On Quasiconvex Conditional Maps
Duality Results and Applications to Finance

Marco Frittelli and Marco Maggis

Università degli Studi di Milano

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Objectives

Recall that $\rho : L_{\mathcal{F}} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ is quasiconvex (QCO) if

$$\rho(\lambda X + (1-\lambda)Y) \leq \max\{\rho(X), \rho(Y)\}, \ \lambda \in [0,1]$$

or equivalently: ρ is (QCO) if all the lower level sets

$$\{X \in L_{\mathcal{F}} \mid \rho(X) \leq c\} \quad \forall\, c \in \mathbb{R}$$

are convex

- Motivations for the study of quasiconvex maps in Mathematical Finance
- Examples of applications
- New results in the dynamic setting.

Notations

- $(\Omega, \mathcal{F}, P)$ is a probability space.
- $L^0 = L^0(\Omega, \mathcal{F}, P)$ is the space of all P.a.s. finite random variables.
- L^∞ is the subspace of all essentially bounded random variables.
- $L_{\mathcal{F}}$ is a TVS of $\mathcal{F}$-measurable random variables.
- We suppose that
$$L^\infty \subseteq L_{\mathcal{F}} \subseteq L^0.$$
- $\{\mathcal{F}_t\}_{t \geq 0}$ will denote a right continuous filtration.

Monetary Risk Measure

Definition

A map $\rho : L_{\mathcal{F}} \to \mathbb{R}$ is called a *monetary risk measure* on $L_{\mathcal{F}}$ if it has the following properties:

(CA) Cash additivity :
$\forall\ X \in L_{\mathcal{F}}$ and $\forall\ c \in \mathbb{R}\ \ \rho(X + c) = \rho(X) - c$.

$(MON \downarrow)$ $\rho(X) \leq \rho(Y)\ \forall\ X, Y \in L_{\mathcal{F}}$ such that $X \geq Y$.

- $\rho(0) = 0$.

Coherent and Convex Risk Measures

A convex risk measures is a monetary risk measure which satisfies:

- (CO) Convexity : For all $\lambda \in [0,1]$ and for all $X, Y \in L_{\mathcal{F}}$ we have that $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$.

A coherent risk measure is a convex risk measure that satisfies:

- (SA) Subadditivity : $\rho(X+Y) \leq \rho(X) + \rho(Y) \;\; \forall\, X, Y \in L_{\mathcal{F}}$.
- (PH) Positive homogeneity : $\rho(\lambda X) = \lambda\rho(X) \;\forall\, X \in L_{\mathcal{F}}$ and $\forall\, \lambda \geq 0$.

Robust representation of monetary risk measures

Coherent Risk Measures (Artzner, Delbaen, Eber, Heath (1997))

$$\rho(X) = \sup_{Q \in \mathcal{P}'} E_Q[-X]$$

where

$$\mathcal{P}' \subseteq \mathcal{P} := \{Q << P,\ Q \text{ probability}\}$$

Convex Risk Measures (Follmer, Schied (2002) - Frittelli, Rosazza (2002))

$$\rho(X) = \sup_{Q \in \mathcal{P}} \{E_Q[-X] - \alpha(Q)\}$$

where α is the penalty function $\alpha : \mathcal{P} \to [0, \infty]$.

As enlighten by Follmer-Schied-Weber (2008), the representation results for risk measures may be used in decision theory for the robust approach to model uncertainty.

