Entropy, Risk and Quadratic BSDEs

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joint work with all math finance community

Running supremum and Entropy

L LogL Doob inequality

Let $L_.$ be a continuous positive exponential submartingale and $\overline{L}_.$ its running supremum.

- L LogL Doob inequality (Protter) gives NS Condition on L_ζ for L to be in $\mathbb{H}^{\mathbf{1}}$.
- Sharp estimates are proposed by P.Harremoës (2010) in discrete time.

Theorem For any $m > 0$, let $v_m(x)$ be the convex function defined on $\mathbb{R}^+$ by $v_m(x) = x - m - m\ln(x)$, $v_1(x) := v(x)$.

- (Doob) If $L_.$ is a u.i. integrable martingale with $L_0 = 1$, then $\max L_T$ is an integrable variable if and only if $\mathbb{E}\big(L_T \ln(L_T)\big) < \infty$
- (Harremoës) Then, the following sharpe inequality holds true

$$\mathbf{v}\big(\mathbb{E}(\overline{\mathbf{L}}_\mathbf{T})\big) \leq \mathbb{E}\big(\mathbf{L}_\mathbf{T} \ln(\mathbf{L}_\mathbf{T})\big),$$

- Assume L to be a positive $(\mathcal{D})$-submartingale. Then

$$\mathbf{v}_\mathbf{m}\big(\mathbb{E}(\overline{\mathbf{L}}_\mathbf{T})\big) - \mathbf{v}_\mathbf{m}(\mathbf{L}_\mathbf{0}) \leq \mathbb{E}\big(\mathbf{L}_\mathbf{T} \ln(\mathbf{L}_\mathbf{T})\big) - \mathbb{E}(\mathbf{L}_\mathbf{T})\mathbb{E}\big(\ln(\mathbf{L}_\mathbf{T})\big), \quad \mathbf{m} = \mathbb{E}(\mathbf{L}_\mathbf{T})$$

The inequality is **sharp**

Sketch of the proof Dellacherie (79) and Harremoës (2010)

- Since L is continuous, and $\overline{L}_s$ only increases on$\{L_. = \overline{L}_.\}$,
$\max L_t = 1 + \int_0^t d\overline{L}_s = 1 + \int_0^t \frac{L_s}{\overline{L}_s} d\overline{L}_s$ and

$$\mathbb{E}\big(\overline{L}_T\big) - 1 = \mathbb{E}\big(L_T \ln(\overline{L}_T)\big) \geq \mathbb{E}\big(L_T(\ln(L_T))^+\big).$$

- The converse is obtained by comparing $\mathbb{E}\big(L_T \ln(\overline{L}_T)\big)$ and $\mathbb{E}\big(L_T \ln(L_T)\big)$ using the convavity of the function ln, around the level $x^* = \mathbb{E}(\overline{L}_T)$.

$$\mathbb{E}\big(L_T \ \ln(\overline{L}_T/L_T)\big) \leq \mathbb{E}\big(L_T(\ln x^* + \frac{1}{x^*}(\frac{\overline{L}_T}{L_T} - x^*)\big) = \ln x^*$$

Then, reorganize the terms in the inequality.

- $\mathbb{E}\big(L_T \ \ln(L_T)\big) - \mathbb{E}(L_T)\mathbb{E}\big(\ln(L_T)\big)$ is the **Shannon entropy** of L

Portfolio, Duality and Incomplete Market

Stochastic Volatility

In option world, use stochastic volatility models with specific uncertainty

$$\frac{dX_t}{X_t} = \mu(t, X_t, Y_t)dt + \sigma(t, X_t, Y_t)dW_t^1,$$
$$dY_t = \eta(t, X_t, Y_t) + \gamma(t, X_t, Y_t)\, dW_t^2$$

where dW^1 and dW^2 are two correlated Brownian motions. γ is the volatility of the volatility.

What does it change ? In fact, everything !

- Perfect replication is not possible any more ;
- Notion of unique price does not exist any longer...

But, such a situation is often the general case.

