Coupling Index and Stocks

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Modeling and managing financial risks
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Outline

1. Introduction
2. Model Specification
3. Calibration
4. Numerical experiments
5. Conclusion

- Handling both an Index and its composing stocks is still a challenging task.
- Standard approach : a model for the stocks (with smile) + a correlation matrix. Then, reconstruct the index local/implied vol. (*Avellaneda Boyer-Olson, Busca, Friz [2002], Lee, Wang, Karim [2003], ...*)
  - Difficulty to retrieve the the index smile (steeper than stock smile) by historical estimation of the correlation matrix .
  - Adjusting the correlation matrix is tedious (keep it positive definite ? implied correlation matrix ?).

Our objective : a new modeling approach allowing for a good fit of both Index and stocks.

- Another viewpoint : a factor model (the index represents the market and influences the stocks).
- (In discrete time) *Cizeau, Potters and Bouchaud [2001]* show that it is possible to capture the essential features of stocks cross-correlations by a simple non-Gaussian one factor model, specially in extreme market conditions : the daily return $r_j(t) = \frac{S_j(t)}{S_j(t-1)} - 1$ of stock j is given by

$$r_j(t) = \beta_j r_m(t) + \epsilon_j(t)$$

where $r_m(t)$ is the market daily return and $\epsilon_j(t)$ is a Student random variable. The regression coefficients β_j are narrowly distributed around 1.

- Our model can be seen as an extension in continuous time.
- Calibration to both index and stocks is feasible and leads to a new correlation structure.

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Consider an Index composed of M stocks $(S^{j,M})_{1\leq j\leq M}$:

$$I_t^M = \sum_{j=1}^{M} w_j S_t^{j,M}$$

where the w_j are positive weights assumed to be constant.
In a risk-neutral world, we specify the following dynamics for the stocks :

$$\forall j \in \{1,\dots,M\},\ \frac{dS_t^{j,M}}{S_t^{j,M}} = (r-\delta_j)dt + \beta_j\,\sigma(t, I_t^M)dB_t + \eta_j(t, S_t^{j,M})dW_t^j \quad (1)$$

- r is the short interest rate.
- $\delta_j \in [0,\infty[$ incorporates both repo cost and dividend yield of the stock j.
- β_j is the usual beta coefficient of the stock j.
- $(B_t)_{t\in[0,T]}, (W_t^1)_{t\in[0,T]}, \dots, (W_t^M)_{t\in[0,T]}$ are independent BMs.
- We assume that the functions $(s_1,\dots,s_M)\in\mathbb{R}^M \mapsto (s_j\sigma(t, \sum_{j=1}^M w_j s_j), s_j\eta_j(t,s_j))_{1\leq j\leq M}$ are Lipschitz continuous and have linear growth unif. in t.
  $\Rightarrow$ Existence and trajectorial uniqueness for the SDE (1).

$$\forall j \in \{1, \dots, M\}, \; \frac{dS_t^{j,M}}{S_t^{j,M}} = (r - \delta_j)dt + \beta_j \, \sigma(t, I_t^M) dB_t + \eta_j(t, S_t^{j,M}) dW_t^j$$

- M-dimensional SDE driven by $M+1$ sources of noise $B, W^1, \dots, W^M$: incomplete market.
- The dynamics of a given stock depends on all the other stocks composing the index through the volatility term $\sigma(t, I_t^M)$.
- The cross-correlations between stocks are not constant but stochastic :

$$\rho_{ij}(t) = \frac{\beta_i \beta_j \sigma^2(t, I_t^M)}{\sqrt{\beta_i^2 \sigma^2(t, I_t^M) + \eta_i^2(t, S_t^{i,M})} \sqrt{\beta_j^2 \sigma^2(t, I_t^M) + \eta_j^2(t, S_t^{j,M})}}$$

Note that they depend not only on the stocks but also on the index.
Nice feature : when the systemic volatility $\sigma(t, I_t^M)$ raises, so does the correlation $\rho_{ij}(t)$.

