \| \Phi_{\text{self}} \|_{L^1} X^{(0)}(t) + \| \Phi_{XY} \|_{L^1} Y^{(0)}(t) \]

\[ Y^{(0)}(t) = M^{(0)}_Y(t) - \| \Phi_{\text{self}} \|_{L^1} Y^{(0)}(t) + \| \Phi_{YX} \|_{L^1} X^{(0)}(t) \]

- Convergence of the martingale part

\[ (M^{(\delta)}_X, M^{(\delta)}_Y) \overset{d}{\to} \sigma \| \Phi_s \|, \| \Phi_{XY} \|, \| \Phi_{YX} \| (W^{(1)}, W^{(2)}) \]

where \( W^{(1)} \) and \( W^{(2)} \) are two independent Brownian motions. (We need \( t \sim t\Phi_{XY}(t) \) and \( t\Phi_{YX}(t) \) in \( L^1 \).)
Scaling limit in dimension 2 (cont.)

- We have in the limit $\delta \to 0$

$$X(\delta) \xrightarrow{d} \frac{\sigma \|\Phi_s\|,\|\Phi_{XY}\|,\|\Phi_{YX}\|}{(1+\|\Phi_s\|)^2 - \|\Phi_{XY}\|\|\Phi_{YX}\|} \left[ (1+\|\Phi_s\|)W^{(1)} + \|\Phi_{XY}\|W^{(2)} \right].$$

and (by symmetry)

$$Y(\delta) \xrightarrow{d} \frac{\sigma \|\Phi_s\|,\|\Phi_{XY}\|,\|\Phi_{YX}\|}{(1+\|\Phi_s\|)^2 - \|\Phi_{XY}\|\|\Phi_{YX}\|} \left[ \|\Phi_{YX}\|W^{(1)} + (1+\|\Phi_s\|)W^{(2)} \right].$$

- Macroscopic correlation formula

$$C(X, Y) = \frac{(\|\Phi_{XY}\| + \|\Phi_{YX}\|)(1 + \|\Phi_s\|)}{\|\Phi_{XY}\|\|\Phi_{YX}\| + (1 + \|\Phi_s\|)^2}$$
The mean Epps effect the dimension 2 model

- Daily "correlation" estimator: \( C_{\Delta t} = \tilde{C}_{\Delta t} / \tilde{C}_0 \)

\[
\tilde{C}_{\Delta t} = \frac{1 \text{day}/\Delta t}{\sum_{n=0}^{\infty} (X((n+1)\Delta t) - X(n\Delta t))(Y((n+1)\Delta t) - Y(n\Delta t))}
\]

- The mean Epps effect

\[
MEpps_{\Delta t} = \frac{E(X(\Delta t)Y(\Delta t))}{\sqrt{E(X(\Delta t)^2)E(Y(\Delta t)^2)}} \tag{1}
\]

with initial condition: \( X(0) = 0 \)

- Closed-form formula for the mean Epps effect when \( \Phi_{X,X}, \Phi_{Y,Y}, \Phi_{X,Y}, \Phi_{Y,X} \) are of the form \( \alpha e^{-\beta x} \)

→ through the explicit computation of the Bartlett spectrum (1963).
Closed form for the mean Epps effect in dimension 2

- General case → too many parameters...
- Reducing the parameters
  - $\mu_X, \mu_Y$
  - $\alpha_{same} = \alpha_{X,X} = \alpha_{X,Y}$,
  - $\alpha_{cross} = \alpha_{X,Y} = \alpha_{Y,X}$,
  - $\beta = \beta_{X,Y} = \beta_{Y,X} = \beta_{X,X} = \beta_{Y,Y}$
Mean Epps effect on 50 hours simulated data

Mean Epps effect on simulated data

$\Delta t$ (seconds)
Mean Epps effect on real data

- Bund 10Y / Bobl 5Y: 41 days, 9-11 AM - Last Traded

![Graph showing the mean Epps effect on real data with Bund 10Y and Bobl 5Y.](image-url)
Mean Epps effect on real data

- Bund 10Y / Bobl 5Y: 41 days, 9-11 AM - Last Traded

 dengan

\[ \alpha_{Bobl} = \alpha_{Bund} \]  no way to perform good fits for the two individual signature plots and the Epps effect at the same time.
The 2d model accounts for 2d microstructure noise but it remains a "first-brick" model for tick-by-tick time-series themselves:

- "Naive" model
  - Arbitrary parametric shape $\phi(t) = \alpha e^{-\beta t}$
  - Fully symmetric constant parameters → clearly not the case at all in the real life!

- tick-by-tick time-series: Arbitrary projection of a complex phenomenon (orderbook dynamics)

Moreover

- "filtering" is even more arbitrary than in the 1d case
  → No reason to use the same filtering rule for each asset
More scaling limits! (in dimension 1)

- **Bachelier** (additive) limit with arbitrary $\mu_+ \neq \mu_-$, $\Phi_+ \neq \Phi_-$
- **Black-Scholes** (multiplicative) limit
- How to – simply– obtain a continuous diffusion process as macroscopic limit
- Toward macroscopic **stochastic volatility diffusion** via a Nelson type argument
The fair/efficient price \( (S_t) \) is a diffusion of the form

\[
dS_t = b_t \, dt + \sigma_t \, dB_t, \quad t \in [0, 1]
\]

but cannot be observed.

What we can observe \( = (Y_1, \ldots, Y_{\Delta-1}) \), where

\[
\text{Law}(Y_k \mid (S_t)_t) = K_{\Delta}(S_{k\Delta}, dx)
\]

\( K_{\Delta}(s, dx) \) Markov kernel.

