Maximization of Recursive Utilities: A Dynamic Maximum Principle Approach

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Outline

Introduction

Utility Maximization: a fundamental concern in Mathematical finance.
Usual assumption: the underlying model is known
Literature:there are 3 approaches

- HJB Approach: Merton (1971): maximizing expected utility from terminal wealth
- Dual Approach: Kramkov and Schachermayer (1999-2001) and many other references.
- BSDE Approach: Schroder and Skiadas (1999, 2003, 2005) study the problem of maximization stochastic differential utility.
  El Karoui Quenez and Peng (2001) study the portfolio consumption problem with a recursive utility with nonlinear constraints on the wealth.

Introduction

Some authors studied the problem of utility maximization under model uncertainty

$$\text{find } \sup_{\pi} \inf_{Q} U(\pi, Q)$$

- $U(\pi, Q)$ is the Q-expected utility.
- π runs through a set of strategies (investment in risky assets, consumption)
- Q runs through a set of models $\mathcal{Q}$

Introduction

- Anderson Hansen and Sargent (2003): They study the problem of robust utility maximization when model uncertainty is penalized by a relative entropy term. They derive formally the HJB equation and they provide the optimal investment behaviour.
- Quenez (2004): dual approach. She considered a set of probability measures called priors and she minimizes over this set.
  Schied(2007): He considered a set of probability measures such that the penalty term is finite and he minimizes over this set the expected utility and the penalty term.

Introduction

- Bordigoni, Matoussi and Schweizer (2007) proved the existence of a unique martingale measure, equivalent to the historical probability measure, that minimizes the sum of the utility and the penalty term.
- Bordigoni (PhD Thesis (2005))solved partially the maximization problem (in the criterion, she maximized the consumption utility or the terminal wealth utility) by deriving First Order Conditions of optimality.

Introduction

In this talk:
$\longrightarrow$ We will study a robust utility maximization problem from terminal wealth and consumption

- We extend the results of El Karoui Quenez and Peng (2001)
- We state a comparison theorem and a dynamic maximum principle
- We prove the existence of an optimal strategy
- We characterize the optimal wealth and consumption rate as the unique solution of a forward-backward system

Problem formulation

- Uncertainty and information : $(\Omega, \mathcal{F}, \mathbb{F}, P)$ over a finite time horizon $[0, T]$.
- The filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfies the usual conditions of right-continuity and P-completeness.
- Possible scenarios given by

$$\mathcal{Q} = \{Q \text{ probability measure such that } Q \ll P \text{ on } \mathcal{F}_T\}$$

- The density process of Q with respect to P is the RCLL P-martingale

$$Z_t^Q = \frac{dQ}{dP}\Big|_{\mathcal{F}_t} = E_P\left[\frac{dQ}{dP}\Big|\mathcal{F}_t\right]$$

Problem formulation

Bordigoni, Matoussi and Schweizer (2007)

$$\begin{aligned} Y_t &= \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_f} \Big(\frac{1}{S_t^\delta} E_Q \Big[\int_t^T \alpha S_s^\delta \hat{U}_s ds + \bar{\alpha} S_T^\delta \bar{U}_T \Big| \mathcal{F}_t \Big] \\ &+ \beta E_Q \Big[\mathcal{R}_{t,T}^\delta(Q) | \mathcal{F}_t \Big] \Big), \end{aligned} \quad (1)$$

$$\mathcal{Q}_f = \{ Q | Q \ll P, Q = P \text{ on } \mathcal{F}_0 \text{ and } H(Q|P) := E_Q[\log \frac{dQ}{dP}] < \infty \},$$

- α and $\bar{\alpha}$ are non negative constants
- $S^\delta = (S_t^\delta := e^{-\int_0^t \delta_s ds})_{0 \le t \le T}$ discount factor
- $\hat{U} = (\hat{U}_t)_{0 \le t \le T}$ are progressively measurable processes, $\bar{U}_T$ is a $\mathcal{F}_T$-measurable random variable.