The index $I_t^M = \sum_{j=1}^M w_j S_t^{j,M}$ satisfies the following SDE

$$
\begin{aligned}
dI_t^M &= rI_t^M dt - \left(\sum_{j=1}^M \delta_j w_j S_t^{j,M}\right) dt \\
&\quad + \underbrace{\left(\sum_{j=1}^M \beta_j w_j S_t^{j,M}\right)} \sigma(t, I_t^M) dB_t + \underbrace{\sum_{j=1}^M w_j S_t^{j,M} \eta_j(t, S_t^{j,M}) dW_t^j}
\end{aligned}
$$

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$$+ \underbrace{\left(\sum_{j=1}^M \beta_j w_j S_t^{j,M}\right)}_{\simeq I_t^M} \sigma(t, I_t^M) dB_t + \underbrace{\sum_{j=1}^M w_j S_t^{j,M} \eta_j(t, S_t^{j,M}) dW_t^j}$$

Our model is inline with *Cizeau, Potters and Bouchaud [2001]* :

- The beta coefficients are narrowly distributed around 1
  $\Rightarrow \sum_{j=1}^M \beta_j w_j S_t^{j,M} \simeq I_t^M$.

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- For large M, we will show that the term $\sum_{j=1}^M w_j S_t^{j,M} \eta_j(t, S_t^j) dW_t^j$ can be neglected.

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$$\Rightarrow r_j = \beta_j r_{I^M} + \eta_j \Delta W^j + \text{drift}$$

where r_j (resp. r_{I^M}) is the log-return of the stock j (resp. the index).

The return of a stock is decomposed into a systemic part driven by the index, which represents the market, and a residual part.

A simplified Model

We look at the asymptotics for a large number M of underlying stocks.
Consider the limit candidate $(I_t)_{t\in[0,T]}$ solution of

$$\frac{dI_t}{I_t} = (r-\delta)dt + \beta\sigma(t,I_t)dB_t; \quad I_0 = I_0^M \tag{2}$$

Theorem 1

Let $p \in \mathbb{N}^*$. If

($\mathcal{H}$1) $\exists K_b$ *s.t.* $\forall (t,s)$, $|\sigma(t,s)| + |\eta_j(t,s)| \leq K_b$
$\exists K_\sigma$ *s.t.* $\forall (t,s_1,s_2)$, $|s_1\sigma(t,s_1) - s_2\sigma(t,s_2)| \leq K_\sigma |s_1 - s_2|$.

then, there exists a constant C_T depending on $\beta, \delta, K_b, K_\sigma$ but not on M such that

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|I_t^M - I_t|^{2p}\right) \leq \quad C_T \max_{1\leq j\leq M}|S_0^{j,M}|^{2p}\left(\left(\sum_{j=1}^M w_j|\beta_j - \beta|\right)^{2p} + \left(\sum_{j=1}^M w_j^2\right)^p + \left(\sum_{j=1}^M w_j|\delta_j - \delta|\right)^{2p}\right)$$

Replacing I^M by I in the dynamics of the j-th stock, one obtains

$$\frac{dS_t^j}{S_t^j} = (r - \delta_j)dt + \beta_j\,\sigma(t, I_t)dB_t + \eta_j(t, S_t^j)dW_t^j$$

Theorem 2

Under the assumptions of Theorem 1 and if

($\mathcal{H}$2) $\exists K_\eta$ *s.t.* $\forall j \leq M,\ \forall (t, s_1, s_2),\ |s_1\eta_j(t, s_1) - s_2\eta_j(t, s_2)| \leq K_\eta |s_1 - s_2|$

$\exists K_{Lip}$ *s.t.* $\forall (t, s_1, s_2),\ |\sigma(t, s_1) - \sigma(t, s_2)| \leq K_{Lip}|s_1 - s_2|$

then, $\forall j \in \{1, \ldots, M\}$*, there exists a constant* $\widetilde{C}_T^j$ *depending on* $\beta, \delta, \beta_j, \delta_j, K_b, K_\sigma, K_\eta, K_{Lip}$ *and* $\max_{1\leq j\leq M} S_0^{j,M}$ *but not on M s.t.*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |S_t^{j,M} - S_t^j|^{2p}\right) \leq \widetilde{C}_T^j \left(\left(\sum_{j=1}^M w_j|\beta_j - \beta|\right)^{2p} + \left(\sum_{j=1}^M w_j^2\right)^p + \left(\sum_{j=1}^M w_j|\delta_j - \delta|\right)^{2p}\right)$$

Three different indexes

Original index : $I_t^M = \sum_{j=1}^M w_j S_t^{j,M}$ where $(S^{j,M})_{1 \le j \le M}$ solves (1).

