A Modelisation of Public Private Parternships with failure time

Caroline HILLAIRET *Monique PONTIER*

Ecole Polytechnique, Paris *IMT, Toulouse.*

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Maitrise d'ouvrage publique (MOP) : the community has commissioned equipment (hospital, prison ....) for its own needs and bear the cost, partly by self and partly by a loan from a bank.

"Contrat de partenariat" (partenariat public-privé: PPP) : community agrees on a period (15-25 years) with the contractor, and is billed rent.
$\sim$ "leasing" purchase, covering three parts: depreciation of equipment, maintenance costs, financial costs.

New formula based on an "ordonnance" of June 17, 2004, amended by the law of 28 July 2008 (see legifrance.gouv.fr)
Justification: Emergency of requested equipment construction or its complexity.

Our aim: to study advantages and disadvantages of this new formula. Here is a particular case of a risk-neutral consortium.

We discuss of the advantages of outsourcing (‘externalisation’) in terms of model parameters and prove that:
externality is interesting only when noise is large enough if we exclude the risk of bankruptcy.
Indeed, this represents a transfer of risk from public to private.

Second part: effects of the introduction of a bankruptcy time (this risk does not seem covered under current legislation).

Third part: adding penalities to be paid by the private consortium in case of failure. In such a case, externality could be interesting in some context as high noise, high reference cost, short maturity, high enough penality.

References

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1 The problem

$(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ filtered probability space: underlying hazard.

Operational Cost $(C_s)_{s\in[0,T]}$ infrastructure maintenance service, $\mathbb{F}$-adapted positive process (euro/time unit):

$$C_s = \theta_0 + \varepsilon_s - e_s - \delta a, \quad s \in [0,T], \tag{1.1}$$

- θ_0 = benchmark cost of the maintenance,
- e_s = effort on the maintenance done at date s to reduce the cost, it is a nonnegative **F**-adapted process,
  a = effort on the construction to improve the infrastructure quality, it is a parameter in $\mathbb{R}^+$,
- δ is the externality, it is a parameter in $\mathbb{R}^+$,
- $(\varepsilon_s)_{s\in[0,T]}$ centered bounded **F**-adapted process, modelizes the random operational risk, $\varepsilon \in [-m, M]$, $dt \otimes d\mathbf{P}$ $a.s.$

This maintenance cost supported by the operator until the maturity T (or to a possible bankruptcy), is financed by a debt at a constant rate D (supported by the operator during the contract period).

To the private operator, the public pays a rent $t(c)$ which is a function of costs c to:

. compensate the private operator,

. cover maintenance costs supported by the private operator.

Hypothesis : the public chooses a linear form rule repayment rule

$t(c) = \alpha - \beta c$, with $\beta \geq -1$, and α such that a.s. $t(C_s) \geq C_s \ \forall s \in [0, T]$.

$t(c) - c$ is a decreasing function of costs C_s, thus an increasing function of efforts e_s. More high is β, more the operator is prompted to make efforts for the maintenance of infrastructure, but at the cost of a higher risk premium α.

Remarque 1.1 Let be $\varepsilon_s \in [-m, M] \quad \forall s \in [0, T]$ ($m > 0$, $M > 0$) condition $t(C_s) \geq C_s \ \ \forall s \in [0, T]$ is satisfied as soon as $\alpha \geq (\beta + 1)(\theta_0 + M)$.

Private consortium's aim: maximize total utility minus the costs:

$$\max_{(a,e)\in[0,+\infty[\times E} \left(\mathbb{E} \left(\int_0^T e^{-rs} \left(U(t(C_s) - C_s, s) - \phi(e_s) \right) ds \right) - \psi(a) \right) \tag{1.2}$$

where $E = \{(e_s)_{s\in[0,T]}\, \mathbb{F}$ adapted s.t. $\forall s \in [0,T]\, e_s \geq 0\, p.s\}$.
Functions ϕ et ψ represent the cost of effort (actually below we only deal with $\phi(x) = \frac{x^2}{2}$ et $\psi(x) = \frac{x^2}{2}$.
U is a utility function which modelize the private part's risk aversion.

