Model-independent lower bound on variance swaps

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Outline

- Definition and use of variance swaps
- Previous work on pricing of variance swaps
- Our results
- Examples
- Conclusion

Realized volatility and variance

Discrete time

- S is a stock paying no dividends
- Given time-steps $0 = t_0 < \cdots < t_{n+1} = T$, the realized variance over $[0, T]$ is

$$V[\mathbf{t}] = \frac{1}{T} \sum_{i=0}^{n} \ln^2\left(\frac{S_{i+1}}{S_i}\right),$$

  where S_i is the value of S at t_i
- The realized volatility is the square root of the realized variance

Realized volatility and variance

Continuous time

- If the volatility of S is constant and equal to σ, then $V[\mathbf{t}] \to \sigma^2$ when $\sup_i t_{i+1} - t_i \to 0$.
- Under general conditions, if S is continuous with volatility σ_s at time s,

$$V[\mathbf{t}] \to \frac{1}{T} \int_0^T \sigma_s^2 \, ds$$

  when $\sup_i t_{i+1} - t_i \to 0$.

Volatility and variance swaps

- A variance swap with maturity T and strike K is a contract that pays the realized variance minus K at time T.
- Example: if the realized variance is $20\%^2$ and $K = 0.03$ the payoff of the variance swap is 0.01.
- A volatility swap is a forward contract on future realized volatility.
- Both instruments can be used to
  - speculate on future volatility levels
  - hedge the volatility exposure of other positions

Notation and assumptions

- F_0 is the forward price of S
- $C(K)$ is the forward call price on S with strike K and maturity T
- Assume $C(K)$ are known for all K
- A fair strike K_{fair} is a strike such that the value of the variance swap with strike K_{fair} is 0
- Define a *log contract* as a contract that pays at T $2T^{-1}\ln(F_0/S_T)$

Pricing of variance swaps when the stock is continuous

- Under a continuity assumption on the stock price (Dupire 1993, Neuberger 1994)

$$K_{\text{fair}} = V_{\text{log}},$$

  where V_{log} is the forward price of the log contract
- Furthermore

$$V_{\text{log}} = \frac{2}{T} \int_{(0,\infty)} \frac{C(K) - \max(0, F_0 - K)}{K^2}$$

- Thus K_{fair} depends only on $C(K)$, $K > 0$.

Consequences of the relation $K_{\text{fair}} = V_{\text{log}}$

- the calculation of the VIX index
- an analytic approximation of K_{fair} in the presence of a volatility skew (Demeterfi, Derman, Kamal & Zou 1999)
- an efficient forecast of future realized volatility (Jiang & Tian 2005, Becker, Clements & McClelland 2009)

Proof of $K_{\text{fair}} = V_{\text{log}}$

Proof.

Assume $r = 0$.
From $e^z = 1 + z + z^2/2 + O(z^3)$,

$$\frac{S_i}{S_{i-1}} \approx 1 + \ln(\frac{S_i}{S_{i-1}}) + \ln^2(\frac{S_i}{S_{i-1}})/2$$

and so

$$\sum_{i=1}^{n+1}(\frac{S_i - S_{i-1}}{S_{i-1}}) \approx \ln(\frac{S_T}{S_0}) + (T/2)V[\mathbf{t}]$$

Thus, price of derivative paying $V[\mathbf{t}]$ is approximately price of log contract. $\square$

Pricing of variance swaps without the continuity assumption

In the presence of jumps, the price of a variance swap with strike V_{log}

- depends on the jumps size and frequency (Demeterfi, Derman, Kamal & Zou 1999, Broadie & Jain 2008, Carr & Wu 2009)
- can be significantly positive or negative

Our objective

- No assumptions on stock behavior
- What is the supremum lower bound V_{inf} on the hedged payoff, at T, of a long position in the realized variance?
- Equivalently, V_{inf} is the supremum strike K such that an investor with a long position in a variance swap with strike K can ensure to have a non-negative payoff at maturity T.

