Hedging Contingent Claims in Incomplete Markets

Non-Quadratic Local Risk-Minimization

Nicolas Millot
nicolas.millot@ecp.fr

BNP Paribas Chair of Quantitative Finance
École Centrale Paris

12th January 2011

Outline

The Problematic

- **Position of the problem** Basic question from the trader (or the risk manager)'s point of view :
  Having sold an option, what initial capital to allocate and which strategy to follow in order to hedge the embedded risk
- **Possible solutions**
  - Static hedging
    Use of a combination of market instruments *approximately* replicating the terminal payout
  - Dynamic hedging
    Continuously rebalance a portfolio of available assets to *produce* the terminal payout

Modelling Framework

Mathematical Framework

We use a probabilistic model to describe the future evolution of assets

- A filtration $(\mathcal{F}_t)_{0 \le t \le T}$ on a probability space $(\Omega, \mathcal{F}, P)$
- N observables S^i which could be assets available for trading, stochastic processes adapted to the filtration $(\mathcal{F}_t)_{0 \le t \le T}$
- A contingent claim H, which given a fixed and finite time horizon, is simply an $\mathcal{F}_T$ measurable random variable

Markets (In)Completeness

Dynamic Hedging

- *Classical* framework : Using self-financing strategies
  - Complete markets : every contingent claim is attainable

$$\begin{aligned} \tilde{H} &= c_0 + \int_0^T \delta_s d\tilde{S} \\ c_0 &= \mathbb{E}^*(\tilde{H}) \end{aligned}$$

  - Incomplete markets : min variance hedging, superhedging, quantile hedging, shortfall minimization, etc.
    e.g.

$$c_0 = \inf\{c | \exists \delta, c + \int_0^T \delta_s d\tilde{S} > \tilde{H} \ \mathcal{P} - ps\}$$

Quadratic Risk-Minimization

Relaxing the self-financing constraint

- Quadratic local risk-minimization :

$$(\delta_k^*, V_k^*) = \text{argmin}\mathbb{E}_k\left(\Delta C_{k+1}(\delta, V)^2\right)$$

  Where $\Delta C_{k+1}(\delta, V) = \Delta V_{k+1} - \delta_k \Delta S_{k+1}$ when there is no transaction costs and no liquidity effects
- Many characterizations available for the optimal strategy, in discrete time and continuous time through martingale orthogonality properties

Beyond Quadratic ?

Quadratic local risk-minimization

- Fruitful framework, many theoretical results, numerical schemes available, see bibliography
- Yet the main interrogation is : what is the rationale behind putting the same weight on gains as on losses ?

Non-Quadratic Local Risk-Minimization

The idea

Relaxing the quadratic hypothesis : using a smooth convex function f to assess risk due to incremental costs

- **One time step setting**

$$(\delta_0^*, V_0^*) = \text{argmin}\mathbb{E}\left(f\left(H - V_0 - \delta_0(S_T - S_0)\right)\right)$$

Characterization of the optimal strategy through first order optimality conditions

$$\begin{aligned} \mathbb{E}\left(f'(H - V_0 - \delta_0(S_T - S_0))\right) &= 0 \\ \mathbb{E}\left(f'(H - V_0 - \delta_0(S_T - S_0))(S_T - S_0)\right) &= 0 \end{aligned}$$

Non-Quadratic Local Risk-Minimization

- **Multiple time steps setting**
  Minimization program working backward : given a contingent claim H, find Φ^*, admissible strategy such that

$$\forall k \in (0, \cdots, T-1),\ \Delta R_k(\Phi) \geq \Delta R_k(\Phi^*) \forall \Phi \text{ admissible,}$$
$$\text{with } \delta_{k+1} = \delta^*_{k+1} \text{ and } \beta_{k+1} = \beta^*_{k+1}$$

- **First order optimality conditions**
  Equivalent characterization : the process $(C^f_k)_{k=1,..,N} = \sum_{i=0}^{k-1} f'(\Delta C_{i+1})$ is a martingale (strongly) orthogonal to the martingale part of the process (S_k)