Simplified limit index : I_t solving $\frac{dI_t}{I_t} = (r-\delta)dt + \beta\sigma(t, I_t)dB_t$

Beware, in general $I_t \neq \sum_{j=1}^M w^j S_t^j$ where

$$\frac{dS_t^j}{S_t^j} = (r-\delta_j)dt + \beta_j \sigma(t, I_t)dB_t + \eta_j(t, S_t^j)dW_t^j$$

Reconstructed index : $\bar{I}_t^M \stackrel{\text{def}}{=} \sum_{j=1}^M w^j S_t^j$

Theorem 3

Under the assumptions of Theorem 2,

$$\mathbb{E}\left(\sup_{0 \le t \le T} |I_t^M - \bar{I}_t^M|^{2p}\right) \le \max_{1 \le j \le M} \widetilde{C}_T^j \left(\sum_{j=1}^M w_j\right)^{2p}$$
$$\times \left(\left(\sum_{j=1}^M w_j|\beta_j - \beta|\right)^{2p} + \left(\sum_{j=1}^M w_j^2\right)^p + \left(\sum_{j=1}^M w_j|\delta_j - \delta|\right)^{2p}\right)$$

In Theorems 1,2 and 3 the upper-bound is small if $\sum_{j=1}^M w_j^2$, $\sum_{j=1}^M w_j|\beta_j - \beta|$ and $\sum_{j=1}^M w_j|\delta_j - \delta|$ are small.

- For uniform weights $w_j = \frac{1}{M}$, $\sum_{j=1}^M w_j^2 = \frac{1}{M}$ is small as soon as M is large.
- Let D be a random variable such that
$$\forall j \in \{1, \dots, M\},\ \mathbb{P}(D = \delta_j) = \frac{w_j}{\sum_{k=1}^M w_k}.$$
One has $\sum_{j=1}^M w_j|\delta_j - \delta| = \left(\sum_{k=1}^M w_k\right) \mathbb{E}|D - \delta| \rightarrow$ the value of the dividend rate δ of the index minimizing $\sum_{j=1}^M w_j|\delta_j - \delta|$ is the median $\delta_\star$ of the random variable D.

- idem for $\sum_{j=1}^M w_j|\beta_j - \beta|$ but, because of the interpretation of the β_j as regression coefficients, we rather have to take $\beta = 1$.

$\sum_{j=1}^M w_j^2$	$(\sum_{j=1}^M w_j\lvert\beta_j - 1\rvert)^2$	$\inf_\beta(\sum_{j=1}^M w_j\lvert\beta_j - \beta\rvert)^2$	$\beta_\star$
0.026	0.0174	0.0173	0.975

TABLE: Example of the Eurostoxx index at December 21, 2007 (M=50). The β_j are estimated on a two year history.

To sum up, under mild assumptions, when the number of underlying stocks is large, our original model may be approximated by

$$\begin{cases} & \dfrac{dI_t}{I_t} = (r - \delta_I)dt + \sigma(t, I_t)dB_t \\ \forall j \in \{1, \ldots, M\}, & \dfrac{dS_t^j}{S_t^j} = (r - \delta_j)dt + \beta_j \sigma(t, I_t)dB_t + \eta_j(t, S_t^j)dW_t^j \end{cases} \quad (3)$$

We end up with

- A local volatility model for the index
- A novel stochastic volatility model for each stock, decomposed into a systemic part driven by the index level and an intrinsic part.

