Analytical formulas for local volatility model with stochastic rates

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Motivation

- Long term callable path dependent equity options have generated new modeling challenges.
- The path dependency requires consistency in the equity asset diffusion.
- The early exercise on long period suggests to take in account interest rates risk.
- Several works has been done in the case of stochastic volatility with interest rates (Piterbarg 2005, Balland 2005, Andreasen 2006 or Haastrecht et al 2008). But, few have considered local volatility model plus stochastic rates (Benhamou et al 2008).

No arbitrage relations

Under the risk neutral probability $\mathbb{Q}$, one has:

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t dW_t^1,$$

$$r_t = f(0,t) - \int_0^t \gamma(s,t).\Gamma(s,t)ds + \int_0^t \gamma(s,t)dB_s,$$

where

- σ_t is the volatility (not nessecary deterministic).
- r_t follows the HJM framework ($\gamma(t,T) = -\partial_T \Gamma(t,T)$).

- The pricing of a European option with final payoff $\varphi(S_T)$ can be reformulated in the forward measures as follows (see Geman et coauthors 95):

$$\mathbb{E}[e^{-\int_0^T r_s ds}\varphi(S_T)] = B(0,T)\mathbb{E}_T[\varphi(F_T^T)]$$

  where (F_t^T) is a martingale under the forward measure $\mathbb{Q}^T$.
- For path dependent options, we have to use as many volatility models as many the maturities used in the options.

Modeling the asset

- We define a model on the discounted price process:

$$S_t^d = e^{-\int_0^t r_s ds} S_t$$

- which is a martingale (under $\mathbb{Q}$). We assume that

$$\frac{dS_t^d}{S_t^d} = \sigma^d(t, S_t^d) dW_t^1.$$

- Equivalently, we study the log discounted process $X_t = \log(S_t^d)$:

$$dX_t = \sigma(t, X_t) dW_t^1 - \frac{\sigma^2}{2}(t, X_t) dt, \quad X_0 = x_0, \tag{1}$$

The interest rates framework

$$r_t = f(0,t) - \int_0^t \gamma(s,t).\Gamma(s,t)ds + \int_0^t \gamma(s,t)dB_s.$$

- We consider Gaussian model for interest rates, by assuming that $\Gamma, \gamma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^n$ are deterministic functions (n is the number of Gaussian factors).
- The Brownian motions W^1 and $B = (B^1, \cdots, B^n)$ are correlated:

$$d\langle W^1, B^i\rangle_t = \rho^{S,r}_{i,t} dt \quad 1 \leq i \leq n.$$

- Hence, the price to compute is now formulated as:

$$A = \mathbb{E}[e^{-\int_0^T r_s ds} h(\int_0^T r_s ds + X_T)]. \tag{2}$$

Black Formula

When the volatility σ is deterministic:

- The equivalent volatility σ^{Black} of $\int_0^T r_s ds + X_T$ is deterministic and is defined by:

$$(\sigma^{Black})^2 T = \int_0^T [\sigma^2(t, x_0) + |\Gamma(t, T)|^2 - 2\sigma(t, x_0)\rho_t^{S,r}.\Gamma(t, T)]dt.$$

- Hence, the call price in such model has a closed formula like the Black Scholes formula with volatility σ^{Black}.
- In tis case, we note $X_T^B \equiv X_T$. This model is the proxy of our approximation.

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Assumptions

- ***Assumption*** *(R_5). The function σ is bounded and of class C^5 w.r.t x. Its derivatives up to order 5 are bounded.* Under (R_5), we set

$$M_0 = max(|\sigma|_\infty, \cdots, |\partial_x^5 \sigma|_\infty),$$
$$M_1 = max(|\partial^1 \sigma|_\infty, \cdots, |\partial_x^5 \sigma|_\infty).$$

- ***Assumption*** *(E). The function σ does not vanish and its oscillation is bounded, meaning $1 \leq \frac{|\sigma|_\infty}{\sigma_{inf}} \leq C_E$ where* $\sigma_{inf} = \inf_{(t,x)\in\mathbb{R}^+\times\mathbb{R}} \sigma(t,x)$.
- ***Assumption*** *(Rho). The asset is not perfectly correlated (positively or negatively) to the interest rate:*

$$|\rho^{S,r}|_\infty = \sup_{t\in[0,T]} |\rho_t^{S,r}| < 1.$$

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Table: Historical correlation between assets and short term interest rate EUR. Period: 23-Sep-2007 to 22-Sep-09.