Notation

- $P(K)$: forward price of put struck at K
- μ is a probability measure on $[0, \infty)$ such that $\mu(\{0\}) = 0$ and $C(K) = \int_{(K,\infty)} (z - K)\, d\mu(z)$
- $I = \{y \geq F_0$ with $C'_-(y) < 0\}$
- For $y \in I$, let $\psi(y)$ be the unique solution to the equation

$$C(y) + (x - y)C'_-(y) = P(x)$$

Preliminary result

Theorem

If

$$\int_{(0,\infty)} \ln^2(x)\, d\mu(x) = \infty$$

then $V_{\text{inf}} = \infty$.

We assume for now on that

$$\int_{(0,\infty)} \ln^2(x)\, d\mu(x) < \infty.$$

V-convex functions

Definition

f is V-convex on $(0,\infty)$ if, for $0 < x < z < y$,

$$\frac{f(x) + \ln^2(x/z) - f(z)}{x - z} \leq \frac{f(y) + \ln^2(y/z) - f(z)}{y - z}.$$

Lemma

Let f be a V-convex function on $(0,\infty)$. For $x, z > 0$,

$$f(x) - f(z) \leq \ln^2(z/x) + f'_+(x)(x - z).$$

Example

The function $f(x) = -\ln^2(x)1_{x\geq 1}$ is V-convex on $(0,\infty)$.

The set of V-convex functions is convex.

Lower bounding technique

Theorem

If f is Lipschitz on $[0,\infty)$, V-convex on $(0,\infty)$ and $f(F_0)=0$, then f is μ-integrable and

$$V_{\text{inf}} \geq -T^{-1}\int_{(0,\infty)} f\,d\mu$$

.

Proof (sketch).

Assume $r=0$.
Let $0=t_0<\cdots<t_{n+1}=T$.
For $1\leq i\leq n+1$, let

$$\xi_i = f'_+(S_{i-1}).$$

Then

$$f(S_{i-1})-f(S_i)\leq \ln^2(S_i/S_{i-1})+\xi_i(S_{i-1}-S_i)$$

and so

$$-f(S_T)\leq T\;V[\mathbf{t}]+\sum_{i=1}^{n+1}\xi_i(S_{i-1}-S_i).$$

$\square$

Further notation

- $b = \max\{x \geq 0 : P(x) = 0\}$
- For $b \leq x < F_0$

  $$\phi(x) = \min\{y \geq F_0 : C(y) + (x - y)C'_+(y) \leq P(x)\}$$

  and

  $$g(x) = 2\int_{(x,F_0)} (x-u)\frac{\ln(\phi(u)/u)}{u(\phi(u)-u)}\, du$$

- For $y \in I$

  $$g(y) = 2\int_{[\psi(y),F_0)} (y-u)\frac{\ln(\phi(u)/u)}{u(\phi(u)-u)}\, du - \ln^2(\frac{y}{\psi(y)})$$

- $g(z) = 0$ if $z \notin [b, F_0) \cup I$

Main result: the general case

Theorem

$$V_{\text{inf}} = -T^{-1} \int_0^\infty g \, d\mu$$

is finite.

- Under general conditions, V_{inf} is the forward price of a European option that pays $-T^{-1} g(S_T)$ at T

Main result: the differentiable case

Theorem

If C is differentiable on (F_0, ∞) then

$$V_{\text{inf}} = T^{-1} \int_I \ln^2 \frac{y}{\psi(y)} \, d\mu(y)$$

- Under general conditions, V_{inf} is the forward price of a European option that pays $T^{-1} \ln^2(S_T / \psi(S_T))$ at T if $S_T \in I$, and 0 otherwise

Comparing V_{inf} and V_{log}

- Let $g_{\text{log}}(z) = 2(\ln(z/F_0) - z/F_0 + 1)$.
- It can be shown that $g(z) \geq g_{\text{log}}(z)$ for $z \geq 0$
- But V_{log} is the forward price of a European option that pays $-T^{-1} g_{\text{log}}(S_T)$ at maturity T and
- V_{inf} is the forward price of a European option that pays $-T^{-1} g(S_T)$ at T.
- Thus $V_{\text{inf}} \leq V_{\text{log}}$ if V_{inf} is finite.