Problem formulation

- $\mathcal{R}^{\delta}_{t,T}(Q)$ is the penalty term

$$\mathcal{R}^{\delta}_{t,T} = \frac{1}{S^{\delta}_t}\int_t^T \delta_s S^{\delta}_s \log \frac{Z^Q_s}{Z^Q_t} ds + \frac{S^{\delta}_T}{S^{\delta}_t} \log \frac{Z^Q_T}{Z^Q_t}.$$

(sum entropy rate and terminal entropy)

- $\beta \in (0, \infty)$ strength of the penalty

Problem formulation

We define the following spaces

$L^0_+(\mathcal{F}_T)$ is the set of non-negative $\mathcal{F}_T-$ measurable random variables

$\mathcal{M}^p_0(P)$ is the space of all P-martingales $M = (M_t)_{0 \leq t \leq T}$ with $M_0 = 0$ and

$$E_P\Big[\sup_{0\leq t\leq T} |M_t|^p\Big] < \infty$$

Problem formulation

$L^{\exp}$ is the space of all $\mathcal{F}_T$-measurable random variables X with

$$E_P\left[\exp\left(\gamma|X|\right)\right] < \infty \qquad \text{for all } \gamma > 0$$

$D_0^{\exp}$ is the space of all progressively measurable processes $X = (X_t)_{0\le t\le T}$ with

$$E_P\left[\exp\left(\gamma \operatorname{ess\,sup}_{0\le t\le T}|X_t|\right)\right] < \infty \qquad \text{for all } \gamma > 0$$

$D_1^{\exp}$ is the space of all progressively measurable processes $X = (X_t)_{0\le t\le T}$ such that

$$E_P[\exp(\gamma \int_0^T |X_s|ds)] < \infty \qquad \text{for all } \gamma > 0$$

Problem formulation

(H1) $0 \leq \delta \leq ||\delta||_\infty$ for some constant $||\delta||_\infty$.
(H2)$\hat{U} \in D_1^{exp}$ and $\bar{U}_T \in L^{exp}$.
Under **(H1)-(H2)**, Bordigoni, Matoussi and Schweizer (2007) prove that there exists a unique probability measure Q^* that attains the minimum in (1).

Problem formulation

They show that

- the dynamics of $(Y_t)_{t\in[0,T]}$ satisfies the following BSDE

$$dY_t = (\delta_t Y_t - \alpha \hat{U}_t)dt + \frac{1}{2\beta} d < M^Y >_t + dM_t^Y \quad (2)$$

$$Y_T = \bar{\alpha}\bar{U}_T \quad (3)$$

- $(Y, M^Y) \in D_0^{exp} \times \mathcal{M}_0^p(P)$ is the unique solution of (2)-(3)
- Y has a recursive relation

$$Y_t = -\beta \log E_P\Big[\exp\Big(\frac{1}{\beta}\int_t^T (\delta_s Y_s - \alpha \hat{U}_s)ds - \frac{1}{\beta}\bar{\alpha}\bar{U}_T\Big)\Big|\mathcal{F}_t\Big]$$

- the density of the probability measure Q^* is given by

$$Z_t^{Q^*} = \mathcal{E}(-\frac{1}{\beta}M_t^Y). \quad (4)$$

Problem formulation

- a financial market consisting of a bond S^0 and d risky assets $S = (S^1, ..., S^d)$.
- $H = ((H_t^1, ..., H_t^n)_{t\in[0,T]})^*$ the investment strategy representing the number of each asset invested in the portfolio and S-integrable.
- $c = (c_t)_{0\leq t\leq T}$ the consumption rate of the investor.

$$\tilde{\mathcal{C}} = \{c = (c_t)_{t\in[0,T]}\, \mathbb{F} - \text{adapted},\ c_t \geq 0\, dt \otimes dP\text{a.s.} \int_0^T c_t dt < \infty\}$$

$$\tilde{\mathcal{H}} = \{H = (H_t)_{t\in[0,T]}\, \mathbb{F} - \text{adapted},\ R^d \text{ valued and } H \in L^2(S)\}$$