Beware ! Our simplified model is not valid for options written on the index together with all its composing stocks since the limit index is no longer an exact, but an approximate, weighted sum of the stocks. Instead, one should consider the reconstructed index $\bar{I}_t^M = \sum_{j=1}^M w_j S_t^j$ or use the original model.

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Calibration of the simplified model

$$\begin{aligned} \frac{dI_t}{I_t} &= (r-\delta_I)dt + \sigma(t,I_t)dB_t \\ \frac{dS_t}{S_t} &= (r-\delta)dt + \beta\,\sigma(t,I_t)dB_t + \eta(t,S_t)dW_t. \end{aligned} \tag{4}$$

- We first fit the index smile : standard calibration of a local volatility model $\longrightarrow \sigma(t,x)$.
- Fitting an individual stock smile is more complicated. The regression coefficient β is estimated historically.

$\Rightarrow$ Our model gives an advantage to the fit of index option prices (index options are usually more liquid than individual stock options).

Calculation of η

Let $v_{loc}(t, K)$ be the square of the local volatility fitting the stock smile given by Dupire's formula [3] :

$$v_{loc}(t, K) = 2\frac{\partial_t C(t, K) + (r - \delta)K\partial_K C(t, K) + \delta C(t, K)}{K^2 \partial^2_{KK} C(t, K)}$$

where $C(t, K)$: market price of the Call option with maturity t and strke K written on S.

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According to *Gyöngy [1986]*, if $\mathbb{E}(\beta^2\sigma^2(t,I_t) + \eta^2(t,S_t)|S_t = K) = v_{loc}(t,K)$, then $\forall T, K > 0,\ \mathbb{E}\left(e^{-rT}(S_T - K)^+\right) = C(T,K)$. Hence we want to compute

$$\eta(t,K) = \sqrt{v_{loc}(t,K) - \beta^2 \mathbb{E}\left(\sigma^2(t,I_t) \,|\, S_t = K\right)} \tag{5}$$

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$$\eta(t,K) = \sqrt{v_{loc}(t,K) - \beta^2 \mathbb{E}\left(\sigma^2(t,I_t)\,|\,S_t = K\right)} \tag{5}$$

- In practice, v_{loc} can be calibrated with the best-fit of a parametric form to the stock market smile.
- Estimating the conditional expectation is more challenging (it depends implicitly on η as it is the case for the law of (S_t, I_t))

Non-parametric estimation of η

If we plug expression of η (5) in the dynamics of the stock we obtain

$$\frac{dS_t}{S_t} = (r-\delta)dt + \beta\sigma(t,I_t)dB_t + \sqrt{v_{loc}(t,S_t) - \beta^2\mathbb{E}\left(\sigma^2(t,I_t)\,|\,S_t\right)}dW_t$$

$$\frac{dI_t}{I_t} = (r-\delta_I)dt + \sigma(t,I_t)dB_t$$

- This SDE is non-linear in the sense of McKean.
- The conditional expectation may be approximated by Kernel estimators of the Nadaraya-Watson type :

$$\mathbb{E}\left(\sigma^2(t,I_t)\,|\,S_t = s\right) \simeq \frac{\sum_{i=1}^{N}\sigma^2(t,I_t^i)K\left(\frac{s-S_t^i}{h_N}\right)}{\sum_{i=1}^{N}K\left(\frac{s-S_t^i}{h_N}\right)}$$

where K is a non-negative kernel s.t. $\int_{\mathbb{R}} K(x)dx = 1$ and $\lim_{N\to\infty} h_N = 0$.