Continuous Time Limit

- Start with a sequence of partitions $\mathcal{P}_n$ of $[0,T]$ tending to the identity
- Define the f-costs process as the following limit, whenever it exists :

$$\lim_{n\to\infty}\sum_{k=1}^{l_n} f'\left(V^{\tau_k^n} - V^{\tau_{k-1}^n} - \delta^{\tau_{k-1}^n}\left(S^{\tau_k^n} - S^{\tau_{k-1}^n}\right)\right)$$

Where convergence is required in ucp topology

The f-costs Process

- When the process S_t satisfies some regularity constraints and the strategies satisfy some admissibility conditions the limit exists and is equal to :

$$C_t^f = f''(0)\left(V_t - V_0 - \int_{0+}^t \delta_{s-} dS_s\right)$$
$$+\frac{f^{(3)}(0)}{2}\left([V,V]_t^c - 2\int_{0+}^t \delta_{s-} d[V,S]_s^c + \int_{0+}^t \delta_{s-}^2 d[S,S]_t^c\right)$$
$$+\sum\nolimits_{0<s\leq t} f'(\Delta V_s - \delta_s \Delta S_s) - f''(0)(\Delta V_s - \delta_s \Delta S_s)$$

Optimal Strategies

Definition of the (pseudo-)Optimality

- By analogy with the discrete time setting, we define optimal strategies as those strategy such that C_t^f is a martingale orthogonal to the martingale part of the S_t process
- When the strategy is assumed to be a Markov process, this allows to express the optimal strategies as solutions of a non linear parabolic PDE or alternatively of a quadratic BSDE

The Markovian setting

The case of stochastic volatility models

- We model the evolution of S through an SDE with stochastic volatility

$$\begin{aligned} dS_t &= \mu_t dt + \sigma_t dW_t^1 \\ d\sigma_t &= \gamma_t dt + \Sigma_t dW_t^2 \end{aligned}$$

  With μ_t, γ_t and Σ_t adapted processes, and $d < W^1, W^2 >_t = \rho dt$

- Markovian strategies

$$\begin{aligned} \delta_t &= \delta(t, S_t, \sigma_t) \\ V_t &= V(t, S_t, \sigma_t) \end{aligned}$$

The f-costs Process

The f-costs process is then

$$
\begin{aligned}
C_t(\Phi) = \quad & \int_0^t \left[f''(0) \left(\frac{\partial V}{\partial u} + \frac{\partial V}{\partial S}\mu_u + \frac{\partial V}{\partial \sigma}\gamma_u + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_u^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\rho\sigma_u\Sigma_u - \delta_u\mu_u \right) \right. \\
+ \quad & \frac{f^{(3)}(0)}{2}\left(\left(\frac{\partial V}{\partial S}\right)^2 \sigma_u^2 + \left(\frac{\partial V}{\partial \sigma}\right)^2 \Sigma_u^2 + 2\frac{\partial V}{\partial S}\frac{\partial V}{\partial \sigma}\rho\sigma_u\Sigma_u \right) \\
- \quad & \left. f^{(3)}(0)\delta_u \left(\frac{\partial V}{\partial S}\sigma_u^2 + \frac{\partial V}{\partial S}\rho\sigma_u\Sigma_u \right) + \frac{f^{(3)}(0)}{2}\delta_u^2\sigma_u^2 \right] du \\
+ \quad & \int_0^t f''(0)\left(\frac{\partial V}{\partial S} - \delta_u \right)\sigma_u dW_u^1 + \int_0^t f''(0)\frac{\partial V}{\partial \sigma}\Sigma_u dW_u^2
\end{aligned}
$$

The Equation Satisfied by δ

Applying to the strategy Φ the second local risk-minimization criterion gives the equation satisfied by the optimal hedge δ

$$\left(\frac{\partial V}{\partial S} - \delta_u\right)\sigma_u^2 + \frac{\partial V}{\partial \sigma}\rho\sigma_u\Sigma_u = 0$$

- $\Rightarrow$ Linearity of the optimal hedge ratio δ with respect to the portfolio value V
- In the case when f is quadratic and $\Sigma = 0$ we have $\delta = \frac{\partial V}{\partial S}$