Given an initial wealth x and a policy $(c, H) \in \tilde{\mathcal{C}} \times \tilde{\mathcal{H}}$, the wealth process at time t is given by:

$$X_t^{x,c,H} = x + \int_0^t H_s dS_s - \int_0^t c_s ds. \tag{5}$$

(H3)(i) $U: \ R_+ \longrightarrow R$ and $\bar{U}: \ R_+ \longrightarrow R$ are C^1 on the sets $\{U < \infty\}$ and $\{\bar{U} < \infty\}$ respectively, strictly increasing and concave.
(ii) U and $\bar{U}$ satisfy the usual Inada conditions i.e. $U^{'}(\infty) = \bar{U}^{'}(\infty) = 0$ and $U^{'}(0) = \bar{U}^{'}(0) = \infty$.

Problem formulation

We define the set $\mathcal{A}(x)$ as the largest convex in $\mathcal{C}(x) \times \mathcal{L}(x)$ (denoted by $\hat{\mathcal{C}}(x) \times \hat{\mathcal{L}}(x)$) where $\mathcal{C}(x) \times \mathcal{L}(x)$consists of all processes $(c, \xi) \in \tilde{\mathcal{C}} \times L^0_+(\mathcal{F}_T)$ such that there exists $H \in \tilde{\mathcal{H}}$ satisfying $X_T^{x,c,H} = \xi$ as well as the families

$$\{\exp\left(\gamma \int_0^T |U(c_t)|dt\right) \,:\, c \in \tilde{\mathcal{C}}\} \tag{6}$$

$$\{\exp\left(\gamma |\bar{U}(\xi)|\right) \,:\, \xi \in L^0_+(\mathcal{F}_T)\}, \tag{7}$$

are uniformly integrable for all $\gamma > 0$.

- We assume that S is a continuous semimartingale with canonical decomposition:

$$S_t = S_0 + M_t + A_t, \ \ t \in [0, T].$$

$< M >$ the sharp bracket process of M, is absolutely continuous with respect to the Lebesgue measure on $[0, T]$

- We define the predictable $d \times d$-matrix valued process $\sigma = (\sigma_t)_{0 \leq t \leq T}$ by:

$$< M >_t = \int_0^t \sigma_u du \ \ t \in [0, T].$$

- We assume that S satisfies the structure condition (terminology of Schweizer (1994)), in the sense that there exists a predictable R^d-valued process $\lambda = (\lambda_t)_{0 \leq t \leq T}$ such that:

$$A_t = \int_0^t \sigma_u \lambda_u du, \ \ t \in [0, T].$$

- We assume that

$$\int_0^T \lambda_t^* \sigma \lambda_t dt < \infty$$

where $*$ stands for the transposition and

$$\sigma_t \ \text{ is definite positive } a.s., \ \text{ for all } \ t \in [0, T].$$

We assume that the market is complete i.e there exists a unique equivalent local martingale measure (ELMM) $\tilde{P}$ for the risky assets S. We denote by $\tilde{Z} = (\tilde{Z}_t)_{t\in[0,T]}$ the density process of $\tilde{P}$ with respect to P.
Under the structure condition, $\tilde{Z}$ is given by

$$\tilde{Z} = \mathcal{E}(-\int \lambda dM).$$

Problem formulation

$$V(x) = \sup_{(c,\xi)\in\hat{\mathcal{A}}(x)} Y_0^{x,c,\xi}, \tag{8}$$

$$\hat{\mathcal{A}}(x) := \left\{(c,\xi) \in \mathcal{A}(x) \text{ s.t. } E_{\tilde{P}}[\xi + \int_0^T c_s ds] \leq x\right\}$$