Utility function $(t,c) \to U(t,c)$:
(i) $U : [0,T]\times]0,+\infty[\to \mathbb{R}$ is continuous,
(ii) $\forall t \in [0,T]$, $U(t,\cdot)$ is strictly increasing and strictly concave,
(iii) continuous derivatives $\frac{\partial}{\partial t}U$, $\frac{\partial}{\partial c}U$, exist on $[0,T]\times]0,+\infty[$.

Public community's aim: maximise the social welfare=difference between social value of the projet minus payements to the consortium, meaning maximize in $\mathcal{A}$:

$$SW : (\alpha, \beta) \mapsto \left(E \left[B_0 + \int_0^T e^{-rs} b(e_s) ds - \left(\int_0^T e^{-rs} t(C_s) ds - C_0 \right) \right] \right) \tag{1.3}$$

$B_0 \geq 0$, social initial social value of the projet,
$b \in C^2(\mathbb{R}^+, \mathbb{R}^+)$, social value rate $b \in C^2(\mathbb{R}^+, \mathbb{R}^+)$ increasing, (e.g. $b(x) = bx,\ b > 0$),
$\mathcal{A} = \{(\alpha, \beta), \alpha \geq 0, \beta \geq -1 \text{ s.t. } t(C_s) \geq C_s\ \forall s \in [0, T]\}$ so that the consortium is able to hedge the costs;
C_0 initial cost, payable by the consortium.

2 Solving the problem without default

2.1 Maximisation of consortium's utility

Proposition 2.1 *We fix social welfare parameters* (α, β). *Then there exists a unique solution* $(\widehat{a}, \widehat{e_s})$ *of the optimisation problem (1.1), solution of the system*

$$\begin{aligned} e_s &= (\beta+1)U'\left(\alpha - (\beta+1)(\theta_0 - e_s - \delta a + \varepsilon_s)\right) \\ a &= \delta \mathbb{E}\left(\int_0^T e^{-rs} e_s ds\right). \end{aligned} \tag{2.1}$$

Moreover the link between these optimal solutions is decreasing: $a \mapsto e_s(a)$: *more effort the consortium has made to build, less effort he has to make to maintain.*

The existence of a solution within the general framework as well as the uniqueness seem difficult to solve, hence we choose linear utility and social welfare:

$$U(x) = \gamma x,\ \ \gamma > 0;\ \ b(x) = b.x, b > 0.$$

The rent rule being fixed, then the consortium's optimal efforts are

$$\begin{cases} \widehat{e_s} & = \gamma(\beta + 1) \quad \forall s \in [0, T] \\ \widehat{a} & = \delta \widehat{e_s} (\int_0^T e^{-rs} ds) = \delta\gamma(\beta + 1) A_T. \end{cases}$$

2.2 Maximization of the community social welfare

The cost has to be a.s. positive $\Rightarrow$ the following constraint

$$C_s \geq 0 \Longleftrightarrow \theta_0 \geq e_s + \delta a - \varepsilon_s, \; p.s.$$

meaning ($\varepsilon_s \geq -m$ p.s.)

$$C_s \geq 0 \Longleftrightarrow \theta_0 - m \geq \gamma(\beta + 1)(1 + \delta^2 A_T), \tag{2.2}$$

linking δ and β and getting $\theta_0 > m$ since $\beta \geq -1$.

Proposition 2.2 *Recall the social welfare (1.3) :*

$$SW(\alpha,\beta) = \left(E\left[B_0 + \int_0^T e^{-rs} b(e_s)ds - \left(\int_0^T e^{-rs} t(C_s)ds - C_0 \right) \right] \right)$$

with $C_s = \theta_0 - \widehat{e_s} - \delta\widehat{a} + \varepsilon_s$ *et* $t(C_s) = \alpha - \beta C_s$*. Suppose*

$$0 \leq \delta^2 \leq \delta^2_{max} = \frac{\theta_0 - 2m - \gamma(b+1)}{\gamma A_T}, \tag{2.3}$$

then the optimal community policy is

$$\widehat{\beta} = \frac{\gamma b + \theta_0 - \gamma(1+\delta^2 \int_0^T e^{-rs} ds)}{2\gamma(1+\delta^2(\int_0^T e^{-rs} ds))} \text{ or } \gamma(1+\delta^2 \int_0^T e^{-rs} ds) = \frac{\gamma b + \theta_0}{2\widehat{\beta}+1},$$

$$\widehat{\alpha} = \frac{\gamma b + \theta_0 + \gamma(1+\delta^2 \int_0^T e^{-rs} ds)}{2\gamma(1+\delta^2(\int_0^T e^{-rs} ds))} (M - b\gamma - \gamma(1+\delta^2 \int_0^T e^{-rs} ds)). \tag{2.4}$$