A system of interacting particles

Replacing the conditional expectation by its non-parametric estimator yield the following system : $\forall 1 \leq i \leq N$,

$$\frac{dS_t^{i,N}}{S_t^{i,N}} = (r-\delta)dt + \beta\,\sigma(t, I_t^i)dB_t^i + \sqrt{v_{loc}(t, S_t^{i,N}) - \overline{\sigma^2}(t, S_t^{i,N})}dW_t^i$$

$$\overline{\sigma^2}(t, S_t^{i,N}) = \frac{\beta^2 \sum_{k=1}^N \sigma^2(t, I_t^k) K\left(\frac{S_t^{i,N} - S_t^{k,N}}{h_N}\right)}{\sum_{k=1}^N K\left(\frac{S_t^{i,N} - S_t^{k,N}}{h_N}\right)}$$

$$\frac{dI_t^i}{I_t^i} = (r-\delta_I)dt + \sigma(t, I_t^i)dB_t^i$$

$(B^i, W^i)_{i\geq 1}$ is a sequence of independent two-dimensional Brownian motions. This $2N$-dimensional linear SDE may be discretized using a simple Euler scheme !

- $K \longrightarrow \mathbb{E}(\sigma^2(t, I_t)|S_t = K)$ may be computed by spatial interpolation of the $(\overline{\sigma^2}(t, S_t^{i,N}))_{1\leq i\leq N} \longrightarrow$ approx of $\eta(t, K)$
- the particle system may be used directly for pricing in the calibrated model : Monte Carlo estimate of the price of the option with payoff h written on $S \longrightarrow \frac{e^{-rT}}{N}\sum_{i=1}^N h(S^{i,N})$.

An acceleration technique

- The simulation of the particle system is time consuming : a global complexity of order $O(nN^2)$ where n is the number of time steps in the Euler scheme.
- A possible acceleration technique : neglect particles which are far away from each other.
- How ? Sort the particles and stop the estimation of the conditional expectation whenever the contribution of a particle is lower than some fixed threshold.
- We lose in precision but we gain much more in computation time.

A similar problem : calibration of a Stochastic Volatility Local Volatility model

$$\begin{cases} dS_t = \eta(t,S_t)f(Y_t)S_t dW_t + rS_t dt \\ dY_t = \alpha(Y_t)dB_t + b(Y_t)dt \end{cases}, \ d < W,B >_t = \rho dt$$

According to *Gyöngy [1986]*, if $\mathbb{E}\left(\eta^2(t,S_t)f^2(Y_t)|S_t = K\right) = v_{loc}(t,K)$ then

$$\forall T,K > 0, \ \mathbb{E}\left(e^{-rT}(S_T - K)^+\right) = C(T,K).$$

One looks for

$$\begin{cases} dS_t = \sqrt{\frac{v_{loc}(t,S_t)}{\mathbb{E}(f^2(Y_t)|S_t)}} f(Y_t)S_t dW_t + rS_t dt \\ dY_t = \alpha(Y_t)dB_t + b(Y_t)dt \end{cases}, \ d < W,B >_t = \rho dt$$

Calibration of the original model

$$\frac{dS_t^{j,M}}{S_t^{j,M}} = (r-\delta_j)dt + \beta_j\,\sigma(t, I_t^M)dB_t + \eta_j(t, S_t^{j,M})dW_t^j \text{ with } I_t^M = \sum_{j=1}^{M} w_j S_t^{j,M}$$

- A perfect calibration of both the index and the individual stocks is complicated... but we can
  - take for σ the calibrated local vol of the index and then calibrate the η_j coefficients in order to fit all the individual stock smiles $\Rightarrow$ the index is not perfectly calibrated but the error should be small (Theorem 1).
  - take for σ and η_j the calibrated coefficients in the simplified model $\Rightarrow$ the index and the stocks are not perfectly calibrated but the error should be small (Theorems 1 and 2).
- We allow for slight errors in the calibration but the additivity constraint is observed.
- Similarly, the reconstructed index $\bar{I}_t^M = \sum_{j=1}^{M} w_j S_t^j$ in the calibrated simplified model does not prefectly fit the market index smile.

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Data :

- Local volatilities of the Eurostoxx index and of Carrefour at December 21, 2007.
- Beta coefficient estimated on a two years history ($\beta = 0.7$).
- Short interest rate and dividend yields as of December 21, 2007.
- Maturity $T = 1$.
- Threshold for the accelerated technique : $\frac{1}{N}$.
- Smoothing parameter : $h_N = N^{-\frac{1}{10}}$.
- Number of time steps for the Euler scheme : $n = 20$.