Stochastic Dynamic Utilities

Definition

A stochastic dynamic utility (SDU)

$$u : \mathbb{R} \times [0, \infty) \times \Omega \to \mathbb{R} \cup \{-\infty\}$$

satisfies the following conditions: for any $t \in [0, +\infty)$ there exists $A_t \in \mathcal{F}_t$ such that $\mathbb{P}(A_t) = 1$ and

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(a) the effective domain, $\mathcal{D}(t) := \{x \in \mathbb{R} : u(x, t, \omega) > -\infty\}$ and the range $\mathcal{R}(t) := \{u(x, t, \omega) \mid x \in \mathcal{D}(t)\}$ do not depend on $\omega \in A_t$; moreover $0 \in int\mathcal{D}(t)$, $E[u(0, t)] < +\infty$ and $\mathcal{R}(t) \subseteq \mathcal{R}(s)$;

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(b) for all $\omega \in A_t$ and $t \in [0, +\infty)$ the function $x \to u(x, t, \omega)$ is strictly increasing on $\mathcal{D}(t)$ and increasing, concave and upper semicontinuous on $\mathbb{R}$.

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(c) $\omega \to u(x,t,\cdot)$ is $\mathcal{F}_t-$measurable for all $(x,t) \in \mathcal{D}(t)\times[0,+\infty)$

Stochastic Dynamic Utilities

We introduce the following useful notation

Notation:

$$\mathcal{U}(t) = \{X \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}) \,|\, u(X,t) \in L^1(\Omega, \mathcal{F}, \mathbb{P})\}.$$

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Related literature:

- Series of papers by Musiela and Zariphopoulou (2006,2008,...);
- Henderson and Hobson (2007);
- Berrier, Rogers and Theranchi (2007);
- El Karoui and Mrad (2010);
- Schweizer and Choulli (2010);
- probably many other...

Conditional Certainty Equivalent

Definition

Let u be a SDU and X be a random variable in $\mathcal{U}(t)$. For each $s \in [0, t]$, the backward Conditional Certainty Equivalent $C_{s,t}(X)$ of X is the random variable in $\mathcal{U}(s)$ solution of the equation:

$$u(C_{s,t}(X), s) = E\left[u(X, t) | \mathcal{F}_s\right].$$

Thus the CCE defines the valuation operator

$$C_{s,t} : \mathcal{U}(t) \to \mathcal{U}(s), \;\; C_{s,t}(X) = u^{-1}\left(E\left[u(X, t) | \mathcal{F}_s\right]\right), s).$$

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Even if u is concave the CCE is not a concave functional, but it is conditionally quasiconcave

Other examples popping up the financial world

Dynamic Risk Measures

$$\rho_{C_T,v_t}(X)(\omega) = ess \inf_{Y \in L^0_{\mathcal{F}_t}} \{v_t(Y,\omega) \mid X + Y \in C_T\}.$$

V@R is also quasiconvex if defined on an opportune distribution set.

Acceptability Indices
Conditional Gain Loss Ratio

$$CGLR(X|\mathcal{G}) = \frac{E_{\mathbb{P}}[X|\mathcal{G}]}{E_{\mathbb{P}}[X^-|\mathcal{G}]}\mathbf{1}_{\{E_{\mathbb{P}}[X|\mathcal{G}]>0\}}.$$

Diversification = Quasiconvexity

Let $\lambda \in \mathbb{R}$, $0 \leq \lambda \leq 1$

- The convexity of $\rho : L_{\mathcal{F}} \to \mathbb{R}$ implies

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y) \leq \rho(X) \vee \rho(Y).$$

- Quasiconvexity alone:

$$\rho(\lambda X + (1-\lambda)Y) \leq \rho(X) \vee \rho(Y)$$

  allows to control the risk of a diversified position.

- As pointed out in [CVMMM09], the principle that diversification should not increase the risk has the mathematical counterpart in QCO, not in convexity .