Conditional on the latent \( (S_t)_t \), the \( Y_i \) are independent.

Popular model: additive microstructure (white) noise

\[
Y_i = S_{i\Delta} + \xi_{i,\Delta}, \quad i = 1, \ldots, \Delta^{-1}, \quad \mathbb{E}[\xi_{i,\Delta}] = 0
\]
Some references

- Latent price approach
  - In statistics: Gloter and Jacod (2001), Munk and Schmiedt-Hieber (2009), Reiβ (2010)
  - In financial econometrics: Ait-Sahalia, Mykland and Zhang (2003 to 2006).
  - And many more... Podolkii, Vetter, Jacod, Mykland, Zhang, Bandi, Russell, Diebold, Strasser, Barndorff-Nielsen, Hansen, Lund, Shepard,

- Other approaches for modelling microstructure noise:
  - Econophysics literature Order book oriented modelling...
Comparison: additive microstructure noise vs. Hawkes

- **Latent price approach.** One observes

\[ Y_{i,\Delta} = S_{i\Delta} + \xi_i^\Delta, \quad \mathbb{E}[\xi_{i,\Delta}] = 0, \quad \mathbb{E}[\xi_{i,\Delta}^2] = \rho^2 > 0, \]

with \( dS_t = \sigma(t)dB_t \).

- Take \( \sigma(t) \equiv \sigma \) for simplicity...

- **This is not a microscopic model** in our terminology!

- Indeed: the observation horizon \([0, 1]\) is fixed **irrespectively** of the sampling observation frequency \( \Delta^{-1} \).
Indeed

- As $\Delta \to 0$, one equivalently observes (in a distributional sense) (Reiß, 2010)

$$Y(dt) = X_t + \rho \Delta^{1/2} \dot{B}(dt), \; t \in [0, 1]$$

- Hence infinite information over fixed time as $\Delta \to 0$.

- In our setting, we can observe continuously $(X(t), t \in [0, 1])$. This observation contains finite information only about $\mu$ and $\Phi$. (Equivalently: one cannot recover $\mu$ nor $\Phi$ from $(X(t), t \in [0, 1])$.)
So?

- How to reconcile both approaches and compare them?
- Recast the additive microstructure noise model into microscopic time, over the horizon \([0, \delta^{-1}]\) with \(\delta \approx 0\).
- In this setting, we have data at (microscopic) times

\[
0, \Delta, 2\Delta, \ldots, n\Delta = \delta^{-1}
\]

- We can compare now additive microstructure noise data \(\{Y_{i\Delta}\}\) and Hawkes data \(\{X^{(\delta)}(i\Delta)\}\), for \(i = 1, \ldots, n\) (same sample size).
Mean “signature plot”

- Recall the time-space renormalization
  \[ X^{(\delta)}(t) := \sqrt{\delta}X(\delta^{-1}t), \quad t \in [0, 1] \]

- Realized volatility
  \[
  V_\Delta \{X^{(\delta)}\} := \sum_{i=1}^{\Delta^{-1}} \left( X^{(\delta)}(i\Delta) - X^{(\delta)}((i-1)\Delta) \right)^2
  \approx \frac{1}{\Delta \delta^{-1}} \mathbb{E}[(X(\Delta \delta^{-1}) - X(0))^2]
  \]

- Mean signature plot
  \[
  \mathcal{V}(t) := \frac{1}{t} \mathbb{E}[(X(t) - X(0))^2]
  \]

- Interpretation
  \[
  \mathcal{V}(\Delta \delta^{-1}) \approx V_\Delta \{X^{(\delta)}\}.
  \]
Comparing signature plots

- Transform the **additive microstructure noise model**
  \[ Y_{i,\Delta} = \sigma B_{i\Delta} + \rho \xi_{i,\Delta} \]
  into
  \[ Y_{i\Delta}^{(\delta)} = \sqrt{\delta} \sigma B_{i\Delta \delta^{-1}} + \rho \sqrt{\delta} \xi_{i,\Delta}. \]

- **Historic volatility approximation** \( V_\Delta \{ Y^{(\delta)} \} \)

\[
\Delta^{-1} \sum_{i=1}^{\Delta^{-1}} (Y_{i\Delta}^{(\delta)} - Y_{(i-1)\Delta}^{(\delta)})^2 \approx \sigma^2 + 2 \rho^2 \delta \Delta^{-1} =: V_{\text{add micro}}^{(\delta^{-1}\Delta)}
\]
Conclusion

- **Additive microstructure** signature plot \( (t = \Delta \delta^{-1}) \):
  \[
  \mathcal{V}_{\text{add micro}}(t) = \sigma^2 + \frac{2\rho^2}{t}
  \]

- **Hawkes** signature plot:
  \[
  \mathcal{V}_{\text{Hawkes}}(t) = \sigma^2 + \sigma^2 \left\{ (1 + \|\Phi\|)^2 - 1 \right\} G(t)
  \]
  with \( G(t) = \frac{1-e^{-(\alpha+\beta)t}}{(\alpha+\beta)t} \sim (\alpha + \beta)^{-1}/t \) (large \( t \)) and with the identification
  \[
  \sigma^2 = \frac{2\mu}{1 - \|\Phi\|} \frac{1}{(1 + \|\Phi\|)^2}
  \]

- \( \mathcal{V}_{\text{add micro}}(t) \) cannot be consistent with empirical data in the regime \( t \approx 0 \) unless \( \rho = \rho(t) \).