Hedging Contingent Claims in Incomplete Markets
Non-Quadratic Local Risk-Minimization

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Outline

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The Problematic

- **Position of the problem** Basic question from the trader (or the risk manager)’s point of view:
  Having sold an option, what initial capital to allocate and which strategy to follow in order to hedge the embedded risk

- **Possible solutions**
  - Static hedging
    Use of a combination of market instruments *approximately* replicating the terminal payout
  - Dynamic hedging
    Continuously rebalance a portfolio of available assets to *produce* the terminal payout
Mathematical Framework

We use a probabilistic model to describe the future evolution of assets

- A filtration \((\mathcal{F}_t)_{0 \leq t \leq T}\) on a probability space \((\Omega, \mathcal{F}, P)\)
- \(N\) observables \(S^i\) which could be assets available for trading, stochastic processes adapted to the filtration \((\mathcal{F}_t)_{0 \leq t \leq T}\)
- A contingent claim \(H\), which given a fixed and finite time horizon, is simply an \(\mathcal{F}_T\) measurable random variable
Markets (In)Completeness

Dynamic Hedging

- *Classical* framework: Using self-financing strategies
- Complete markets: every contingent claim is attainable

\[
\tilde{H} = c_0 + \int_0^T \delta_s d\tilde{S}
\]

\[
c_0 = \mathbb{E}^*(\tilde{H})
\]

- Incomplete markets: min variance hedging, superhedging, quantile hedging, shortfall minimization, etc.

  e.g.

  \[
c_0 = \inf\{c | \exists \delta, c + \int_0^T \delta_s d\tilde{S} > \tilde{H} \mathcal{P} - ps\}
\]
Relaxing the self-financing constraint

- Quadratic local risk-minimization:

\[
(\delta_k^*, V_k^*) = \arg\min \mathbb{E}_k \left( \Delta C_{k+1}(\delta, V)^2 \right)
\]

Where \( \Delta C_{k+1}(\delta, V) = \Delta V_{k+1} - \delta_k \Delta S_{k+1} \) when there is no transaction costs and no liquidity effects.

- Many characterizations available for the optimal strategy, in discrete time and continuous time through martingale orthogonality properties.
Beyond Quadratic?

**Quadratic local risk-minimization**

- Fruitful framework, many theoretical results, numerical schemes available, see bibliography
- Yet the main interrogation is: what is the rationale behind putting the same weight on gains as on losses?
Non-Quadratic Local Risk-Minimization

The idea
Relaxing the quadratic hypothesis: using a smooth convex function $f$ to assess risk due to incremental costs

- **One time step setting**

$$(\delta_0^*, V_0^*) = \arg\min_{\delta_0} \mathbb{E} \left( f(H - V_0 - \delta_0(S_T - S_0)) \right)$$

Characterization of the optimal strategy through first order optimality conditions

$$\mathbb{E} \left( f'(H - V_0 - \delta_0(S_T - S_0)) \right) = 0$$
$$\mathbb{E} \left( f'(H - V_0 - \delta_0(S_T - S_0))(S_T - S_0) \right) = 0$$
Non-Quadratic Local Risk-Minimization

- **Multiple time steps setting**
  Minimization program working backward: given a contingent claim $H$, find $\Phi^*$, admissible strategy such that

  $$\forall k \in (0, \cdots, T - 1), \Delta R_k(\Phi) \geq \Delta R_k(\Phi^*) \forall \Phi \text{ admissible},$$

  with $\delta_{k+1} = \delta^*_{k+1}$ and $\beta_{k+1} = \beta^*_{k+1}$

- **First order optimality conditions**
  Equivalent characterization: the process

  $$(C_{k}^f)_{k=1,\ldots,N} = \sum_{i=0}^{k-1} f'(\Delta C_{i+1})$$

  is a martingale (strongly) orthogonal to the martingale part of the process $(S_k)$
Continuous Time Limit

- Start with a sequence of partitions $\mathcal{P}_n$ of $[0, T]$ tending to the identity
- Define the $f$-costs process as the following limit, whenever it exists:

$$
\lim_{n \to \infty} \sum_{k=1}^{l_n} f'(V_{\tau_k}^n - V_{\tau_{k-1}}^n - \delta_{\tau_{k-1}}^n (S_{\tau_k}^n - S_{\tau_{k-1}}^n))
$$

Where convergence is required in ucp topology
The $f$-costs Process

- When the process $S_t$ satisfies some regularity constraints and the strategies satisfy some admissibility conditions the limit exists and is equal to:

$$C^f_t = f''(0) \left( V_t - V_0 - \int_{0+}^t \delta_s dS_s \right)$$

$$+ \frac{f^{(3)}(0)}{2} \left( [V, V]^c_t - 2 \int_{0+}^t \delta_s d[V, S]^c_s + \int_{0+}^t \delta^2_s d[S, S]^c_t \right)$$

$$+ \sum_{0 < s \leq t} f'(\Delta V_s - \delta_s \Delta S_s) - f''(0)(\Delta V_s - \delta_s \Delta S_s)$$
Definition of the (pseudo-)Optimality

