Model-independent lower bound on variance swaps

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Outline

Definition and use of variance swaps

Previous work on pricing of variance swaps

Our results

Examples

Conclusion
S is a stock paying no dividends

Given time-steps $0 = t_0 < \cdots < t_{n+1} = T$, the realized variance over $[0, T]$ is

$$V(t) = \frac{1}{T} \sum_{i=0}^{n} \ln^2 \left( \frac{S_{i+1}}{S_i} \right),$$

where $S_i$ is the value of $S$ at $t_i$

The realized volatility is the square root of the realized variance
If the volatility of $S$ is constant and equal to $\sigma$, then
$V[t] \to \sigma^2$ when $\sup_i t_{i+1} - t_i \to 0$.

Under general conditions, if $S$ is continuous with volatility $\sigma_s$ at time $s$,
$V[t] \to \frac{1}{T} \int_0^T \sigma_s^2 ds$
when $\sup_i t_{i+1} - t_i \to 0$. 
Volatility and variance swaps

- A variance swap with maturity $T$ and strike $K$ is a contract that pays the realized variance minus $K$ at time $T$.
- Example: if the realized variance is $20\%^2$ and $K = 0.03$ the payoff of the variance swap is $0.01$.
- A volatility swap is a forward contract on future realized volatility.
- Both instruments can be used to
  - speculate on future volatility levels
  - hedge the volatility exposure of other positions
Notation and assumptions

- $F_0$ is the forward price of $S$
- $C(K)$ is the forward call price on $S$ with strike $K$ and maturity $T$
- Assume $C(K)$ are known for all $K$
- A fair strike $K_{\text{fair}}$ is a strike such that the value of the variance swap with strike $K_{\text{fair}}$ is 0
- Define a log contract as a contract that pays at $T$
  \[ 2T^{-1} \ln\left(\frac{F_0}{S_T}\right) \]
Pricing of variance swaps when the stock is continuous

- Under a continuity assumption on the stock price (Dupire 1993, Neuberger 1994)

\[ K_{\text{fair}} = V_{\text{log}}, \]

where \( V_{\text{log}} \) is the forward price of the log contract

- Furthermore

\[ V_{\text{log}} = 2 \int_{(0,\infty)} \frac{C(K) - \max(0, F_0 - K)}{K^2} \]

- Thus \( K_{\text{fair}} \) depends only on \( C(K) \), \( K > 0 \).
Consequences of the relation $K_{\text{fair}} = V_{\log}$

- the calculation of the VIX index
- an analytic approximation of $K_{\text{fair}}$ in the presence of a volatility skew (Demeterfi, Derman, Kamal & Zou 1999)
- an efficient forecast of future realized volatility (Jiang & Tian 2005, Becker, Clements & McClelland 2009)
Proof of $K_{\text{fair}} = V_{\log}$

Proof.
Assume $r = 0$.
From $e^z = 1 + z + z^2/2 + O(z^3)$,

$$\frac{S_i}{S_{i-1}} \approx 1 + \ln\left(\frac{S_i}{S_{i-1}}\right) + \ln^2\left(\frac{S_i}{S_{i-1}}\right)/2$$

and so

$$\sum_{i=1}^{n+1} \left(\frac{S_i - S_{i-1}}{S_{i-1}}\right) \approx \ln\left(\frac{S_T}{S_0}\right) + (T/2)V[t]$$

Thus, price of derivative paying $V[t]$ is approximately price of log contract. \qed
Pricing of variance swaps without the continuity assumption

In the presence of jumps, the price of a variance swap with strike \( V_{\log} \)

- depends on the jumps size and frequency (Demeterfi, Derman, Kamal & Zou 1999, Broadie & Jain 2008, Carr & Wu 2009)
- can be significantly positive or negative
Our objective

- No assumptions on stock behavior
- What is the supremum lower bound $V_{\text{inf}}$ on the hedged payoff, at $T$, of a long position in the realized variance?
- Equivalently, $V_{\text{inf}}$ is the supremum strike $K$ such that an investor with a long position in a variance swap with strike $K$ can ensure to have a non-negative payoff at maturity $T$. 