What kind of answer may we bring to such a problem ?

Robust Hedging

Let us now suppose the interest rate $r = 0$.

• The option problem is still a **target** problem C_T, to be hedged by an admissible portfolio

$$V_T(\pi, \delta) = \pi + \int_0^T \delta_t . dX_t$$

• Investment decisions,(control parameter) (δ_t) are taken from available information at time t $(\delta_t \in \mathcal{F}(X_s, Y_s); s \leq t)$.
Some other constraints (size, sign...) may also be imposed

• Let $\mathcal{V}_T$ be the set of all derivatives, replicable at time T by an admissible portfolio. The 0-price of such derivatives $\widehat{C}_T$ is the unique value of replicating portfolios

Super-Replication

Definition: Super-replicating C_T is finding the smallest derivative $\widehat{C}_T \in \mathcal{V}_T$ which dominates C_T a.s..

- The $\widehat{C}_T$ replicating portfolio, $V^{\widehat{C}}$, is (def) the C_T **robust hedge**.
- The super-replication price, $\widehat{C}_0$ is the $\widehat{C}_T$ price $= V_0^{\widehat{H}}$.

Duality

Let $\mathcal{Q}_T$ the dual space of $\mathcal{V}_T$, that is the set of probabilities $\mathcal{Q}_T = \{Q, E_Q(V_T) = V_0 \;\; V_T \in \mathcal{V}_T\}$. The **main result** is the dual characterization (Hahn-Banach)

$$\widehat{C}_0 = \sup_{Q \in \mathcal{Q}_T} \mathbb{E}_Q(C_T).$$

More generally, if $\mathcal{V}_T$ is rich enough, $\widehat{C}_t = \text{esssup}_{Q \in \mathcal{Q}_T} \mathbb{E}_Q(C_T|\mathcal{F}_t)$ is a $\mathcal{Q}_T$-supermartingale, and a $\mathcal{Q}_T$-martingale if and only if $C_T \in \mathcal{V}_T$.

(EKQ, CviKa, Foell-Kram,Delb-Scha,Yan.+...100..)

Super-price with Stochastic Volatility Model

The super-replication price essentially depends on the values set of stochastic volatility:

- If it is R^+, then the super-replicating derivative of $h(X_T)$ is $\widehat{h}(X_T)$ where $\widehat{h}$ is the **concave envelope** of h ; the replicating strategy is the **trivial one**: buying $\widehat{h}'_x(x_0)$ stocks and holding them till maturity.
- If the volatility is bounded (up and down relatively to 0), the super-replication price is a (not depending on y) solution of

$$\widehat{h}'_t(t,x) + \frac{1}{2}\sup_y(\sigma^2(t,x,y)\widehat{h}''_{xx}(t,x)) = 0, \quad \widehat{h}(T,x) = h(x) \qquad (1)$$

- When h is convex, $\widehat{h}(t,x)$ is convex and the super-replication price is the one calculated with the **upper volatility** (in y).

(Avellaneda, Pham, Touzi+...100..)

Coherent and Convex Risk Measures

When super-replicating is too expensive, the trader has to **measure** his market risk exposure. The traditional measure is the **variance** of the replicating error.

Characterization of Risk Measure

• (Artz,Delb.97) have shown that **sub-additive and homogeneous, cash invariante** risk measures are an average estimation of losses w.r. to a probability measures family :

$$\rho(X) = \sup_{Q \in \mathcal{Q}_T} \mathbb{E}_Q(-X).$$

• **convex** monetary risk measures (FoelSc 02), by adding a penalization term on probability density

$$\rho(X) = \sup_{Q \in \mathcal{Q}_T} \mathbb{E}_Q(-X) - \alpha(Q)$$

• Dynamic risk measures (the same+Delbaen+Bion Nadal+BSDEs 2003-..)

