Model-independent lower bound on variance swaps

Nabil Kahale\textsuperscript{1}

\textsuperscript{1}ESCP Europe
Paris

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Outline

Definition and use of variance swaps

Previous work on pricing of variance swaps

Our results

Examples

Conclusion
- \( S \) is a stock paying no dividends
- Given time-steps \( 0 = t_0 < \cdots < t_{n+1} = T \), the realized variance over \([0, T]\) is

\[
V[t] = \frac{1}{T} \sum_{i=0}^{n} \ln^2 \left( \frac{S_{i+1}}{S_i} \right),
\]

where \( S_i \) is the value of \( S \) at \( t_i \)
- The realized volatility is the square root of the realized variance
If the volatility of $S$ is constant and equal to $\sigma$, then $V(t) \rightarrow \sigma^2$ when $\sup_i t_{i+1} - t_i \rightarrow 0$.

Under general conditions, if $S$ is continuous with volatility $\sigma_s$ at time $s$, $V(t) \rightarrow 1 \int_0^T \sigma_s^2 \, ds$ when $\sup_i t_{i+1} - t_i \rightarrow 0$. 
A variance swap with maturity $T$ and strike $K$ is a contract that pays the realized variance minus $K$ at time $T$.

Example: if the realized variance is $20\%^2$ and $K = 0.03$ the payoff of the variance swap is 0.01.

A volatility swap is a forward contract on future realized volatility.

Both instruments can be used to
- speculate on future volatility levels
- hedge the volatility exposure of other positions
Notation and assumptions

- $F_0$ is the forward price of $S$
- $C(K)$ is the forward call price on $S$ with strike $K$ and maturity $T$
- Assume $C(K)$ are known for all $K$
- A fair strike $K_{\text{fair}}$ is a strike such that the value of the variance swap with strike $K_{\text{fair}}$ is 0
- Define a log contract as a contract that pays at $T$
  \[ 2T^{-1} \ln\left(\frac{F_0}{S_T}\right) \]
Pricing of variance swaps when the stock is continuous

- Under a continuity assumption on the stock price (Dupire 1993, Neuberger 1994)

\[ K_{\text{fair}} = V_{\text{log}}, \]

where \( V_{\text{log}} \) is the forward price of the log contract

- Furthermore

\[ V_{\text{log}} = \frac{2}{T} \int_{(0, \infty)} \frac{C(K) - \max(0, F_0 - K)}{K^2} \]

- Thus \( K_{\text{fair}} \) depends only on \( C(K), K > 0 \).
Consequences of the relation $K_{\text{fair}} = V_{\log}$

- the calculation of the VIX index
- an analytic approximation of $K_{\text{fair}}$ in the presence of a volatility skew (Demeterfi, Derman, Kamal & Zou 1999)
- an efficient forecast of future realized volatility (Jiang & Tian 2005, Becker, Clements & McClelland 2009)
Proof of $K_{\text{fair}} = V_{\text{log}}$

Proof.
Assume $r = 0$.
From $e^z = 1 + z + z^2/2 + O(z^3)$,
\[
\frac{S_i}{S_{i-1}} \approx 1 + \ln\left(\frac{S_i}{S_{i-1}}\right) + \frac{\ln^2\left(\frac{S_i}{S_{i-1}}\right)}{2}
\]
and so
\[
\sum_{i=1}^{n+1} \left(\frac{S_i - S_{i-1}}{S_{i-1}}\right) \approx \ln\left(\frac{S_T}{S_0}\right) + \frac{(T/2)}{V[t]}
\]
Thus, price of derivative paying $V[t]$ is approximately price of log contract. □
In the presence of jumps, the price of a variance swap with strike $V_{\log}$
- depends on the jumps size and frequency (Demeterfi, Derman, Kamal & Zou 1999, Broadie & Jain 2008, Carr & Wu 2009)
- can be significantly positive or negative
Our objective

- No assumptions on stock behavior
- What is the supremum lower bound $V_{\text{inf}}$ on the hedged payoff, at $T$, of a long position in the realized variance?
- Equivalently, $V_{\text{inf}}$ is the supremum strike $K$ such that an investor with a long position in a variance swap with strike $K$ can ensure to have a non-negative payoff at maturity $T$. 
- $P(K)$: forward price of put struck at $K$ 
- $\mu$ is a probability measure on $[0, \infty)$ such that $\mu(\{0\}) = 0$ and $C(K) = \int_{[K, \infty)} (z - K) \, d\mu(z)$ 
- $I = \{y \geq F_0 \text{ with } C'_-(y) < 0\}$ 
- For $y \in I$, let $\psi(y)$ be the unique solution to the equation 

$$C(y) + (x - y)C'_-(y) = P(x)$$
Theorem

If

\[ \int_{(0, \infty)} \ln^2(x) \, d\mu(x) = \infty \]

then \( V_{\text{inf}} = \infty \).

We assume for now on that

\[ \int_{(0, \infty)} \ln^2(x) \, d\mu(x) < \infty. \]
V-convex functions

**Definition**

$f$ is V-convex on $(0, \infty)$ if, for $0 < x < z < y$, 

\[
\frac{f(x) + \ln^2(x/z) - f(z)}{x - z} \leq \frac{f(y) + \ln^2(y/z) - f(z)}{y - z}.
\]

**Lemma**

*Let $f$ be a V-convex function on $(0, \infty)$. For $x, z > 0$, 

\[
f(x) - f(z) \leq \ln^2(z/x) + f'(x)(x - z).
\]*

**Example**

The function $f(x) = -\ln^2(x)1_{x \geq 1}$ is V-convex on $(0, \infty)$.