Economic motivations for Quasiconvexity

- Risk management:

  Diversification principle $\leftrightarrow$ Quasiconvexity

- In economic theory:

  convexity of preferences over acts $\leftrightarrow$ uncertainty aversion i.e.:

  if $\overline{X}$ and $\overline{Y}$ are preferred to $\overline{Z}$ then
  any mixture $\Lambda\overline{X} + (1-\Lambda)\overline{Y}$ is also preferred to $\overline{Z}$.

  and leads to quasiconcavity of utility functionals

General Results on Quasiconvex Conditional Maps

Dual representation for QCO real valued maps

As a straightforward application of the Hahn-Banach Theorem:

Proposition (Volle 98)

Let E be a locally convex topological vector space and E^* be its topological dual space. If $f : E \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ is LSC and QCO then

$$f(x) = \sup_{x^* \in E^*} R(x^*(x), x^*),$$

where $R : \mathbb{R} \times E^* \to \overline{\mathbb{R}}$ is defined by

$$R(m, x^*) := \inf \{ f(x) \mid x \in E \text{ such that } x^*(x) \geq m \}.$$

Dual representation of STATIC (QCO) cash-subadditive risk measures

Proposition (Cerreia-Maccheroni-Marinacci-Montrucchio, 2009)

A function $\rho : L^\infty \to \overline{\mathbb{R}}$ is QCO cash-subadditive MON ($\downarrow$) if and only if

$$\begin{aligned} \rho(X) &= \max_{Q \in ba_+(1)} R(E_Q[-X], Q), \\ R(m, Q) &= \inf \{\rho(\xi) \mid \xi \in L^\infty \text{ and } E_Q[-\xi] = m\} \end{aligned}$$

where $R : \mathbb{R} \times ba_+(1) \to \overline{\mathbb{R}}$ and $R(m, Q)$ is the reserve amount required today, under the scenario Q, to cover an expected loss m in the future.

Questions

Let $\rho : L_{\mathcal{F}} \to \overline{\mathbb{R}}$ be MON ($\downarrow$) and QCO, and set:

$$\mathcal{P} := \left\{ Q \in (L_{\mathcal{F}})^*_+ \mid Q(1) = 1 \right\}$$

1 Under which assumptions on $L_{\mathcal{F}}$ and under which continuity property of ρ do we have the dual representation

$$\rho(X) = \sup_{Q \in \mathcal{P}} R(E_Q[-X], Q),$$

where

$$R(m, Q) := \inf_{\xi \in L_{\mathcal{F}}} \{\rho(\xi) \mid E_Q[-\xi] \geq m\} \ ?$$

2 Is it possible to identify a class $\mathcal{S}$ of maps $S : \mathbb{R} \times \mathcal{P} \to \overline{\mathbb{R}}$ such that: $\rho : L_{\mathcal{F}} \to \overline{\mathbb{R}}$ is MON ($\downarrow$), QCO and "continuous" if and only if

$$\rho(X) = \sup_{Q \in \mathcal{P}} S(E_Q[-X], Q),$$

with $S \in \mathcal{S}$. ?

3 (Complete duality) For the class $\mathcal{L}$ of functions

$$\rho : L_{\mathcal{F}} \to \overline{\mathbb{R}}$$

that are:

- MON ($\downarrow$),
- QCO,
- "continuous"

is it possible to identify a class $\mathcal{R}$ of maps

$$R : \mathbb{R} \times \mathcal{P} \to \overline{\mathbb{R}}$$

such that there is a complete duality between $\mathcal{R}$ and $\mathcal{L}$?

On Complete Duality in the QCO setting

Definition

There is a complete duality between a class $\mathcal{R}$ of maps

$$R : \mathbb{R} \times \mathcal{P} \to \overline{\mathbb{R}}$$

and a class $\mathcal{L}$ of functions

$$\rho : L_{\mathcal{F}} \to \overline{\mathbb{R}}$$

if for every $\rho \in \mathcal{L}$ the only $R \in \mathcal{R}$ such that

$$\rho(X) = \sup_{Q \in \mathcal{P}} R(E_Q[-X], Q)$$

is given by

$$R(m, Q) = \inf_{\xi \in L_{\mathcal{F}}} \{\rho(\xi) \mid E_Q[-\xi] \geq m\};$$

and conversely for every $R \in \mathcal{R}$ there is a unique $\rho \in \mathcal{L}$ satisfying the above equations.