- By analogy with the discrete time setting, we define optimal strategies as those strategies such that $C_t^f$ is a martingale orthogonal to the martingale part of the $S_t$ process.
- When the strategy is assumed to be a Markov process, this allows to express the optimal strategies as solutions of a non-linear parabolic PDE or alternatively of a quadratic BSDE.
The Markovian setting

The case of stochastic volatility models

- We model the evolution of $S$ through an SDE with stochastic volatility

$$dS_t = \mu_t dt + \sigma_t dW^1_t$$
$$d\sigma_t = \gamma_t dt + \Sigma_t dW^2_t$$

With $\mu_t$, $\gamma_t$ and $\Sigma_t$ adapted processes, and
$$d < W^1, W^2 >_t = \rho dt$$

- Markovian strategies

$$\delta_t = \delta(t, S_t, \sigma_t)$$
$$V_t = V(t, S_t, \sigma_t)$$
The $f$-costs Process

The $f$-costs process is then

$$
C_t(\Phi) = \int_0^t \left[ f''(0) \left( \frac{\partial V}{\partial u} + \frac{\partial V}{\partial S} \mu_u + \frac{\partial V}{\partial \sigma} \gamma_u + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma_u \Sigma_u - \delta_u \mu_u \right) \right.

+ \frac{f^{(3)}(0)}{2} \left( \left( \frac{\partial V}{\partial S} \right)^2 \sigma_u^2 + \left( \frac{\partial V}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial V}{\partial S} \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right)

- f^{(3)}(0) \delta_u \left( \frac{\partial V}{\partial S} \sigma_u^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) + \frac{f^{(3)}(0)}{2} \delta_u \sigma_u^2 \right] du

+ \int_0^t f''(0) \left( \frac{\partial V}{\partial S} - \delta_u \right) \sigma_u dW_u^1 + \int_0^t f''(0) \frac{\partial V}{\partial \sigma} \Sigma_u dW_u^2
$$
Applying to the strategy $\Phi$ the second local risk-minimization criterion gives the equation satisfied by the optimal hedge $\delta$

\[
\left( \frac{\partial V}{\partial S} - \delta_u \right) \sigma_u^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u = 0
\]

- $\Rightarrow$ Linearity of the optimal hedge ratio $\delta$ with respect to the portfolio value $V$
- In the case when $f$ is quadratic and $\Sigma = 0$ we have $\delta = \frac{\partial V}{\partial S}$
Applying to the strategy $\Phi$ the first local risk-minimization criterion gives the PDE satisfied by the portfolio value $V$

\[
\begin{align*}
  f''(0) \left( \frac{\partial V}{\partial u} + \frac{\partial V}{\partial S} \mu_u + \frac{\partial V}{\partial \sigma} \gamma_u + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma_u \Sigma_u - \delta_u \mu_u \right) \\
  + \frac{f^{(3)}(0)}{2} \left( \left( \frac{\partial V}{\partial S} \right)^2 \sigma_u^2 + \left( \frac{\partial V}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial V}{\partial S} \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\
  - f^{(3)}(0) \delta_u \left( \frac{\partial V}{\partial S} \sigma_u^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) + \frac{f^{(3)}(0)}{2} \delta_u^2 \sigma_u^2 &= 0
\end{align*}
\]

- $\Rightarrow$ Non-linearity due to the term in $f^{(3)}(0)$, quadratic growth in the gradient
- In the case when $f$ is quadratic and $\Sigma = 0$ we find the $B&S$ PDE
Optimality or Pseudo-Optimality?

- **Small Perturbations** A small perturbation is a bounded admissible strategy $\Delta \Phi = (\beta, \delta)$ such that $\beta_T = 0$ and $\delta_T = 0$

- **Local Risk along a Partition**

\[
r_f^\tau[\phi, \Delta](t, \omega) = \sum_{t_i, t_{i+1} \in \tau} \frac{\Delta R_{t_i}(\phi + \Delta|_{[t_i, t_{i+1}})(\omega) - \Delta R_{t_i}(\phi)(\omega)}{t_{i+1} - t_i} 1_{[t_i, t_{i+1}[}(t)
\]
Locally Risk-Minimizing Strategies

Locally Risk-Minimizing Strategies An admissible strategy $\phi$ is called locally risk-minimizing for the option $H$ if for every small perturbation $\Delta$ and every increasing sequence of partitions $(\tau_n)_{n \in \mathbb{N}}$ tending to the identity, we have

$$\liminf_{n \to \infty} r^{\tau_n}[\phi, \Delta] \geq 0 \text{ } \mathcal{P} - a.e.$$ 

In a Markovian framework with a market described by Itô processes we can show that optimality coincides with pseudo-optimality.
Conclusion and Outlooks

Conclusion

- Extending the results of quadratic local risk-minimization to asymmetric risk, more realistic
- Allowing for several characterizations/representations of the optimal strategies with numerical methods available
- Flexibility of the methodology so it can apply to real markets

Outlooks

- Incorporating liquidity costs into the picture: forthcoming paper
- Define suitable approach to value the option from the incurred costs
- More numerical studies to be done for analysing the impact of different choices of risk functions
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References II