Entropic Risk measure

A typical example is entropic risk measure

$$\rho_{\text{ent}}(X) = \sup_{Q \in \mathcal{Q}_T} \left(\mathbb{E}_Q(-X) - \frac{1}{\gamma} H(Q|P)\right)$$

$$H(Q|P) = \mathbb{E}_P\left(\frac{dQ}{dP} \ln\left(\frac{dQ}{dP}\right)\right)$$

$$\rho_{\text{ent}}(X) = \frac{1}{\gamma} \text{Ln}(\mathbb{E}(\exp(-\gamma \xi_T)))$$

Risk measures and Reserve Price

- A trader willing to relax the super-replication assumption is looking for the **smallest portfolio**, generating an acceptable loss.
- The initial value of this portfolio is called the *reserve price.*
- Mean-variance and entropic problems have now a complete solution
- More surprisingly (because of non-convexity), this also holds for the quantile hedging problem (FoLe,99).
- Sub-products of portfolio optimisation in incomplete markets.

(Fritelli, RoEK, CvKa or Scha, Carmona book (Indifference Pricing) .

Hedging in Practice

In a continuous arbitrage-free complete market, the Black-Scholes framework states that
The price of a derivative is the present value of the dynamical replicating portfolio

• In a real market, practitioners use basket of vanilla trading instruments such that very liquid Call or Put options to hedge complex options. With static strategies, a perfect hedge will not be possible regardless of risk aversion and market parameters, but transaction costs are reduced.

• Such strategies can be viewed as a particular case of constrained strategies in continuous framework.

The goal of the risk adverse investor is to maximize the expected utility of the global static portfolio

Calibration

The goal of the calibration is

- to identify parameters of underlying assets dynamics
- by adjusting the parameters of the distribution in such way that the fit between theoretical prices and market prices is "perfect".

In other words, the problem is to select a probability measure $\mathbb{Q}^\star$ such that, for a given contingent claims family $\{C^i_{T_j}; i \in I, j \in J\}$

$$\mathbb{E}_{\mathbb{Q}^\star}[C^i_{T_j} | \mathcal{F}_t] = C^{i,j}_{obs}(t)$$

where $C^{i,j}_{obs}(t)$ is the market price of the contingent claim $C^i_{T_j}$.
Ex : Avellaneda propose to adjust the distribution via Monte Carlo simulation based on prior distribution. (Weighted Monte Carlo)

Pricing in Constrained Market

The initial wealth of the static strategy includes the premium of the contingent claim B. But how is defined this premium ? .
Let be $X_t^{x,\pi}$ the present value of constrained portfolio tailored by the investor. Different ways are proposed to pricing the derivative **B**.

Super-replication

-Nek-Quenez(93), Cvitanic and Karatzas (95), Foellmer-Kramkov(97)
Price= smallest initial amount of wealth such that there exists a super-replicating portfolio

$$p^*(t, B) = \inf\{X_t^{x,\pi}; X_T^{x,\pi} \geq B\}$$

Minimization of the quadratic error

– Duffie-Richardson(92), Foellmer-Schweitzer(93), Gourieroux, Laurent, Pham (1998) and many others........

$$\inf_{x,\pi} \mathbb{E}[(X_t^{x,\pi} - B)^2]$$

The price is given by the minimal investment.

Pricing via Utility Maximization

• In 1989, Hodges and Neuberger(1989) link the price of a contingent claim with the opportunity for the investor to deliver this option.

• Given a utility function U, let $\widehat{U}(x, B)$ be the maximal expected utility for a portfolio with initial wealth x, if a contingent claim B is delivered.They define a preference relation by

$$(x, B) \succ (x', B') \Longleftrightarrow V(x, B) > V(x', B')$$

• The price of the claim is defined in such a way that the investor is indifferent to deliver the option or not, that is

$$(x + p, B) \sim (x, 0) \quad \Longleftrightarrow \quad \mathbf{V(x + p, B) = V(x, 0)}$$

• M.Davis, Karatzas and Kou (95-97) Avelanedda, define the price in reference to the marginal utility of the wealth.