Evenly Quasiconvex functions (EVQCO)

Definition

(Fenchel, 1952) A set C is Evenly Convex if it is the intersection of open half spaces.

Note: both open convex sets and closed convex sets are evenly convex.

Definition

A function $\rho : L_{\mathcal{F}} \to \mathbb{R}$ is Evenly Quasiconvex if all the lower level sets

$$\{X \in L_{\mathcal{F}} \mid \rho(X) \leq c\}\,,\; c \in \mathbb{R},$$

are evenly convex.

Lemma

If $\rho : L_{\mathcal{F}} \to \mathbb{R}$ is LSC and QCO then it is EVQCO
If $\rho : L_{\mathcal{F}} \to \mathbb{R}$ is USC and QCO then it is EVQCO

Literature in the STATIC case ($\rho : L_{\mathcal{F}} \to \mathbb{R}$)

- Marinacci et al. (2009) provides solutions to all these three questions, under fairly general conditions, for MON ($\uparrow$) Evenly Quasiconcave real valued maps, hence covering both cases of maps $\rho : L_{\mathcal{F}} \to \mathbb{R}$ that are:

  MON ($\downarrow$), QCO and LSC
  MON ($\downarrow$), QCO and USC

- More recently, Drapeau and Kupper (2010) provide similar solutions to these questions, under different assumptions on the vector space $L_{\mathcal{F}}$, for maps $\rho : L_{\mathcal{F}} \to \mathbb{R}$ that are:

  MON ($\downarrow$), QCO and LSC

The conditional setting: let $\mathcal{G} \subseteq \mathcal{F}$ (or $\mathcal{F}_s \subseteq \mathcal{F}_t$, $s < t$)

A map

$$\pi : L(\Omega, \mathcal{F}, P) \to L(\Omega, \mathcal{G}, P)$$

is quasiconvex (QCO) if $\forall X, Y \in L(\Omega, \mathcal{F}, P)$ and for all $\mathcal{G}$-measurable r.v. Λ, $0 \leq \Lambda \leq 1$,

$$\pi(\Lambda X + (1 - \Lambda) Y) \leq \pi(X) \vee \pi(Y);$$

On question 1: the main message

- Convex case
  From the static representation (Follmer-Schied; Frittelli - Rosazza (2002))
  $$\rho(X) = \sup_{Q \in \mathcal{P}} \{E_Q[-X] - \alpha(Q)\}$$
  to the conditional one (Detlefsen-Scandolo (2005))
  $$\rho_{\mathcal{G}}(X) = ess \sup_{Q \in \mathcal{P}_{\mathcal{G}}} \{E_Q[-X \mid \mathcal{G}] - \alpha_{\mathcal{G}}(Q)\}$$
- Quasiconvex case
  From the static representation (Marinacci et al. (2009))
  $$\rho(X) = \sup_{Q \in \mathcal{P}} \{R(E_Q[-X], Q)\}$$
  to the conditional one (Frittelli-M. (2009))
  $$\rho_{\mathcal{G}}(X) = ess \sup_{Q \in \mathcal{P}_{\mathcal{G}}} \{R_{\mathcal{G}}(E_Q[-X \mid \mathcal{G}], Q)\}$$

Notations: vector space approach

- $L^p_{\mathcal{F}} := L^p(\Omega, \mathcal{F}, P)$, $p \in [0, \infty]$.
- $L_{\mathcal{F}} := L(\Omega, \mathcal{F}, P) \subseteq L^0(\Omega, \mathcal{F}, P)$ is a lattice of $\mathcal{F}$ measurable random variables.
- $L_{\mathcal{G}} := L(\Omega, \mathcal{G}, P) \subseteq L^0(\Omega, \mathcal{G}, P)$ is a lattice of $\mathcal{G}$ measurable random variables.
- $L^*_{\mathcal{F}} = (L_{\mathcal{F}}, \geq)^*$ is the order continuous dual of $(L_{\mathcal{F}}, \geq)$, which is also a lattice.