Moneyness ($\frac{K}{S_0}$)	0.5	0.7	0.9	1	1.1	1.2	1.5	2
Error : $\lvert\widehat{\sigma}_{simul} - \widehat{\sigma}_{exact}\rvert$	36	8	2	1	2	9	32	56

TABLE: Error (in bp) on the implied volatility of Carrefour with $N = 200000$ particles.

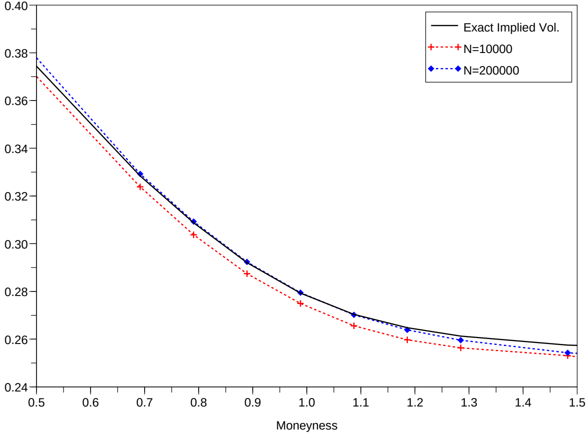
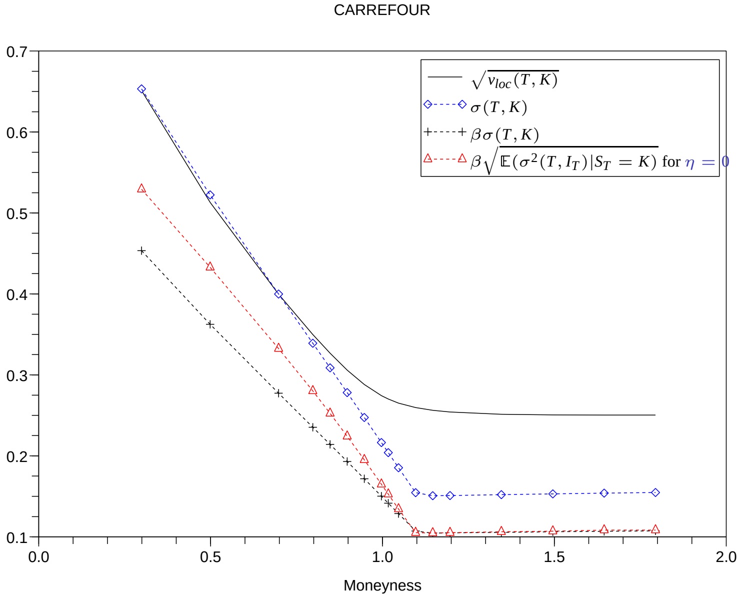


FIGURE: Convergence of the implied volatility of carrefour obtained with non-parametric estimation.


CARREFOUR
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.5
1.0
1.5
2.0
Moneyness
$\sqrt{v_{loc}(T,K)}$
$\sigma(T,K)$
$\beta\sigma(T,K)$
$\beta\sqrt{\mathbb{E}(\sigma^2(T,I_T)|S_T=K)}$ for $\eta=0$