$Y^{x,c,\xi} = (Y_t^{x,c,\xi})_{0\leq t\leq T}$ is given by

$$dY_t^{x,c,\xi} = (\delta_t Y_t^{x,c,\xi} - \alpha U(c_t))dt + \frac{1}{2\beta} d < M^{x,c,\xi} >_t + dM_t^{x,c,\xi} \tag{9}$$

$$Y_T^{x,c,\xi} = \bar{\alpha}\bar{U}(X_T^{x,c,\xi}) \tag{10}$$

Proposition

If the utility functions U and $\bar{U}$ are power functions, then $\mathcal{C}(x) \times \mathcal{L}(x)$ is a convex set.

Comparison theorem

Theorem

Let $(\hat{U}^1, \bar{U}^1_T)$ *and* $(\hat{U}^2, \bar{U}^2_T)$ *be standard parameters of BSDE* (2) *satisfying Assumptions* **(H1)-(H2)** *with*

$$\hat{U}^1_t \leq \hat{U}^2_t \; dt \otimes dP \text{ a.s.} \tag{11}$$

$$\bar{U}^1_T \leq \bar{U}^2_T \; dP \text{ a.s.} \tag{12}$$

Let (Y^1, M^{Y^1}) *and* (Y^2, M^{Y^2}) *be the associated solutions, then we have almost surely for any time* $t \in [0, T]$

$$Y^1_t \leq Y^2_t.$$

Dynamic Maximum principle

Proposition

We assume **(H1)** *and* **(H2)**. *Let* $(c,\xi) \in \hat{\mathcal{A}}(x)$ *and* $(c^n,\xi^n)_{n\in\mathbf{N}}$ *a sequence of admissible strategies.*
(i) If $\xi^n \searrow \xi$ *dP a.s. and* $c_t^n \searrow c_t$, $0 \le t \le T$, $dt \otimes dP$ *a.s. when n goes to infinity, then* $Y^{x,c^n,\xi^n} \searrow Y^{x,c,\xi}$ *when n goes to infinity.*
(ii) If $\xi^n \nearrow \xi$ *dP a.s. and* $c_t^n \nearrow c_t$, $0 \le t \le T$, $dt \otimes dP$ *a.s. when n goes to infinity, then* $Y^{x,c^n,\xi^n} \nearrow Y^{x,c,\xi}$ *when n goes to infinity.*

Dynamic Maximum principle

Let ν be a positive constant, we consider the following consumption-investment problem

$$\sup_{(c,\xi)\in\mathcal{A}(x)} J(x,c,\xi,\nu), \tag{13}$$

where the functional J is defined on $\mathcal{A}(x)$ by

$$J(x,c,\xi,\nu) = Y_0^{x,c,\xi} + \nu\big(x - E_{\tilde{P}}(\xi + \int_0^T c_t dt)\big) \tag{14}$$

Dynamic Maximum principle

Lemma

Under Assumption **(H1)**, *we have*

$$\sup_{(c,\xi)\in\mathcal{A}(x)} J(x,c,\xi,\nu) < \infty \tag{15}$$

Dynamic Maximum principle

We recall the following result of convex analysis

Proposition

We assume **(H1)** *and* **(H3)**.
(i) There exists a constant ν^* *such that*

$$V(x) = \sup_{(c,\xi)\in\mathcal{A}(x)} J(x, c, \xi, \nu^*). \tag{16}$$

(ii) If the maximum is attained in (8) *by* (c^*, ξ^*)*, then it is attained in* (13) *by* (c^*, ξ^*) *with* $E_{\tilde{P}}\left[\xi^* + \int_0^T c_t^* dt\right] = x$.
(iii) Conversely, If there exist a constant ν^* *and* $(c^*, \xi^*) \in \mathcal{A}(x)$ *that achieve the maximum in* (13) *with* $E_{\tilde{P}}\left[\xi^* + \int_0^T c_t^* dt\right] = x$ *, then the maximum is attained in* (8) *by* (c^*, ξ^*) .