- $P(K)$: forward price of put struck at $K$
- $\mu$ is a probability measure on $[0, \infty)$ such that $\mu(\{0\}) = 0$ and $C(K) = \int_{(k, \infty)}(z - K)\, d\mu(z)$
- $I = \{y \geq F_0 \text{ with } C_-(y) < 0\}$
- For $y \in I$, let $\psi(y)$ be the unique solution to the equation
  \[
  C(y) + (x - y)C_-(y) = P(x)
  \]
Theorem

If

$$\int_{(0,\infty)} \ln^2(x) \, d\mu(x) = \infty$$

then $V_{\text{inf}} = \infty$.

We assume for now on that

$$\int_{(0,\infty)} \ln^2(x) \, d\mu(x) < \infty.$$
V-convex functions

Definition
$f$ is V-convex on $(0, \infty)$ if, for $0 < x < z < y$,

$$
\frac{f(x) + \ln^2(x/z) - f(z)}{x - z} \leq \frac{f(y) + \ln^2(y/z) - f(z)}{y - z}.
$$

Lemma
Let $f$ be a V-convex function on $(0, \infty)$. For $x, z > 0$,

$$
f(x) - f(z) \leq \ln^2(z/x) + f'_+(x)(x - z).
$$

Example
The function $f(x) = -\ln^2(x)1_{x \geq 1}$ is V-convex on $(0, \infty)$.

The set of V-convex functions is convex.
Lower bounding technique

Theorem

If $f$ is Lipschitz on $[0, \infty)$, $V$-convex on $(0, \infty)$ and $f(F_0) = 0$, then $f$ is $\mu$-integrable and

$$V_{\text{inf}} \geq -T^{-1} \int_{(0,\infty)} f \, d\mu.$$

Proof (sketch).

Assume $r = 0$.

Let $0 = t_0 < \cdots < t_{n+1} = T$.

For $1 \leq i \leq n + 1$, let

$$\xi_i = f'(S_{i-1}).$$

Then

$$f(S_{i-1}) - f(S_i) \leq \ln^2(S_i/S_{i-1}) + \xi_i(S_{i-1} - S_i)$$

and so

$$-f(S_T) \leq T \, V[t] + \sum_{i=1}^{n+1} \xi_i(S_{i-1} - S_i).$$

$\square$
Further notation

- \( b = \max \{ x \geq 0 : P(x) = 0 \} \)
- For \( b \leq x < F_0 \)
  \[ \phi(x) = \min \{ y \geq F_0 : C(y) + (x - y)C'(y) \leq P(x) \} \]
  and
  \[ g(x) = 2 \int_{(x,F_0)} (x - u) \frac{\ln(\phi(u)/u)}{u(\phi(u) - u)} \, du \]
- For \( y \in I \)
  \[ g(y) = 2 \int_{(\psi(y),F_0)} (y - u) \frac{\ln(\phi(u)/u)}{u(\phi(u) - u)} \, du - \ln(\frac{y}{\psi(y)}) \]
- \( g(z) = 0 \) if \( z \notin [b,F_0) \cup I \)
Main result: the general case

Theorem

\[ V_{\text{inf}} = -T^{-1} \int_{0}^{\infty} g \, d\mu \]

is finite.

- Under general conditions, \( V_{\text{inf}} \) is the forward price of a European option that pays \( -T^{-1}g(S_T) \) at \( T \)
Theorem

If $C$ is differentiable on $(F_0, \infty)$ then

$$V_{\text{inf}} = T^{-1} \int_I \ln^2 \frac{y}{\psi(y)} d\mu(y)$$

Under general conditions, $V_{\text{inf}}$ is the forward price of a European option that pays $T^{-1} \ln^2 (S_T/\psi(S_T))$ at $T$ if $S_T \in I$, and 0 otherwise.
Comparing $V_{\text{inf}}$ and $V_{\text{log}}$