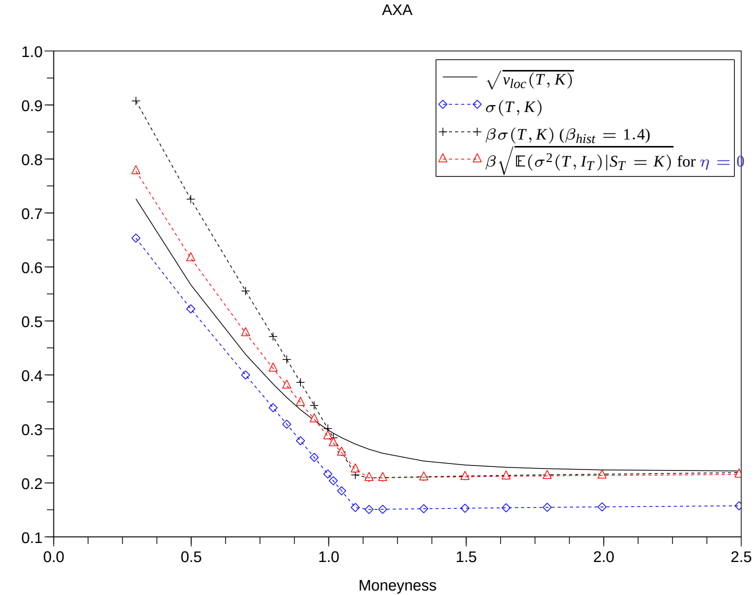

AXA
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.5
1.0
1.5
2.0
2.5
Moneyness
$\sqrt{v_{loc}(T,K)}$
$\sigma(T,K)$
$\beta\sigma(T,K)$ ($\beta_{hist} = 1.4$)
$\beta\sqrt{\mathbb{E}(\sigma^2(T,I_T)|S_T = K)}$ for $\eta = 0$

Illustration of Theorems 1, 2 and 3

1. The original model

$$\forall j \in \{1,\ldots,M\}, \quad \begin{array}{l} \frac{dS_t^{j,M}}{S_t^{j,M}} = rdt + \sigma(t, I_t^M)dB_t + \eta(t, S_t^{j,M})dW_t^j \\ I_t^M = \sum_{j=1}^M w_j S_t^{j,M}. \end{array}$$

2. The simplified model

$$\forall j \in \{1,\ldots,M\}, \quad \begin{array}{l} \frac{dS_t^j}{S_t^j} = rdt + \sigma(t, I_t)dB_t + \eta(t, S_t^j)dW_t^j \\ \frac{dI_t}{I_t} = rdt + \sigma(t, I_t)dB_t. \end{array}$$

Reconstructed index $\overline{I}_t^M = \sum_{i=1}^M w_i S_t^i$.

3. The constant-correlation market model

$$\forall j \in \{1,\ldots,M\}, \quad \begin{array}{l} \frac{dS_t^j}{S_t^j} = rdt + \sqrt{v_{loc}(t, S_t^j)}d\widetilde{W}_t^j \\ \forall i \neq j, d < \widetilde{W}^i, \widetilde{W}^j >_t = \rho dt. \end{array}$$

- M, I_0, and $w_1, \ldots, w_M \rightarrow$ values for the Eurostoxx index at December 21, 2007.
- $S_0^1 = \ldots = S_0^M = 53.$
- $r = 0.045$
- $\sigma(t,i) \rightarrow$ calibrated local vol of the Eurostoxx.
- We choose an arbitrary parametric form for the vol coefficient η.
- We evaluate v_{loc} s.t. the constant-correl market model with local vol $\sqrt{v_{loc}}$ yields the same implied vol for individual stocks as the simplified model $\rightarrow v_{loc}(t,s) = \eta^2(t,s) + \mathbb{E}(\sigma^2(t,I_t)|S_t^1 = s)$.
- We fix the correlation coefficient ρ s.t. the constant-correl market model and the simplified one yield the same ATM implied vol for the index.
- Number of simulated paths : 100 000.

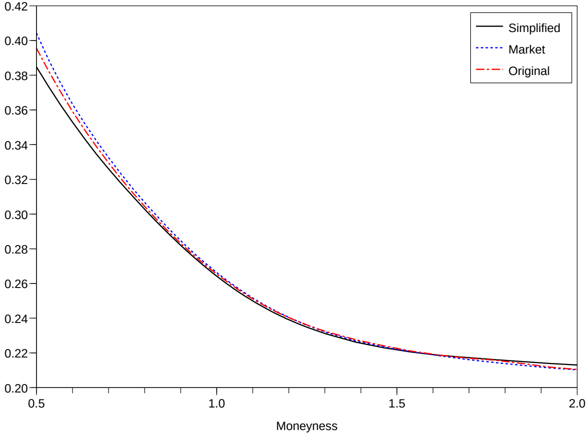


FIGURE: Implied volatility of an individual stock at $T = 1$.