Exponential Utility

$$U(x) = -\alpha \exp(-\gamma x) \quad x \in R$$

Motivation

• Given that the risk-aversion coefficient γ is constant, the complexity of the problem is reduced.
• Unlike of power utilities, this utility allows portfolios with negative values
• by duality , relationship with entropy
• for small γ, mean-variance optimization
• for large γ, the asymptotic problem is related to the control problem of the maximal loss and super-replication

Drawbacks

Optimality does not depend on the initial wealth.

Static Problem : Portfolio

• Based on a (stochastic) zero-coupon bond $B(t,T)$ with maturity T, and d liquid instruments $(C_t^i)_{i=1}^d$, for instance very liquid Call and Put options with different strikes and maturities. They are used for hedging and calibration. The initial wealth is x.

• A static strategy $[y = (y^1, y^2, ..., y^d), \alpha]$ where y^i is the holding of the trading asset i, with $y^i > 0$ for long position in C^i, $y^i < 0$ for short position.

• The present value of the portfolio is :

– At time 0,

$$x = \sum_{i=1}^{d} y^i C_0^i + \alpha B(0,T)$$

– At maturity$T$,

$$X_T^{(x,y)} - B = \sum_{i=1}^{d} y^i C_T^i + \alpha - B$$
$$= \frac{x}{B(0,T)} + \sum_{i=1}^{d} y^i \left(C_T^i - \frac{C_0^i}{B(0,T)}\right) - B$$

since the strategy is static and self-financing. In short,

$$\boxed{X_T^{(x,y)} = \widehat{x} + < y, \widehat{C}_T >}$$

We assume

- no linear redundancy on the set of trading instruments
- The target is non-attainable

Primal Optimization Problem

The Program $\mathcal{P}_\gamma$: to maximize w.r. y

$$\boxed{-V(x,B) = \max_y \Big(-\mathbb{E}[\exp(-\gamma(\widehat{x}+ < y, \widehat{C}_T > -B))]\Big)}$$

We assume that the Laplace transform is defined in the neighborhood of 0.

$$\Phi(y,\gamma) = \mathbb{E}[\exp(- < y, \widehat{C}_T > -\gamma B)]$$

We introduce the family of probability measures $\mathbb{R}_y$

$$\boxed{\frac{d\mathbb{R}_y}{d\mathbb{P}} = \frac{\exp(-\gamma < y, \widehat{C}_T > +\gamma B)}{\mathbb{E}[\exp(-\gamma < y, \widehat{C}_T > +\gamma B)]}}$$

Necessary conditions

The following properties are equivalent :

- y^* is optimal
- $V(x, B) = \mathbb{E}[\exp(-\gamma(\widehat{x}+ < y^*, \widehat{C}_T > -B))]$
- $\mathbb{E}_{\mathbb{R}^*}(C_T^i) = \frac{C_0^i}{B(0,T)}, \quad i = 1....d$, for $\mathbb{R}^* = \mathbb{R}_{y^*}$ is “forward-neutral"
- the covariance matrix $(\mathrm{cov}_{\mathbb{R}^*}(C_T^i, C_T^j))$ is strictly positive.
- $\mathbb{R}^*$ is the **minimal entropy measure** in the class of R satisfying $\mathbb{E}_{\mathbb{R}^*}(C_T^i) = \frac{C_0^i}{B(0,T)}, \forall i$
- Not enough to be sure that $\mathbb{R}^*$ is a martingale-measure.(Weighted MC, Avellaneda)

Dual formulation, Entropy and Game versus the market

Dual formulation,

Let us denote by ν a probability measure a.continuous w.r. to $\mathbb{P}$, and the relative entropy by

$$\mathcal{H}(\nu) = \mathbb{E}_\nu[\mathrm{Ln}(\frac{d\nu}{d\mathbb{P}})] = \mathbb{E}_\mathbb{P}[\frac{d\nu}{d\mathbb{P}}\mathrm{Ln}(\frac{d\nu}{d\mathbb{P}})] \geq 0$$

$$\boxed{\frac{1}{\gamma}\mathrm{Ln}\mathbb{E}[\exp(-\gamma X)] = \sup_\nu\{-\mathbb{E}_\nu(X) - \frac{1}{\gamma}\mathbb{E}_\nu[\mathrm{Ln}(\frac{d\nu}{d\mathbb{P}})]\}}$$