Dynamic Maximum principle

Theorem

We assume **(H1)** *and* **(H3)**. *Let* $(c^*, \xi^*) \in \mathcal{A}(x)$ *be the optimal consumption and investment strategy for* (13). *Let* $(Y^{x,c^*,\xi^*}, M^{x,c^*,\xi^*})$ *be the solution for the BSDE* (9)-(10). *Then the following maximum principle holds:*

$$\begin{aligned} \bar{\alpha} Z_T^* \exp\left(-\int_0^T \delta_u du\right) \bar{U}'(\xi^*) &= \nu^* \tilde{Z}_T \; dP \; a.s. \\ \alpha Z_t^* \exp\left(-\int_0^t \delta_u du\right) U'(c_t^*) &= \nu^* \tilde{Z}_t, \; 0 \le t \le T \; dt \otimes dP \; a.s. \end{aligned}$$

where $Z_t^* = \mathcal{E}(-\frac{1}{\beta} M_t^{x,c^*,\xi^*})$ *and* $\tilde{Z}_t = \mathcal{E}(-\lambda M_t)$, $0 \le t \le T$.

Existence of the optimum Strategy

Lemma

The sets $\mathcal{A}(x)$ and $\hat{\mathcal{A}}(x)$ are closed for the topology of convergence in measure.

Lemma

The functional J is strictly concave and upper-semicontinuous.

Existence of the optimum Strategy

Theorem

Let Assumptions **(H1)** *and* **(H3)** *hold. There exists a unique solution* $(c^*, \xi^*) \in \hat{\mathcal{C}}(x) \times \hat{\mathcal{H}}(x)$ *of* (13).

Existence of the optimum Strategy

The dual function $\tilde{V}$ defined on $(0, \infty)$ by

$$\tilde{V}(\nu) = \sup_{(c,\xi)\in\mathcal{A}(x)} \left\{ Y_0^{x,c,\xi} - \nu E_{\tilde{P}}[\xi + \int_0^T c_t dt] \right\}$$

Existence of the optimum Strategy

Theorem

(1) We have the conjugate duality relation

$$V(x) = \min_{\nu > 0} \left\{ \tilde{V}(\nu) + \nu x \right\}, \ \forall x > 0$$

(2) Let ν^* be such that equality (16) *holds. Let (c^*, ξ^*) be the solution of the optimization problem* (13)*, then the dual functional $\tilde{V}$ is differentiable at ν^* and*

$$\tilde{V}'(\nu^*) = -E_{\tilde{P}}(\xi^* + \int_0^T c_t^* dt) \tag{17}$$

(3) The consumption-investment strategy (c^*, ξ^*) is the unique solution of (8).

Forward-Backward system

The optimal terminal wealth ξ^* and the optimal consumption c_t^* are given by we

$$
\begin{aligned}
c_t^* &= I\Big(\frac{\nu^*}{\alpha}\exp\big(\int_0^t \delta_u du\big)\tilde{Z}_t Z_t^{*-1}\Big),\ 0 \le t \le T\ dt \otimes dP\ a.s.\\
\xi^* &= J\Big(\frac{\nu^*}{\bar{\alpha}}\exp\big(\int_0^T \delta_u du\big)\tilde{Z}_T Z_T^{*-1}\Big)\ dP\ a.s.
\end{aligned}
$$

where J is the inverse of $(\bar{U})'$ and I is the inverse of U'.