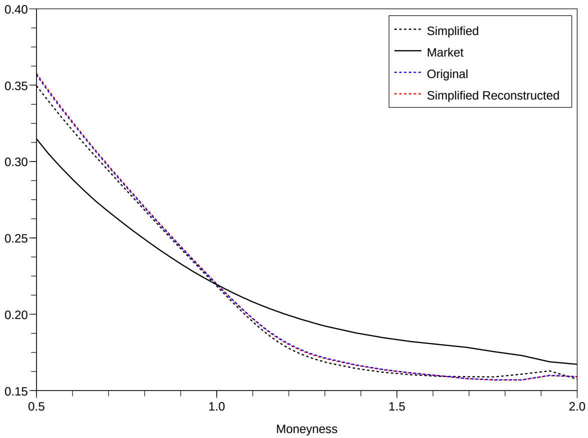


FIGURE: Implied volatility of the index at $T = 1$.

Moneyness ($\frac{K}{I_0}$)	0.5	0.8	0.9	1	1.05	1.1	1.3	1.55	2
$\left\|\widehat{\sigma}_{simplified} - \widehat{\sigma}_{original}\right\|$	81	22	16	14	17	20	24	11	17
$\left\|\widehat{\sigma}_{reconstruct} - \widehat{\sigma}_{original}\right\|$	10	5	4	2	1	2	4	1	0

TABLE: Difference (in bp) between the index (resp. the reconstructed index) implied volatility obtained with the simplified model and the one obtained with the original model.

Pricing of a worst-of option : payoff $\left(\min_{1\le j\le M}\frac{S_1^j}{S_0^j}-K\right)^+$

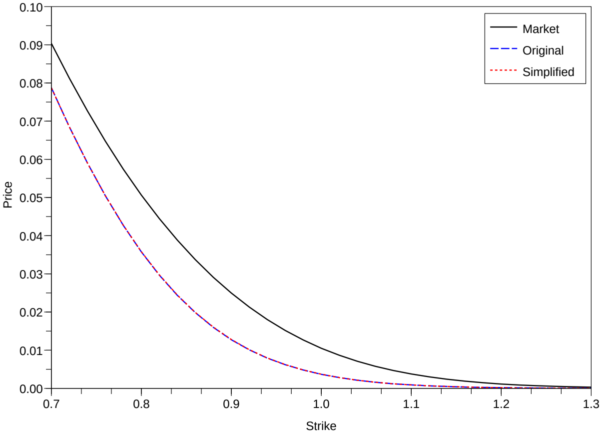


FIGURE: Worst-of price.

Outline

1 Introduction

2 Model Specification

3 Calibration

4 Numerical experiments

5 Conclusion

- We have introduced a new model for describing the joint evolution of an index and its composing stocks.
- The index induces some feedback on the dynamics of its stocks.
- For large number of underlying stocks, the model reduces to a local vol model for the index and to a stochastic vol for each individual stock with volatility driven by the index.
- We favor the fit of the index smile.
- We have proposed a simulation based approach allowing to fit both the index and the stocks smiles.

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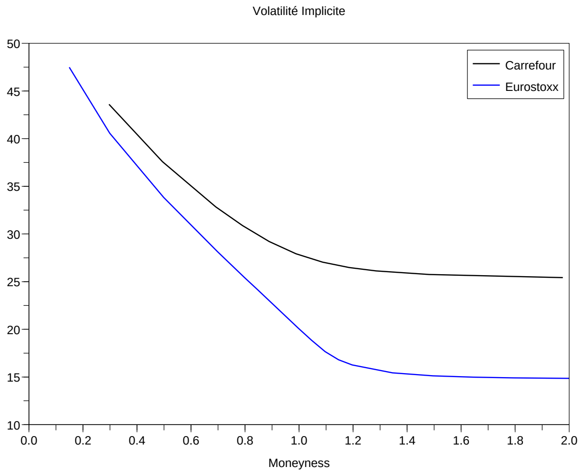

FIGURE: Index smile is steeper than stocks smile.