Hypothesis (2.3) $\Rightarrow$ the benchmark cost θ_0 is bounded from below, otherwise negative costs can occur.

$\widehat{\beta}$ is a decreasing function of outsourcing coefficient δ, satisfying

$$\frac{b\gamma + m}{\theta_0 - 2m - b\gamma} < \widehat{\beta} \leq \frac{b\gamma + \theta_0 - \gamma}{2\gamma} \quad :$$

the upper limit corresponds to $\delta = 0$, no outsourcing,
the lower limit corresponds to δ maximum (2.3).
The study of the impact of the outsourcing δ on the welfare can be made through the influence of $\beta \mapsto SW(\alpha(\beta), \beta)$ where we replace δ depending on β using (2.4).

Proposition 2.3 *When $b\gamma - \frac{\gamma^2}{b\gamma+\theta_0} \leq M$ i.e. in case of noise high enough, the social welfare is optimal for maximum outsourcing,* $\hat{\delta} = \sqrt{\frac{\theta_0 - 2m - \gamma(b+1)}{\gamma A_T}}$.
In case of lower noise, the social welfare is optimal for $\hat{\delta} = \delta_{max}$ if $SW(\beta_{max}) < SW(\beta_{min})$, $\hat{\delta} = 0$ if $SW(\beta_{max}) > SW(\beta_{min})$.

In summary, the outsourcing is interesting only when noise is large enough when we exclude the risk of bankruptcy.

Indeed, this represents a transfer of risk from the community to the consortium.

3 Introduction of a failure time without penalty

Linear utilities: $U(x) = \gamma x,\ \gamma > 0;\ b(x) = bx,\ b > 0.$
We suppose process C is a semi-martingale:
$dC_s = (\theta_0 - e_s - \delta a)ds + \sigma dW_s$, (Brownian motion W)
Rent rule: $dt(C_s) = \alpha ds - \beta dC_s$.
Failure time τ when the consortium can not refund the debt anymore (rate $D : dD_s = De^{-rs}ds$ to deduce of utility as the costs).

Notation :

$$A_t := \int_0^t e^{-rs}ds, \text{ when } r = 0,\ A_t = t,$$

3.1 Utility maximisation for the consortium

$$(e,a,\tau)\mapsto \mathbb{E}\left(\int_0^{\tau\wedge T} e^{-rs}(\gamma[dt(C_s)-dC_s-dD_s]-\frac{1}{2}e_s^2)\right)-\frac{1}{2}a^2=$$

$$\mathbb{E}\left(\int_0^{\tau\wedge T} e^{-rs}[\gamma(\alpha-D)-\gamma(\beta+1)(\theta_0-e_s-\delta a)-\frac{1}{2}e_s^2]ds\right)-\frac{1}{2}a^2$$

Proposition 3.1 *In the case of a risk neutral consortium, under the hypothesis: the initial effort can't depend of the time failure (unknown on initial time), then the consortium's optimal effort is*

$$\begin{cases} \widehat{e_s} & =\gamma(\beta+1)1_{[0,\tau\wedge T]}(s) \\ \widehat{a} & =\delta\gamma(\beta+1)A_T. \end{cases}$$

3.2 Failure time definition and community's optimisation problem

Fund for initial funding is $F = \int_0^T e^{-rs} D ds = DA_T$.
The consortium must be able to hedge costs plus debt, meaning $dt \otimes d\mathbb{P}$ p.s. $F + t(C_t) - C_t - D_t \geq 0$, or

$$Y_t = F + \int_0^t e^{-rs}(\alpha - D - (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T))ds - \int_0^t e^{-rs}(\beta+1)\sigma dW_s \geq 0$$

thus $\tau = \inf\{t : \; Y_t < 0\}$:

$$\tau = \inf\{t : \; \int_0^t e^{-rs}(\beta+1)\sigma dW_s > F + (\alpha - D - (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T))A_t\}.$$

Notation: $K := D + (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T))$.