Forward-Backward system

Theorem

Assume that Assumptions **(H1)-(H3)** *hold. Let* $(Y, M^Y) \in \mathcal{D}_0^{exp} \times \mathcal{M}_0^p$, $(c^*, \xi^*) \in \hat{\mathcal{A}}(x)$ *and* Z^Y *a density of a probability measure. Then* Y *coincides with the optimal value process given by* Y^{x,c^*,ξ^*}, (c, X_T) *coincide with* (c^*, ξ^*) *given by* (18)-(18) *and* Z^Y *coincides with the density of the minimizing measure* Z^* *if and only if there exists a unique solution of the following forward-backward system*

$$\begin{cases} dX_t = H_t dS_t - c_t dt & X_0 = x \\ dY_t = (\delta_t Y_t - \alpha U(c_t))dt + \frac{1}{2\beta} d < M^Y >_t + dM_t^Y & Y_T = \bar{\alpha}\bar{U}(X_T) \\ dZ_t^Y = -\frac{1}{\beta} Z_t^Y dM_t^Y & Z_0^Y = 1 \end{cases}$$

Forward-Backward system

Corollary

Assume that Assumptions **(H1)-(H3)** *. Let $Y \in \mathcal{D}_0^{exp}$, $Z = (Z_t)_{t\in[0,T]}$ R^d-valued adapted process satisfying $E[\int_0^T |Z_t|^2 dt] < \infty$, $(c^*, \xi^*) \in \hat{\mathcal{A}}(x)$ and Z^Y a density of a probability measure. Then Y coincides with the optimal value process given by Y^{x,c^*,ξ^*}, (c, X_T) coincide with (c^*, ξ^*) given by* (18)-(18) *and Z^Y coincides with the density of the minimizing measure Z^* if and only if there exists a unique solution of the following forward-backward system*

$$
\begin{cases}
dX_t = H_t dS_t - c_t dt & X_0 = x \\
dY_t = (\delta_t Y_t - \alpha U(c_t))dt + \frac{1}{2\beta}|Z_t|^2)dt + Z_t dW_t & Y_T = \bar{\alpha}\bar{U}(X_T) \\
dZ_t^Y = -\frac{1}{\beta} Z_t^Y dM_t^Y & Z_0^Y = 1
\end{cases}
$$

Example

If $\delta \equiv 0$, $\alpha = 0$ and $\bar{\alpha} = 1$ then from the recursive relation , we obtain

$$Y_0^{x,\xi} = -\beta \log E_P\Big[\exp\Big(-\frac{1}{\beta}\bar{U}(\xi)\Big)\Big],$$

Our stochastic control problem is related to the problem

$$V^{rm}(x) := \sup_{\xi \in \mathcal{X}(x)} E_P\Big[-\exp\Big(-\frac{1}{\beta}\bar{U}(\xi)\Big)\Big]$$

where
$\mathcal{X}(x) = \{\xi \geq 0\,, \xi = x + \int_0^T H_t dS_t,\ \ H \in L(S)$ and $E_{\tilde{P}}[\xi] \leq x\}$.

Example

From Kramkov and Schachermayer (1999), the optimal terminal wealth is given by

$$\xi^* = I^{rm}(y\tilde{Z}_T) \tag{18}$$

where $U^{rm}(x) = -\exp\left(-\frac{1}{\beta}\bar{U}(x)\right)$, $I^{rm}(x) = ((U^{rm})')^{-1}(x)$ and $y = (V^{rm})'(x)$. In the case of power utility function i.e. $\bar{U}(x) = \frac{x^\gamma}{\gamma}$, we have $J(x) = x^{\frac{1}{\gamma-1}}$. We obtain

$$I^{rm}(y\tilde{Z}_T) = J\left(\nu^* \tilde{Z}_T Z_T^{*-1}\right) \ a.s. \tag{19}$$

Example

In a markovian context and by the dynamic programming principle, equality (19) holds for all $t \in [0, T]$ and so we deduce the following equalities

$$\nu^* = (\bar{U})'(I^{rm}(y)), \tag{20}$$
$$Z_T^* = \frac{1}{(\bar{U})'(I^{rm}(y\tilde{Z}_T))}\nu^*\tilde{Z}_T \tag{21}$$

From equality (18), we deduce that

$$V(x) = Y_0^{x,\xi^*} = -\beta \log E_P\Big[\exp\Big(-\frac{1}{\beta}\bar{U}(\xi^*)\Big)\Big]$$