Standing assumptions on the spaces

1. $L_{\mathcal{F}}$ (resp. $L_{\mathcal{G}}$) satisfies the property $1_{\mathcal{F}}$ (resp $1_{\mathcal{G}}$):

$$X \in L_{\mathcal{F}} \text{ and } A \in \mathcal{F} \Longrightarrow (X\mathbf{1}_A) \in L_{\mathcal{F}}. \qquad (1_{\mathcal{F}})$$

2. $(L_{\mathcal{F}}, \sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}}))$ *is a locally convex TVS.*

   This condition requires that the order continuous dual $L^*_{\mathcal{F}}$ is rich enough to separate the points of $L_{\mathcal{F}}$.

3. $L^*_{\mathcal{F}} \hookrightarrow L^1(\Omega, \mathcal{F}, P)$

4. $L^*_{\mathcal{F}}$ satisfies the property $1_{\mathcal{F}}$

Conditions on $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$

Let $X_1, X_2 \in L_{\mathcal{F}}$

(MON ($\uparrow$)) $X_1 \leq X_2 \Longrightarrow \pi(X_1) \leq \pi(X_2)$

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(τ-LSC) the lower level set

$$\mathcal{A}_Y = \{X \in L_{\mathcal{F}} \mid \pi(X) \leq Y\}$$

is τ closed for each $\mathcal{G}$-measurable Y

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(τ-USC) the strictly lower level set

$$\mathcal{B}_Y = \{X \in L_{\mathcal{F}} \mid \pi(X) < Y\}$$

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(REG) $\forall A \in \mathcal{G}$, $\pi(X_1\mathbf{1}_A + X_2\mathbf{1}_A^C) = \pi(X_1)\mathbf{1}_A + \pi(X_2)\mathbf{1}_A^C$

The dual representation of conditional quasiconvex maps

Theorem (1 - solution to Question 1)

If $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ *is MON (↑), QCO, REG and either* $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$*-LSC or* $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$*-USC then*

$$\pi(X) = ess \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} R(E_Q[X|\mathcal{G}], Q)$$

where

$$R(Y, Q) := ess \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q Y\}, \;\; Y \in L_{\mathcal{G}}$$

$$\mathcal{P} =: \left\{ \frac{dQ}{dP} \mid Q << P \text{ and } Q \text{ probability} \right\}$$

Exactly the same representation of the real valued case, but with conditional expectations.

Solution to Question 2 for the (LSC) case

Define the class:

$$\mathcal{S} := \left\{ S : L^0_{\mathcal{G}} \times L^*_{\mathcal{F}} \to \bar{L}^0_{\mathcal{G}} \text{ such that } S(\cdot, \xi') \text{ is MON } (\uparrow), \text{ REG and CFB} \right\}$$

Note: Any map $S : L^0_{\mathcal{G}} \times L^*_{\mathcal{F}} \to \bar{L}^0_{\mathcal{G}}$ such that $S(\cdot, \xi')$ is MON ($\uparrow$) and REG is automatically QCO in the first component.

Theorem (2)

The map $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is MON($\uparrow$), QCO, REG and $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$-LSC if and only if there exists $S \in \mathcal{S}$ such that

$$\pi(X) = \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} S\left(E\left[\frac{dQ}{d\mathbb{P}} X | \mathcal{G} \right], Q \right).$$

On the $L^0(\mathcal{G})$-Module (Filipovic Kupper Vogelpoth 2009)

- $L^0(\mathcal{G})$ equipped with the order of a.s. dominance is a lattice ordered ring.
- For every $\varepsilon \in L^0_{++}(\mathcal{G})$ define the ball $B_\varepsilon = \{Y \in L^0(\mathcal{G}) \mid |Y| \leq \varepsilon\}$ centered in $0 \in L^0(\mathcal{G})$, which gives the neighborhood basis of 0.
- Endowed with this topology, $(L^0(\mathcal{G}), |\cdot|)$ is not a TVS, but it is a topological $L^0(\mathcal{G})$-module, in the sense of the following:

Definition

A **topological $L^0(\mathcal{G})$-module** (E, τ) is an $L^0(\mathcal{G})$-module E endowed with a topology τ such that the module operations

(i) $(E, \tau) \times (E, \tau) \to (E, \tau)$, $(X_1, X_2) \mapsto X_1 + X_2$,

(ii) $(L^0(\mathcal{G}), |\cdot|) \times (E, \tau) \to (E, \tau)$, $(Y, X_2) \mapsto YX_2$

are continuous w.r.t. the corresponding product topologies.

On the $L^0(\mathcal{G})$-Module $L^p_{\mathcal{G}}(\mathcal{F})$ (FKV 2009)

For every $p \geq 1$ let:

$$L^p_{\mathcal{G}}(\mathcal{F}) =: \{X \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \mid \|X|\mathcal{G}\|_p \in L^0(\Omega, \mathcal{G}, \mathbb{P})\}$$

where $\|\cdot|\mathcal{G}\|_p : \bar{L}^0_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0_+(\mathcal{G})$

$$\|X|\mathcal{G}\|_p =: \begin{cases} \lim_{n\to\infty} E[|X|^p \wedge n|\mathcal{G}]^{\frac{1}{p}} & \text{if } p < +\infty \\ ess.\inf\{Y \in \bar{L}^0(\mathcal{G}) \mid Y \geq |X|\} & \text{if } p = +\infty \end{cases}$$

Then $(L^p_{\mathcal{G}}(\mathcal{F}), \|\cdot|\mathcal{G}\|_p)$ is an $L^0(\mathcal{G})$-normed module having the product structure:

$$L^p_{\mathcal{G}}(\mathcal{F}) = L^0(\mathcal{G})L^p(\mathcal{F}) = \{YX \mid Y \in L^0(\mathcal{G}),\ X \in L^p(\mathcal{F})\}$$

On the dual elements of $L^p_{\mathcal{G}}(\mathcal{F})$

The dual elements can be identified with conditional expectations

For $p \in [1, +\infty)$, any $L^0(\mathcal{G})$-linear continuous functional

$$\mu : L^p_{\mathcal{G}}(\mathcal{F}) \to L^0(\mathcal{G})$$

can be identified with a random variable $Z \in L^q_{\mathcal{G}}(\mathcal{F})$, $\frac{1}{p} + \frac{1}{q} = 1$, s.t.

$$\mu(\cdot) = E[Z \cdot |\mathcal{G}].$$

Define the set of normalized dual elements by:

$$\mathcal{P}^q = \left\{ \frac{dQ}{dP} \in L^q_{\mathcal{G}}(\mathcal{F}) \mid Q \text{ probability, } E\left[\frac{dQ}{dP}|\mathcal{G}\right] = 1 \right\}$$

The class $\mathcal{R}$ for the complete duality

Define the class $\mathcal{R}$ of maps $K : L^0(\mathcal{G}) \times \mathcal{P}^q \to \bar{L}^0(\mathcal{G})$ with:

- K is increasing in the first component.
- $K(Y\mathbf{1}_A, Q)\mathbf{1}_A = K(Y, Q)\mathbf{1}_A$ for every $A \in \mathcal{G}$.
- $\inf_{Y \in L^0(\mathcal{G})} K(Y, Q) = \inf_{Y \in L^0(\mathcal{G})} K(Y, Q')$ for every $Q, Q' \in \mathcal{P}^q$.
- K is $\diamond$-evenly $L^0(\mathcal{G})$-quasiconcave: for every $(\bar{Y}, \bar{Q}) \in L^0(\mathcal{G}) \times \mathcal{P}^q$, $A \in \mathcal{G}$ and $\alpha \in L^0(\mathcal{G})$ such that $K(\bar{Y}, \bar{Q}) < \alpha$ on A, there exists $(\bar{V}, \bar{X}) \in L^0_{++}(\mathcal{G}) \times L^p_{\mathcal{G}}(\mathcal{F})$ with