Proposition 3.2 *For any $r \geq 0$, the default time being postponed as longer as possible, the consortium optimal policy $(\widehat{e_s}, \widehat{a})$ being established, the PPP contract requires nonnegative (in expectation) operational cost and rent.*
Then the optimal rent rule and the optimal externality are

$$\alpha^* = D, \ -1 < \beta^* = \frac{\theta_0}{\gamma} - 1, \ \delta^* = 0. \tag{3.1}$$

Outsourcing is not optimal in this case.

Interpretation: if no penalty in case of a default, the community optimal policy is to outsource the less possible (and MOP are better and more secure than PPP).
Furthermore, at the optimum, $\mathbf{E}(C_s) = 0$ and the rent is $t(C_s) - C_s = D - \frac{\theta_0}{\gamma} C_s$, $\mathbf{E}(t(C_s) - C_s) = D \Rightarrow$ the rent coincides, in expectation, to the refund of the consortium debt.

In conclusion, whatever the interest rate is, outsourcing is NEVER optimal if we consider the possibility that the consortium defaults and if no penalty is administered in case of default.

Given the maturity of PPP contract, it is obvious that we have to take into account the possibility of default. We will now focus on finding some case where outsourcing is attractive if a penalty is administered in case of default.

4 Penalty in case of default with $r = 0$

Here we add in Section 3 model a penalty $\rho V(T-t)^+$ that the consortium should pay in case of default, whereas the community receives the compensation $\rho' V(T-t)^+$.
Natural constraint $\rho V \leq D$, denote $\rho := \rho' + \varepsilon$ where εV is used to pay the liquidation cost:

$$\rho V \leq D \ ; \ \rho = \rho' + \varepsilon, \ \varepsilon > 0. \tag{4.1}$$

We consider the rent $dt(C_s) = \alpha ds - \beta dC_s$, thus the consortium optimal policy remains the following.

Proposition 4.1 *Considering the rent dynamic* $dt(C_s) = \alpha ds - \beta dC_s$ *and the operational cost dynamic* $dC_s = (\theta_0 - e_s - \delta a)ds + \sigma dW_s$*, the consortium optimal policy is*

$$\hat{e}_t = \gamma(\beta + 1)1_{[0,\tau]}(t), \;\; \hat{a} = \gamma(\beta + 1)\delta T.$$

The default time is now defined as

$$\tau = \inf\{t : \;(\beta + 1)\sigma W_t > DT + (\alpha - K)t - \rho V(T - t)^+\} \wedge T\}.$$

Thus $\tau = \tilde{\tau} \wedge T$ with

$$\tilde{\tau} := \inf\{t : \;(\beta + 1)\sigma W_t > DT + (\alpha - K + \rho V)t - \rho VT\}.$$

4.1 Constraints on the parameters

As in Section 3, we choose to postponed the default as longer as possible, for both consortium and community interest :

$$\alpha \geq K - \rho V = D + (\beta + 1)(\theta_0 - \gamma(\beta + 1)(1 + \delta^2 T)) - \rho V.$$

Using the fact that the operational cost and the rent are nonnegative, we precise the constraints on the parameters.

Proposition 4.2 *We assume that the operational cost and the rent are nonnegative (in expectation) and we choose the bigger externality satisfying this assumption. Then the optimal parameters* α, β, δ *satisfy the following constraints :*

$$\begin{aligned} \gamma(\beta+1)(1+\delta^2 T) &= \theta_0, && (4.2) \\ \gamma(\beta+1) &\leq \theta_0, && (4.3) \\ 0 \leq D - \rho V &\leq \alpha < D + b\gamma(\beta+1) - \rho' V, && (4.4) \end{aligned}$$

This last interval is not empty since $\rho > \rho'$ *and* $\gamma(\beta+1) \geq 0$.