Game versus the Market

$$\boxed{\frac{1}{\gamma}V_\gamma(x,B) = \inf_y \sup_\nu\{\mathbb{E}_\nu(B - X_T^{(x,y)}) - \frac{1}{\gamma}\mathcal{H}(\nu)\}}$$

The market chooses the worst case scenario that is the loses maximizing "distribution".

Static Forward-neutral formulation

Let be $\mathcal{Q}^s$ the set of equivalent (static) forward-neutral probability measures

$$\mathcal{Q}^s = \{\mathbb{Q} \sim \mathbb{P}; \mathbb{E}_{\mathbb{Q}}(X_T^{(x,y)}) = \frac{x}{B(0,T)}\}$$

$$\boxed{\frac{1}{\gamma}\mathrm{Ln}V_\gamma(0,B) = \sup_{\mathbb{Q}\in\mathcal{Q}^s}\{\mathbb{E}_{\mathbb{Q}}(B) - \frac{1}{\gamma}\mathcal{H}(\mathbb{Q})\}}$$

Derivatives Pricing

Pricing via calibration

Two steps (Avellaneda,-Douady, Davis, Karatzas- Kou)

• first, by solving the problem without contingent claim and denote by $\mathbb{R}^\star$ the optimal static risk-neutral probability measure

• Then, by pricing any contingent claim via risk-neutral point of view

$$\Pi_t(B) = \mathbb{E}_{\mathbb{R}^\star}[B|\mathcal{F}_t]$$

• the hedge portfolio is the portfolio with minimum variance w.r. to $\mathbb{R}^\star$

Pricing via utility maximization

To find p s.t.

$$V_\gamma(x+p, B) = V_\gamma(x, 0)$$

or equivalently

$$\boxed{\Pi_t(B) = \sup_{\mathbb{Q}\in\mathcal{Q}^s}(-\frac{1}{\gamma}\mathcal{H}(\mathbb{Q})\} - \sup_{\mathbb{Q}\in\mathcal{Q}^s}\{\mathbb{E}_{\mathbb{Q}}(B) - \frac{1}{\gamma}\mathcal{H}(\mathbb{Q})\}}$$

General Dynamic risk measures : Technically difficult extension of these results (Many papers+J.Bion-Nadal+....

Volatility control : Not in the same family of problems as not associated with equivalent martingale-measures.
Hence use **G- expectation, second order BSDEs, weak control problems.**

Optimal Portfolio and "Hedging" of the Claim

• The previous identities hold if an optimal portfolio, or equivalently an optimal probability measure exists.

• Optimal portfolio and optimal forward-neutral probability measure are related by

$$X_T^{\star,B} = B - \frac{1}{\gamma}\left[\mathrm{Ln}V_\gamma(x,B) + \mathrm{Ln}[\frac{d\mathbb{R}^{\star,\gamma}}{d\mathbb{P}}]\right]$$

• The hedging portfolio can be viewed as

$$X_T^{\star,B} - X_T^{\star,0} = B - \Pi_0(B) - \frac{1}{\gamma}\mathrm{Ln}[\frac{d\mathbb{R}^{\star,\gamma,B}}{d\mathbb{R}^{\star,\gamma,0}}]$$

Asymptotic or Risk Sensitive Analysis

Asymptotic Analysis

- The function $\frac{1}{\gamma}\mathrm{Ln}V_\gamma(0,B)$ is increasing w.r to γ.
- If the set $\mathcal{Q}^s$ is not empty,

(in particular if the trading assets are priced w.r.to the assumption of no arbitrage opportunity),

$$\boxed{\lim_\gamma(\frac{1}{\gamma}\mathrm{Ln}V_\gamma(0,B)) = \sup_{\mathbb{Q}\in\mathcal{Q}^s}\{\mathbb{E}_\mathbb{Q}(B)\}}$$