A constant implied volatilities example

Theorem
If the implied volatility is constant and equal to σ for maturity T and all strikes, then $V_{\text{inf}} = \sigma^2 - c\sigma^3\sqrt{T} + O(\sigma^4)$ as $\sigma \to 0$, where

$$c \approx 0.8721.$$

σ	10%	15%	20%	25%	30%	35%
$\sqrt{V_{\text{inf}}}$	9.782%	14.510%	19.129%	23.641%	28.044%	32.340%
$\sigma - \frac{c}{2}\sigma^2\sqrt{T}$	9.782%	14.509%	19.128%	23.637%	28.038%	32.329%

Table: $\sqrt{V_{\text{inf}}}$ and its approximation when $T = 0.25$.

A discrete set of strikes example

Strike	Implied Volatility	Call price	μ	g	$g_{\log}$
35			0.000000	-0.5770	-0.8062
40	37%	60.199502	0.000002	-0.4725	-0.6386
45	36%	55.224448	0.000015	-0.3832	-0.5025
50	35%	50.249468	0.000088	-0.3066	-0.3913
55	34%	45.274923	0.000394	-0.2411	-0.3002
60	33%	40.302340	0.001411	-0.1855	-0.2257
65	32%	35.336777	0.004169	-0.1387	-0.1651
70	31%	30.391954	0.010436	-0.0998	-0.1164
75	30%	25.499053	0.022568	-0.0681	-0.0779
80	29%	20.718432	0.042685	-0.0431	-0.0483
85	28%	16.150173	0.071128	-0.0242	-0.0266
90	27%	11.935778	0.104687	-0.0109	-0.0117
95	26%	8.242208	0.135842	-0.0030	-0.0031
100	25%	5.224458	0.154436	0.0000	0.0000
105	24%	2.975040	0.152160	-0.0019	-0.0019
110	23%	1.482628	0.127838	-0.0081	-0.0084
115	22%	0.626216	0.089568	-0.0179	-0.0190
120	21%	0.215409	0.050812	-0.0304	-0.0334
125	20%	0.057392	0.022458	-0.0447	-0.0512
130	19%	0.011107	0.007359	-0.0597	-0.0723
135	18%	0.001437	0.001677	-0.0743	-0.0963
140	17%	0.000111	0.000245	-0.0865	-0.1231
145	16%	0.000004	0.000021	-0.0947	-0.1524
150			0.000001	-0.0970	-0.1841

Table: The spot price is \$100, implied volatilities are given for European call options with maturity $T = 0.25$ and strikes ranging from \$40 to \$145, spaced \$5 apart. There are no dividends and the continuously compounded risk-free interest rate with maturity T is 2%.

A discrete set of strikes example (cont.)

ΔK	0.1	1	5
V_{inf}	$(23.955\%)^2$	$(23.967\%)^2$	$(24.263\%)^2$
V_{log}	$(25.267\%)^2$	$(25.280\%)^2$	$(25.608\%)^2$

Table: Variance rates in the presence of skew. Strikes range in the interval $[40, 200]$.

ΔK	0	0.1	1	5
V_{inf}	$(23.641\%)^2$	$(23.641\%)^2$	$(23.653\%)^2$	$(23.951\%)^2$
V_{log}	$(25.000\%)^2$	$(25.000\%)^2$	$(25.014\%)^2$	$(25.344\%)^2$

Table: Variance rates with flat volatility. Strikes range in the interval $[40, 200]$.

Conclusion

- We have given a semi-explicit expression for V_{inf}
- We made no continuity assumptions on S
- We made no assumptions on time division
- A simple approximation for V_{inf} has been given when the implied volatilities for maturity T are constant
- A numerical example with a discrete set of strikes and a volatility skew has been treated

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