Corollary 4.3 *With the choice of a maximal externality, the consortium optimal effort can be written with respect to* (θ_0, δ, T) *:*

$$\hat{e}_t = \frac{\theta_0}{1+\delta^2 T} 1_{[0,\tau]}(t), \;\; \hat{a} = \frac{\theta_0}{1+\delta^2 T}\delta T.$$

In this case $dC_s = \sigma dW_s$ *on* $[0, \tau]$.

4.2 Maximisation of the social welfare

To emphasize the dependency on β, we now denotes $\tilde{\tau}$ by

$$\tau_\beta := \inf\{t : \ (\beta+1)\sigma W_t > (D-\rho V)T + (\alpha - D + \rho V)t\}.$$

Remark $\beta \mapsto \tau_\beta$ is decreasing. Using $(T-\tau)^+ = T - \tilde{\tau}\wedge T$, we express the social welfare SW as a function of β.

Lemma 4.4 *Up to an additive constant, the social welfare is the sum of two functions of* $\beta+1$: $f(\beta+1) = b\gamma(\beta+1)\mathbf{E}[\tilde{\tau}\wedge T]$

$$= b\gamma \int_0^\infty t\wedge T \frac{(D-\rho V)T}{\sigma\sqrt{2\pi t^3}} \exp[-\frac{1}{2t}\left(\frac{(D-\rho V)T - (\alpha - D + \rho V)t}{(\beta+1)\sigma}\right)^2]dt$$

and $g(\beta+1) = [D - \alpha - \rho' V]\mathbf{E}[\tau_\beta \wedge T]$.

The following proposition gives the community optimal policy.

Proposition 4.5 *We assume (4.1), the default is postponed as longer as possible, the consortium optimal policy* $(\widehat{e_s}, \widehat{a})$ *being established, the PPP contract requires nonnegative (in expectation) operational cost and rent* $\Rightarrow$
optimal rent rule and optimal externality are

(i) if $D - \alpha - \rho' V \leq 0$, $\widehat{\beta} = \frac{\theta_0}{\gamma} \Rightarrow$ *same conclusions as in the case with no penalty (3.1).*

(ii) if $D - \alpha - \rho' V > 0$, *we choose* $\widehat{\alpha} = D - \rho V$ *(this does not contradict (ii) since* $\rho' < \rho$*) and we denote* $A = \gamma \frac{(D-\rho V)\sqrt{T}}{\theta_0 \sigma}$. *Then for a "small" A, there exists an optimal* β *strictly less than* $\frac{\theta_0}{\gamma}$, *thus the optimal externality* $\hat{\delta}$ *is strictly positive.*

Remark 4.6 *This condition "*$A = \gamma\frac{(D-\rho V)\sqrt{T}}{\theta_0 \sigma}$ *small enough" is satisfied if*

- the noise level is high (large σ*),*

- the benchmark cost θ_0 *is high,*

- the maturity T *is short,*

- $D - \rho V$ *is small, that is the penalty* ρ *is large enough.*

In this section that better modelizes the reality, we show that outsourcing is attractive for the community in case of high uncertainty or high noise, short maturity, high benchmark operational cost or a sufficently high penalty in case of default.

Three models of PPP contracts have been studied in this paper :

- The first one assumes that there is no default risk and that the contract does not end before maturity.

- The second one introduces the default risk of the consortium, without any compensation for the community in case of an unreciprocated contract breaking-off.

- The third one also considers the default risk of the consortium, and the consortium has to pay penalty in case of default, the community receiving a part of this penalty as a compensation.

In the second model, whatever is the discount rate (positive or zero), the community optimal policy is to give up for outsourcing.

In the first model, outsourcing is optimal if the noise level around the maintenance benchmark cost is higher than a threshold: this corresponds to a risk transfer from the community to the consortium.
Remark that this threshold is an increasing function of the benchmark cost and of the coefficient of the consortium utility.

Similarly, in the third model with penalty in case of default, outsourcing is optimal is the randomness is high enough, or if the contract maturity is short, if the benchmark cost or the penalty are high enough.

Thank you for your attention !!