Asymptotic Game

The limit is the super-replicating price given by the static constrained strategies.

$$\sup_{\mathbb{Q}\in\mathcal{Q}^s}\{\mathbb{E}_\mathbb{Q}(B)\} = \inf_y \mathrm{esssup}_\omega\{(B(\omega) - < y, \widehat{C}_T(\omega) >)^+\}$$

or equivalently for the optimal strategy y^*

$$\boxed{\sup_{\mathbb{Q}\in\mathcal{Q}}\{\mathbb{E}_{\mathbb{Q}}(B)\} + < y^*, \widehat{C}_T(\omega) > \geq B(\omega)} \quad \forall\omega$$

and that is the smallest dominating static portfolio.

Dynamic Constrained strategies

Let us consider the same problem with continuous dynamic strategies.

Typical set-up for continuous-time asset pricing :

- One Bond with short rate r_t

$$dS_t^0 = S_t^0 \, r_t \, dt,$$

- $d \geq k$ risky securities are continuously traded, with No Arbitrage Opportunities

$$dS_t^i = S_t^i [b_t^i \, dt + \sum_{j=1}^{k} \sigma_t^{i,j} \, dW_t^j], b_t - r_t \mathbf{1} = \sigma_t \, \eta_t, \quad dt \otimes \mathbb{P} \; a.s,$$

- A k-dimensional Wiener process $W = (W^1, \ldots, W^d)^*$

$\sigma^i = (\sigma^{i,j})_{j=1}^k$, b^i and r are adapted and bounded.

- A self-financing portfolio strategy is a continuously traded portfolio

$$dX_t = r_t X_t dt + \sum_{i=1}^{d} y_t^i S_t^i \sum_{j=1}^{k} \sigma_t^{i,j} \left(dW_t^j + \eta_t^j\right)$$

Examples of constraints

- Limitation on the trading assets : $\pi_t^i = y_t^i S_t^i = 0 \quad \forall i \geq n$
- Limitation on the amounts to be traded : $|\pi_t^i| \leq K^i$
- More generally, the vector π_t is supposed in a convex space K.
- The bonds $B(t, T)$ satisfy the constraint with volatility $\sigma(t, T)$.

Optimization Problem

From above, the optimization problem is equivalent to

$$\frac{1}{\gamma}\mathrm{Ln}V_\gamma(x,B) = -\widehat{x} + \inf_{\pi}\{\frac{1}{\gamma}\mathrm{Ln}\mathbb{E}[\exp - \gamma(X_T^{x,\pi} - \widehat{x} - B)]\}$$

$$\frac{1}{\gamma}\mathrm{Ln}V_\gamma(x,B) = -\widehat{x} + \sup_{\mathbb{Q}\in\mathcal{Q}}\{\mathbb{E}_{\mathbb{Q}}(B) - \frac{1}{\gamma}\mathcal{H}(\mathbb{Q})\}$$

where $\mathcal{Q}$ is now the set of probability measures such that all constrained portfolio $\frac{X_t^{x,\pi}}{B(t,T)}$ are $\mathbb{Q}$-martingales.

Complete Market

$\mathcal{Q}$ is reduced to one element : the forward-neutral probability $\mathbb{Q}^T$, with state price density density

$$dH_t^\theta = H_t^\theta[\theta_t^* dW_t], \quad H_0^0 = 1$$

where $\theta = \eta + \sigma(., T)$.

$$\boxed{\frac{1}{\gamma}\text{Ln}V_\gamma(0, B) = -\widehat{x} + \mathbb{E}_{\mathbb{Q}^T}[B - \frac{1}{2\gamma}\int_0^T |\theta_t|^2 dt]}$$

Incomplete Market

$\mathcal{Q}$ is the probability set with martingale density H^ν such that

$$< \nu_s, \pi_s >= 0, \quad \forall \pi_s$$

$$\boxed{\frac{1}{\gamma}\mathrm{Ln}V_\gamma(x,B) = -\widehat{x} + \mathbb{E}_\nu[B - \frac{1}{2\gamma}\int_0^T |\theta_t + \nu_t|^2 dt]}$$

from classical result about entropy.