- Let $g_{\text{log}}(z) = 2(\ln(z/F_0) - z/F_0 + 1)$.
- It can be shown that $g(z) \geq g_{\text{log}}(z)$ for $z \geq 0$.
- But $V_{\text{log}}$ is the forward price of a European option that pays $-T^{-1}g_{\text{log}}(S_T)$ at maturity $T$.
- $V_{\text{inf}}$ is the forward price of a European option that pays $-T^{-1}g(S_T)$ at $T$.
- Thus $V_{\text{inf}} \leq V_{\text{log}}$ if $V_{\text{inf}}$ is finite.
Theorem

If the implied volatility is constant and equal to \( \sigma \) for maturity \( T \) and all strikes, then \( V_{inf} = \sigma^2 - c\sigma^3\sqrt{T} + O(\sigma^4) \) as \( \sigma \to 0 \), where

\[
c \approx 0.8721.
\]

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>10%</th>
<th>15%</th>
<th>20%</th>
<th>25%</th>
<th>30%</th>
<th>35%</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{V_{inf}} )</td>
<td>9.782%</td>
<td>14.510%</td>
<td>19.129%</td>
<td>23.641%</td>
<td>28.044%</td>
<td>32.340%</td>
</tr>
<tr>
<td>( \sigma - \sqrt{V_{inf}} )</td>
<td>9.782%</td>
<td>14.509%</td>
<td>19.128%</td>
<td>23.637%</td>
<td>28.038%</td>
<td>32.329%</td>
</tr>
</tbody>
</table>

Table: \( \sqrt{V_{inf}} \) and its approximation when \( T = 0.25 \).
A discrete set of strikes example

<table>
<thead>
<tr>
<th>Strike</th>
<th>Implied Volatility</th>
<th>Call price</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\sigma_{log}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>37%</td>
<td>55.199502</td>
<td>0.000002</td>
<td>-0.5770</td>
<td>-0.9062</td>
</tr>
<tr>
<td>40</td>
<td>36%</td>
<td>56.224448</td>
<td>0.000015</td>
<td>-0.5838</td>
<td>-0.9025</td>
</tr>
<tr>
<td>45</td>
<td>35%</td>
<td>51.248868</td>
<td>0.000088</td>
<td>-0.5066</td>
<td>-0.8123</td>
</tr>
<tr>
<td>50</td>
<td>35%</td>
<td>46.274520</td>
<td>0.000379</td>
<td>-0.2417</td>
<td>-0.3020</td>
</tr>
<tr>
<td>55</td>
<td>34%</td>
<td>40.302540</td>
<td>0.001411</td>
<td>-0.1855</td>
<td>-0.2557</td>
</tr>
<tr>
<td>60</td>
<td>33%</td>
<td>35.338777</td>
<td>0.004189</td>
<td>-0.1387</td>
<td>-0.1651</td>
</tr>
<tr>
<td>65</td>
<td>32%</td>
<td>30.391854</td>
<td>0.010436</td>
<td>-0.0999</td>
<td>-0.1164</td>
</tr>
<tr>
<td>70</td>
<td>31%</td>
<td>25.499063</td>
<td>0.022558</td>
<td>-0.0681</td>
<td>-0.0779</td>
</tr>
<tr>
<td>75</td>
<td>30%</td>
<td>20.718432</td>
<td>0.042895</td>
<td>-0.0431</td>
<td>-0.0483</td>
</tr>
<tr>
<td>80</td>
<td>29%</td>
<td>16.150172</td>
<td>0.071128</td>
<td>-0.0342</td>
<td>-0.0295</td>
</tr>
<tr>
<td>85</td>
<td>28%</td>
<td>11.935778</td>
<td>0.104687</td>
<td>-0.0119</td>
<td>-0.0117</td>
</tr>
<tr>
<td>90</td>
<td>27%</td>
<td>8.242028</td>
<td>0.135842</td>
<td>-0.0030</td>
<td>-0.0031</td>
</tr>
<tr>
<td>95</td>
<td>26%</td>
<td>5.224508</td>
<td>0.164436</td>
<td>-0.0090</td>
<td>-0.0090</td>
</tr>
<tr>
<td>100</td>
<td>25%</td>
<td>2.972460</td>
<td>0.192860</td>
<td>-0.0019</td>
<td>-0.0019</td>
</tr>
<tr>
<td>105</td>
<td>24%</td>
<td>1.682628</td>
<td>0.227328</td>
<td>-0.0081</td>
<td>-0.0084</td>
</tr>
<tr>
<td>110</td>
<td>23%</td>
<td>0.922616</td>
<td>0.268508</td>
<td>-0.0179</td>
<td>-0.0180</td>
</tr>
<tr>
<td>115</td>
<td>22%</td>
<td>0.315609</td>
<td>0.308112</td>
<td>-0.0304</td>
<td>-0.0304</td>
</tr>
<tr>
<td>120</td>
<td>21%</td>
<td>0.087392</td>
<td>0.325458</td>
<td>-0.0447</td>
<td>-0.0452</td>
</tr>
<tr>
<td>125</td>
<td>20%</td>
<td>0.021511</td>
<td>0.338859</td>
<td>-0.0597</td>
<td>-0.0601</td>
</tr>
<tr>
<td>130</td>
<td>19%</td>
<td>0.001437</td>
<td>0.351677</td>
<td>-0.0743</td>
<td>-0.0743</td>
</tr>
<tr>
<td>135</td>
<td>18%</td>
<td>0.000101</td>
<td>0.363485</td>
<td>-0.0886</td>
<td>-0.0886</td>
</tr>
<tr>
<td>140</td>
<td>17%</td>
<td>0.000021</td>
<td>0.374425</td>
<td>-0.1027</td>
<td>-0.1027</td>
</tr>
<tr>
<td>145</td>
<td>16%</td>
<td>0.000004</td>
<td>0.384521</td>
<td>-0.1164</td>
<td>-0.1164</td>
</tr>
<tr>
<td>150</td>
<td>15%</td>
<td>0.000001</td>
<td>0.393750</td>
<td>-0.1297</td>
<td>-0.1297</td>
</tr>
</tbody>
</table>