Asset	Historical correlation
ADIDAS	18.32%
BELGACOM	4.09%
CARREFOUR	7.08%
DAIMLER	-0.94%
DANONE	7.23%
LVMH	4.53%
NOKIA	4.37%
PHILIPS	5.23%

The approximation formula

Keep in mind that the Greeks have the following definition:

$$\text{Greek}^l(Z) = \partial^l_{x=0}(\mathbb{E}_T(Z+x))$$

Theorem

*(**Second order approximation price formula**).*
Assume that the model fulfills (R_5), (E) and (Rho), and that the payoff h is a call-put option, we prove that:

$$\begin{aligned}\mathbb{E}[e^{-\int_0^T r_s ds} h(\int_0^T r_s ds + X_T)] = B(0,T)(\mathbb{E}_T[h(\int_0^T r_s ds + X_T^B)] \\ + \sum_{i=1}^{3} \alpha_{i,T} \text{Greek}_i^h(\int_0^T r_s ds + X_T^B) \\ + \text{Resid}_2),\end{aligned}$$

- where

$$\begin{aligned}
\alpha_{1,T} =& -\int_0^T (\rho_t^{S,r}.\Gamma(t,T)\sigma(t,x_0) - \frac{\sigma^2(t,x_0)}{2})(\int_t^T a_s \partial_x^1 \sigma(s,x_0)ds)dt,\\
\alpha_{2,T} =& -\alpha_{1,T} - \alpha_{3,T},\\
\alpha_{3,T} =& \int_0^T a_t \sigma(t,x_0)(\int_t^T a_s \partial_x^1 \sigma(s,x_0)ds)dt,\\
a_t =& \sigma(t,x_0) - \rho_t^{S,r}.\Gamma(t,T).
\end{aligned}$$

- and the error is estimated by:

$$\begin{aligned}
|\mathrm{Resid}_2| \leq & C(\|h^{(1)}(\int_0^T r_s ds + X_T^B)\|_2 + \sup_{v\in[0,1]} \|h^{(1)}(\int_0^T r_s ds + vX_T + (1-v)X_T^B)\|_2)\\
& \frac{M_0}{\sigma_{inf}\sqrt{1-|\rho^{S,r}|_\infty^2}} M_1 M_0^2 (\sqrt{T})^3.
\end{aligned}$$

Example

In the case of

- homogeneous volatility $\sigma(t,x)=\sigma(x)$
- Hull and White stochastic rate (volatilility of volatility ξ and mean reversion κ)

one gets:

$$
\begin{aligned}
\alpha_{1,T} =& \frac{e^{-2\kappa T}\sigma(x_0)\sigma^{(1)}(x_0)}{4\kappa^4}\big(2\rho^2\xi^2 + 2e^{\kappa T}\rho(\kappa\sigma(2\kappa T+1) + 2\rho(\kappa T-1)\xi)\xi \\
&+ e^{2\kappa T}\left(\sigma^2 T^2\kappa^4 + \rho\sigma(\kappa T(3\kappa T-2)-2)\xi\kappa + 2\rho^2(\kappa T-1)^2\xi^2\right),\\
\alpha_{2,T} =& -\alpha_{1,T} - \alpha_{3,T},\\
\alpha_{3,T} =& \frac{e^{-2\kappa T}\sigma(x_0)\sigma^{(1)}(x_0)\left(\rho\xi + e^{\kappa T}\left(\sigma T\kappa^2 + \rho T\xi\kappa - \rho\xi\right)\right)^2}{2\kappa^4}.
\end{aligned}
$$

Plan of the proof

- Expand the model X_T around the proxy model X_T^B.
- Perform a Taylor expansion for the payoff h around the proxy model by assuming h smooth enough.
- Estimate the corrections as a Greeks using Malliavin calculus technique.
- Upper bound the errors using a suitable choice of the Brownian motion used for the Malliavin differentiation and the estimates of the inverse of the Malliavin covariance.
- use a regularisation method in order to prove the approximation formula for call-put option.

Proxy model

How to expand the model X_T around the proxy model X_T^B?

- Suitable parameterisation:

$$dX_t^\epsilon = \epsilon(\sigma(t, X^\epsilon)dW_t - \frac{\sigma^2(t, X^\epsilon)}{2}dt), X_0^\epsilon = x_0$$

  so that $X_t^\epsilon|_{\epsilon=0} + \frac{\partial(X_t^\epsilon)}{\partial\epsilon}|_{\epsilon=0} = X_t^B$ and $X_t^\epsilon|_{\epsilon=1} = X_t$.