The PDE Satisfied by V

Applying to the strategy Φ the first local risk-minimization criterion gives the PDE satisfied by the portfolio value V

$$
\begin{aligned}
f''(0)\left(\frac{\partial V}{\partial u}+\frac{\partial V}{\partial S}\mu_u+\frac{\partial V}{\partial \sigma}\gamma_u+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_u^2+\frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\Sigma_u^2+\frac{\partial^2 V}{\partial S\partial\sigma}\rho\sigma_u\Sigma_u-\delta_u\mu_u\right)& \\
+\frac{f^{(3)}(0)}{2}\left(\left(\frac{\partial V}{\partial S}\right)^2\sigma_u^2+\left(\frac{\partial V}{\partial \sigma}\right)^2\Sigma_u^2+2\frac{\partial V}{\partial S}\frac{\partial V}{\partial \sigma}\rho\sigma_u\Sigma_u\right)& \\
-f^{(3)}(0)\delta_u\left(\frac{\partial V}{\partial S}\sigma_u^2+\frac{\partial V}{\partial S}\rho\sigma_u\Sigma_u\right)+\frac{f^{(3)}(0)}{2}\delta_u^2\sigma_u^2 &= 0
\end{aligned}
$$

- $\Rightarrow$ Non-linearity due to the term in $f^{(3)}(0)$, quadratic growth in the gradient
- In the case when f is quadratic and $\Sigma = 0$ we find the $B\&S$ PDE

Optimality or Pseudo-Optimality ?

- **Small Perturbations** A small perturbation is a bounded admissible strategy $\Delta\Phi = (\beta, \delta)$ such that $\beta_T = 0$ and $\delta_T = 0$
- **Local Risk along a Partition**

$$r_f^{\tau}[\phi, \Delta](t, \omega) = \sum_{t_i, t_{i+1} \in \tau} \frac{\Delta R_{t_i}(\phi + \Delta|_{[t_i, t_{i+1}(})(\omega) - \Delta R_{t_i}(\phi)(\omega)}{t_{i+1} - t_i} 1_{[t_i, t_{i+1}(}(t)$$

Locally Risk-Minimizing Strategies

- **Locally Risk-Minimizing Strategies** An admissible strategy ϕ is called locally risk-minimizing for the option H if for every small perturbation Δ and every increasing sequence of partitions $(\tau_n)_{n\in\mathbb{N}}$ tending to the identity, we have

$$\liminf_{n\to\infty} r^{\tau_n}[\phi,\Delta] \geq 0 \ \mathcal{P} - a.e.$$

- In a Markovian framework with a market described by Itô processes we can show that optimality coincides with pseudo-optimality

Conclusion and Outlooks

Conclusion

- Extending the results of quadratic local risk-minimization to asymmetric risk, more realistic
- Allowing for several characterizations/representations of the optimal strategies with numerical methods available
- Flexibility of the methodology so it can apply to real markets

Outlooks

- Incorporating liquidity costs into the picture : forthcoming paper
- Define suitable approach to value the option from the incurred costs
- More numerical studies to be done for analysing the impact of different choices of risk functions

References I

- F. Abergel and N. Millot, Non quadratic local risk-minimization for hedging contingent claims in incomplete markets, to be published
- F. Abergel and N. Millot, Non Quadratic Locally Risk-Minimizing Strategies : Incorporating Liquidity Costs, forthcoming paper
- A. Diop, Convergence of some random functionals of discretized semimartingales
- N. El Karoui, S. Peng and M. C. Quenez, Backward Stochastic Differential Equations in Finance, *Mathematical Finance*, Vol. 7, No. 1 (January 1997), pp. 1-71

References II

H. Follmer and M. Schweizer, Option Hedging for Semimartingales, *Stochastic Processes and their Applications 37*, 339-363

D. Heath, E. Platen and M. Schweizer, A Comparison of Two Quadratic Approaches to Hedging in Incomplete Markets, *Mathematical Finance 11*, 2001, 385-413