The set of V-convex functions is convex.
Theorem

If $f$ is Lipschitz on $[0, \infty)$, $V$-convex on $(0, \infty)$ and $f(F_0) = 0$, then $f$ is $\mu$-integrable and

$$V_{\inf} \geq -T^{-1} \int_{(0, \infty)} f \, d\mu.$$ 

Proof (sketch).

Assume $r = 0$.

Let $0 = t_0 < \cdots < t_{n+1} = T$.

For $1 \leq i \leq n+1$, let

$$\xi_i = f'(S_{i-1}).$$

Then

$$f(S_{i-1}) - f(S_i) \leq \ln^2(S_i/S_{i-1}) + \xi_i(S_{i-1} - S_i)$$

and so

$$-f(S_T) \leq T \cdot V[t] + \sum_{i=1}^{n+1} \xi_i(S_{i-1} - S_i).$$
Further notation

- \( b = \max\{x \geq 0 : P(x) = 0\}\)
- For \( b \leq x < F_0 \)
  \[
  \phi(x) = \min\{y \geq F_0 : C(y) + (x - y)C'_+(y) \leq P(x)\}
  \]
  and
  \[
  g(x) = 2 \int_{(x,F_0)} (x - u) \frac{\ln(\phi(u)/u)}{u(\phi(u) - u)} \, du
  \]
- For \( y \in I \)
  \[
  g(y) = 2 \int_{[\psi(y),F_0]} (y - u) \frac{\ln(\phi(u)/u)}{u(\phi(u) - u)} \, du - \ln^2\left(\frac{y}{\psi(y)}\right)
  \]
- \( g(z) = 0 \) if \( z \notin [b, F_0) \cup I \)
Main result: the general case

Theorem

\[ V_{\text{inf}} = -T^{-1} \int_{0}^{\infty} g \, d\mu \]

is finite.

- Under general conditions, \( V_{\text{inf}} \) is the forward price of a European option that pays \( -T^{-1}g(S_T) \) at \( T \).
Main result: the differentiable case

**Theorem**

If $C$ is differentiable on $(F_0, \infty)$ then

$$V_{\text{inf}} = T^{-1} \int_l \ln \frac{y}{\psi(y)} d\mu(y)$$

Under general conditions, $V_{\text{inf}}$ is the forward price of a European option that pays $T^{-1} \ln^2(S_T/\psi(S_T))$ at $T$ if $S_T \in l$, and 0 otherwise.
Let $g_{\log}(z) = 2(\ln(z/F_0) - z/F_0 + 1)$.

It can be shown that $g(z) \geq g_{\log}(z)$ for $z \geq 0$.

But $V_{\log}$ is the forward price of a European option that pays $-T^{-1}g_{\log}(S_T)$ at maturity $T$ and $V_{\inf}$ is the forward price of a European option that pays $-T^{-1}g(S_T)$ at $T$.

Thus $V_{\inf} \leq V_{\log}$ if $V_{\inf}$ is finite.
Theorem
If the implied volatility is constant and equal to $\sigma$ for maturity $T$ and all strikes, then $V_{\text{inf}} = \sigma^2 - c\sigma^3\sqrt{T} + O(\sigma^4)$ as $\sigma \to 0$, where

$$c \approx 0.8721.$$
Table: The spot price is $100, implied volatilities are given for European call options with maturity $T = 0.25$ and strikes ranging from $40$ to $145$, spaced $5$ apart. There are no dividends and the continuously compounded risk-free interest rate with maturity $T$ is 2%.
A discrete set of strikes example (cont.)

<table>
<thead>
<tr>
<th>ΔK</th>
<th>0.1</th>
<th>1</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{inf}$</td>
<td>(23.955%)^2</td>
<td>(23.967%)^2</td>
<td>(24.263%)^2</td>
</tr>
<tr>
<td>$V_{log}$</td>
<td>(25.267%)^2</td>
<td>(25.280%)^2</td>
<td>(25.608%)^2</td>
</tr>
</tbody>
</table>

**Table:** Variance rates in the presence of skew. Strikes range in the interval [40, 200].

<table>
<thead>
<tr>
<th>ΔK</th>
<th>0</th>
<th>0.1</th>
<th>1</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{inf}$</td>
<td>(23.641%)^2</td>
<td>(23.641%)^2</td>
<td>(23.653%)^2</td>
<td>(23.951%)^2</td>
</tr>
<tr>
<td>$V_{log}$</td>
<td>(25.000%)^2</td>
<td>(25.000%)^2</td>
<td>(25.014%)^2</td>
<td>(25.344%)^2</td>
</tr>
</tbody>
</table>

**Table:** Variance rates with flat volatility. Strikes range in the interval [40, 200].
Conclusion

- We have given a semi-explicit expression for $V_{inf}$
- We made no continuity assumptions on $S$
- We made no assumptions on time division
- A simple approximation for $V_{inf}$ has been given when the implied volatilities for maturity $T$ are constant
- A numerical example with a discrete set of strikes and a volatility skew has been treated