$$\bar{Y}\bar{V} + E\left[\bar{X}\frac{d\bar{Q}}{dP}|\mathcal{G}\right] < Y\bar{V} + E\left[\bar{X}\frac{dQ}{dP}|\mathcal{G}\right] \text{ on } A$$

  for every (Y, Q) such that $K(Y, Q) \geq \alpha$ on A.
- the set $\mathcal{K} = \left\{K(E[X\frac{dQ}{dP}|\mathcal{G}], Q) \mid Q \in \mathcal{P}^q\right\}$ is upward directed for every $X \in L^p_{\mathcal{G}}(\mathcal{F})$.

Complete duality (solution to Question 3)

By applying the separation theorem in $L^0(\mathcal{G})$-normed module (FKV2009) - which directly provides the existence of a dual element in terms of a conditional expectation - and the idea of the proof in the static case (as in CVMMM2009) we get:

Theorem (3)

The map $\pi : L^p_{\mathcal{G}}(\mathcal{F}) \to L^0(\mathcal{G})$ is an evenly quasiconvex regular risk measure - i.e. it satisfies MON($\uparrow$), REG and EVQCO - if and only if

$$\pi(X) = \sup_{Q \in \mathcal{P}^q} R\left(E\left[\frac{dQ}{dP}X|\mathcal{G}\right], Q\right)$$

with

$$R(Y, Q) = \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \left\{\pi(\xi) \mid E\left[\frac{dQ}{dP}\xi|\mathcal{G}\right] = Y\right\}$$

unique in the class $\mathcal{R}$.

Idea of the proof for the vector space

The proof is non standard and relies on an approximation argument. The steps are the follows:

[I] We represent $\pi^{\Gamma}(X) := \sum_{A\in\Gamma} \{ \sup_A \pi(X) \} \mathbf{1}_A, = H^{\Gamma}(X)$ where

$$H^{\Gamma}(X) = \sup_{Q} \inf_{\xi \in L_{\mathcal{F}}} \left\{ \pi^{\Gamma}(\xi) | E_Q[\xi|\mathcal{F}_s] \geq_Q E_Q[X|\mathcal{F}_s] \right\}$$

[II] We deduce $\pi(X) = \inf_{\Gamma} H^{\Gamma}(X)$.

[III] We approximate $H(X)$ with $K(X, Q_\varepsilon)$ on a set A_ε of probability arbitrarily close to 1

[IV] We need a key *uniform* result to show that element Q_ε does not depend on the partition.

Idea of the proof for modules

- **Dual representation**: follows the proof given by Volle 1998 applying the Hahn Banach separation theorem for modules (Filipovic et al. 2010).

- **Uniqueness**: matches the proof given by Marinacci et al. (2009)

List of topics under investigation

- Conditions under which the *sup* in the dual representation is attained
- The EVQCO conditional case for maps defined on TVS (instead of on L^0-modules)
- g-Expectation with g QCO and its dual representations
- Local approximations of the Conditional Certainty Equivalent
- Time Consistency of Dynamic QCO maps.

References

1. *Dual representation of Quasiconvex Conditional maps*,
Joint with **Frittelli, M. (2009)**, resubmitted after first revision.
2. *Conditional Certainty Equivalent*,
Joint with **Frittelli, M. (2010)**, to appear on the International Journal of Theoretical and Applied Finance.
3. *Complete Duality for Quasiconvex Risk Measures on L^0-Modules of the L^p type*,
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4. *On quasiconvex conditional maps*,
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1. *Dual representation of Quasiconvex Conditional maps,*
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THANK YOU FOR YOUR ATTENTION!!!