The price via utility is

$$\begin{aligned} p(0,B) &= B(0,T)\sup_\nu\{\mathbb{E}_\nu[B - \frac{1}{2\gamma}\int_0^T |\theta_t + \nu_t|^2 dt]\} \\ &- B(0,T)\sup_\nu\{\mathbb{E}_\nu[-\frac{1}{2\gamma}\int_0^T |\theta_t + \nu_t|^2 dt]\} \end{aligned}$$

Dynamic Point of view : Put

$$U_t^{B,\nu} = \mathbb{E}[H^\nu_{t,T}(B - \frac{1}{2\gamma}\int_t^T |\theta_u + \nu_u|^2 du)|\mathcal{F}_t], \quad Y_t^B = \text{esssup}(U_t^{B,\nu})$$

Since $H_t^\nu U_t^{B,\nu} + \frac{1}{2\gamma}\int_0^t |\theta_u + \nu_u|^2 du$ is a martingale, there exists $Z_t^{B,\nu}$ such that $(U_t^{B,\nu}, Z_t^{B,\nu})$ **is sol of the Linear BSDE** with $\mathbf{U_T^{B,\nu} = B}$,

$$-dU_t^{B,\nu} = dt f_\nu(t, U_t^{B,\nu}, Z_t^{B,\nu}) - (Z_t^{B,\nu})^* dW_t$$

$$f_\nu(t,y,z) = (-\frac{1}{2\gamma}|\theta_t + \nu_t|^2 - (\theta_t + \nu_t)^* z)$$

Backward Stochastic Differential Equations

Pardoux-Peng(1989), El Karoui, Peng-Quenez(1997)

$$\boxed{-dY_t = f(t, Y_t, Z_t)dt - Z_t^* dW_t, \quad Y_T = B}$$

A solution is a par of adapted processes (Y, Z).
Existence and uniqueness hold
• in $\mathbb{H}^2$ for f unif.Lipschitz.
• in $\mathbb{H}_\infty \otimes \mathbb{H}^2$ for f_y with quadratic growth in z and f_z with linear growth (Kobylansky 98)
• A comparison theorem holds
(similar to the maximum principle for PDE)

Quadratic BSDE for the Seller Price

Formally, by the comparison theorem, the log of the value function of the optimization problem satisfies,

$$\begin{aligned} -dY_t^B &= \text{esssup}(U_t^{B,\nu}) \\ &= f(t, Y_t^B, Z_t^B)dt - Z_t^B dW_t, \quad Y_T^B = B \end{aligned}$$

$$f(t,y,z) = \text{esssup}_{\nu \in K}\left(-\frac{1}{2\gamma}|\theta_t + \nu_t|^2 - (\theta_t + \nu_t)^{*z}\right)$$

When K is a vector space

$$\boxed{f(t,y,z) = -\frac{1}{2\gamma}||\theta_t - \pi\theta_t||^2 + (\pi\theta_t - \theta_t)^\star z + \frac{\gamma}{2}||\pi z||^2}$$

where π is the projection operator on the orthogonal space $K^\star$

Seller price BSDE

Recall that $p_t = Y_t^0 - Y_t^B$

$$-dp_t = -r_t p_t - (\theta_t + \nu_t^0)^\star Z_t + \frac{\gamma}{2}||\pi_t Z_t||^2 - Z_t^\star dW_t, \quad p_T = B$$

The price process satisfies (if comparison holds true)

- The map $B \Rightarrow p_t^\gamma(B)$ is increasing and convex
- The price is increasing w.r. to γ

$$\lim_{\gamma \to 0} p_t^\gamma(B) = \text{calibration price}$$

$$\lim_{\gamma \to +\infty} p_t^\gamma(B) = \text{superreplication price}$$