- Hence using a Taylor approximation for the model:

$$X_t = X_t^B + \frac{\partial^2(X_t^\epsilon)}{2\partial\epsilon^2}|_{\epsilon=0} + \cdots$$

Perform a Taylor expansion for the payoff h around the proxy model (X_t^B):

$$
\begin{aligned}
\mathbb{E}[e^{-\int_0^T r_s ds} h(\int_0^T r_s ds + X_T)] &= \mathbb{E}[e^{-\int_0^T r_s ds} h(\int_0^T r_s ds + X_T^B + \frac{\partial^2 (X_T^\epsilon)}{2\partial \epsilon^2}|_{\epsilon=0} + \cdots)] \\
&= \mathbb{E}[e^{-\int_0^T r_s ds} h(\int_0^T r_s ds + X_T^B)] \\
&+ \mathbb{E}[e^{-\int_0^T r_s ds} h^{(1)}(\int_0^T r_s ds + X_T^B) \frac{\partial^2 (X_T^\epsilon)}{2\partial \epsilon^2}|_{\epsilon=0}] \\
&+ Resid_2 \\
&= B(0,T)\mathbb{E}_T[h(\int_0^T r_s ds + X_T^B)] \\
&+ \mathbb{E}[e^{-\int_0^T r_s ds} h^{(1)}(\int_0^T r_s ds + X_T^B) \frac{\partial^2 (X_T^\epsilon)}{2\partial \epsilon^2}|_{\epsilon=0}] \\
&+ Resid_2
\end{aligned}
$$

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Greeks identification

$$\mathbb{E}[e^{-\int_0^T r_s ds} h^{(1)}(\int_0^T r_s ds + X_T^B) \frac{\partial^2 (X_T^\epsilon)}{2\partial\epsilon^2}|_{\epsilon=0}] = B(0,T)(\sum_{i=1}^{3} \alpha_{i,T} \mathrm{Greek}_h^{(i)}(\int_0^T r_s ds + X_T^B)$$

How we can achieve that? This technique can be seen as an inverse procedure used in the literature about integration by parts formula and Malliavin calculus (Fournie et al 99).

Estimates of the error

In $Resid_2$, there is terms which constains the second derivative $h^{(2)}$ of the payoff function while the payoff h of interest is only one time differentiable?

- **Lemma** Assume (E), (Rho) and (R_{k+1}) for a given $k \geq 1$. Let Z belong to $\cap_{p\geq 1}\mathbb{D}^{k,p}$. For any $v \in [0,1]$, there exists a random variable Z_k^v in any $Ł_p$ $(p \geq 1)$ such that for any function $l \in C_0^\infty(\mathbb{R})$, we have

$$\mathbb{E}_T[l^{(k)}(\int_0^T r_s ds + vX_T + (1-v)X_T^B)Z] = \mathbb{E}_T[l(v\int_0^T r_s ds + X_T + (1-v)X_T^B)Z_k^v].$$

Moreover, we have $||Z_k^v||_p \leq C\frac{||Z||_{k,2p}}{(\sqrt{1-|\rho_\infty^{S,r}|^2}\sigma_{inf}\sqrt{T})^k}$, uniformly in v and the constants C is an increasing constant on the bounds of the model.

Extensions

- The error of estimation is analyzed for other payoffs (smooth, digitals): the more the payoff is smooth, the more the error is small.
- Extensions to third order formula.
- Extension to stochastic convenience yield. This can be seen as an extension to Gibson Schwartz model to handle local volatility functions for example:

$$
\begin{aligned}
\frac{dS_t}{S_t} &= (r_t - y_t)dt + \sigma dW_t^1, \\
dy_t &= \kappa(\alpha_t - y_t)dt + \xi_t dW_t^2, \\
d\langle W^1, W^2\rangle_t &= \rho_t dt.
\end{aligned}
$$

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Numerical application

We consider the one factor Hull and White model for interest rates, the CEV diffusion for the spot and constant correlation ρ. Then,

$$\gamma(t,T) = \xi e^{-\kappa(T-t)}, \sigma(t,x) = \nu e^{(\beta-1)x}.$$

As a benchmark, we use Monte Carlo methods with a variance reduction technique (3×10^6 simulations using Euler scheme with 50 time steps per year). Parameters: $\beta = 0.8$, $\nu = 0.2$, $\xi = 0.7\%$, $\kappa = 1\%$, $\rho = 15\%$ and $x_0 = 0$.

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Table: Implied Black-Scholes volatilities for the second order formula, the third order formula and the Monte Carlo simulations for maturity $T = 10Y$

Relative Strikes	30%	60%	100%	160%	220%
Second Order formula	22.99%	22.16%	21.25%	20.38%	19.77%
Third Order formula	23.54%	22.25%	21.27%	20.40%	19.84%
MC with control variate	23.66%	22.32%	21.34%	20.47%	19.91%
MC-	22.87%	22.18%	21.28%	20.43%	19.87%
MC+	24.37%	22.47%	21.40%	20.51%	19.94%

MC- and MC+ are the bounds of the 95%-confidence interval of the Monte Carlo estimator

Conclusion

- Efficient modelisation for the hybrid model(local volatility plus stochastic rates).
- Non asymptotic estimates expressed by all the model parameters and analysed according to the payoff smoothness.
- Accurate and fast analytical formulas for the price of European options.

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Thank you for your attention!