Table: The spot price is $100, implied volatilities are given for European call options with maturity $T = 0.25$ and strikes ranging from $40$ to $145$, spaced $5$ apart. There are no dividends and the continuously compounded risk-free interest rate with maturity $T$ is 2%.
A discrete set of strikes example (cont.)

<table>
<thead>
<tr>
<th>$\Delta K$</th>
<th>0.1</th>
<th>1</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{\text{inf}}$</td>
<td>$(23.955%)^2$</td>
<td>$(23.967%)^2$</td>
<td>$(24.263%)^2$</td>
</tr>
<tr>
<td>$V_{\text{log}}$</td>
<td>$(25.267%)^2$</td>
<td>$(25.280%)^2$</td>
<td>$(25.608%)^2$</td>
</tr>
</tbody>
</table>

**Table:** Variance rates in the presence of skew. Strikes range in the interval $[40, 200]$.

<table>
<thead>
<tr>
<th>$\Delta K$</th>
<th>0</th>
<th>0.1</th>
<th>1</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{\text{inf}}$</td>
<td>$(23.641%)^2$</td>
<td>$(23.641%)^2$</td>
<td>$(23.653%)^2$</td>
<td>$(23.951%)^2$</td>
</tr>
<tr>
<td>$V_{\text{log}}$</td>
<td>$(25.000%)^2$</td>
<td>$(25.000%)^2$</td>
<td>$(25.014%)^2$</td>
<td>$(25.344%)^2$</td>
</tr>
</tbody>
</table>

**Table:** Variance rates with flat volatility. Strikes range in the interval $[40, 200]$. 
Conclusion

- We have given a semi-explicit expression for $V_{inf}$
- We made no continuity assumptions on $S$
- We made no assumptions on time division
- A simple approximation for $V_{inf}$ has been given when the implied volatilities for maturity $T$ are constant
- A numerical example with a discrete set of strikes and a volatility skew has been treated


