Model independent bounds for options pricing
A stochastic control approach

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Outline

Unspecified volatility process

- $\Omega = \{\omega \in C(\mathbb{R}_+) : \omega(0) = 0\}$,
- B coordinate process, $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$, $\mathbb{P}_0$: Wiener measure

Suppose that B is only known to be a continuous local martingale with quadratic variation $\langle B \rangle$ a.c. wrt Lebesgue. Let

$$\mathcal{P} := \left\{ \mathbb{P}_0 \circ \left(\int_0^\cdot \sigma_t dB_t \right)^{-1} : \ \sigma \ \mathbb{F}-\text{prog. meas.}, \ \int_0^T |\sigma_t|^2 dt < \infty \right\}$$

- Zero interest rate, and risky asset defined by :

$$dS_t = S_t dB_t, \ \ \mathcal{P} - \text{q.s.}$$

where $\mathcal{P}-$quasi-surely means $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}$

Model-free bounds

• Super-hedging and Sub-hedging problems of $\mathcal{F}_T-$meas. r.v. ξ

$$U := \inf\left\{X_0 :\ \exists\, H \in \mathcal{H} : X_0 + \int_0^T H_t dB_t \geq \xi,\ \mathcal{P}-\text{q.s.}\right\}$$

$$L := \sup\left\{X_0 :\ \exists\, H \in \mathcal{H} : X_0 + \int_0^T H_t dB_t \leq \xi,\ \mathcal{P}-\text{q.s.}\right\}$$

• the portfolio $H \in \mathcal{H}$ does not depend on a particular $\mathbb{P} \in \mathcal{P}$...

• Denis-Martini 2005 and Peng 2007 for the bounded volatility case $(\underline{\sigma} \leq \sigma_. \leq \overline{\sigma})$

Dual formulation

Consider the superhedging problem

$$U_0 := \inf\left\{X_0 : \; X_0 + \int_0^T H_t dB_t \geq \xi, \; \mathcal{P}-\text{q.s. for some } H \in \mathcal{H}_0\right\}$$

where

$$\mathcal{H}_0 := \left\{H : \; H \in \mathbb{H}^2_{loc}(\mathbb{P}) \text{ and } X^H \geq \text{Mart}^{\mathbb{P}}, \; \forall \mathbb{P} \in \mathcal{P}\right\}$$

Theorem (Soner, T., Zhang 2010) For all $\xi \in \mathcal{L}^\infty_{\mathcal{P}}$:

$$U_0 = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi]$$

and existence holds for the problem U_0

Bounds with no further information

For $\xi = g(S_T)$, we find

$$U_0(\xi) = g^{\text{conc}}(S_0) \quad \text{and} \quad L_0(\xi) = g^{\text{conv}}(S_0)$$

and the corresponding hedging strategy H^* is of type Buy-and-Hold

$\Longrightarrow$ dynamic hedge does not help to reduce the superhedging cost...

Model-free bounds with more information

• Suppose that prices of $T-$maturity call options for all possible strikes $c(k), k \geq 0$ are observed and tradable. Then the map

$$k \longmapsto c(k) := \mathbb{E}[(S_T - k)^+]$$

characterizes the distribution $S_T \sim_{\mathbb{P}} \mu$ by $\mu([k,\infty)) = -c'(k)$

• The no-arbitrage bounds can be improved to

$$U(\mu) := \inf\left\{X_0 : \exists\, H \in \mathcal{H},\ \lambda \in \Lambda : X_T^{H,\lambda} \geq \xi,\ \mathcal{P}-\text{q.s.}\right\}$$

$$L(\mu) := \sup\left\{X_0 : \exists\, H \in \mathcal{H},\ \lambda \in \Lambda : X_T^{H,\lambda} \leq \xi,\ \mathcal{P}-\text{q.s.}\right\}$$

where $\Lambda = \{\text{bdd measurable functions}\}$ and

$$X_T^{H,\lambda} := X_0 + \int_0^T H_t dB_t + \lambda(S_T) - \mu(\lambda)$$

$$\text{“} = X_0 + \int_0^T H_t dB_t + \int \lambda''(k)[(S_T - k)^+ - c(k)]dk \text{”}$$

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where $\Lambda = \{\text{bdd measurable functions}\}$ and

$$\begin{aligned} X_T^{H,\lambda} &:= X_0 + \int_0^T H_t dB_t + \lambda(S_T) - \mu(\lambda) \\ \text{``} &= X_0 + \int_0^T H_t dB_t + \int \lambda''(k)[(S_T - k)^+ - c(k)]dk \text{''} \end{aligned}$$

Duality and stochastic control

- Notice that

$$U(\mu) = \inf_{\lambda\in\Lambda} \inf\left\{X_0 : \exists\, H \in \mathcal{H}, X_T^H \geq \xi - \lambda(S_T) + \mu(\lambda),\ \mathcal{P} - \text{q.s.}\right\}$$

$$L(\mu) = \sup_{\lambda\in\Lambda} \sup\left\{X_0 : \exists\, H \in \mathcal{H}, X_T^H \leq \xi - \lambda(S_T) + \mu(\lambda),\ \mathcal{P} - \text{q.s.}\right\}$$

- Then, the previous duality implies that

$$\begin{aligned} U(\mu) &:= \inf_{\lambda\in\Lambda} \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\xi - \lambda(S_T) + \mu(\lambda)\Big] \\ L(\mu) &:= \sup_{\lambda\in\Lambda} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\xi - \lambda(S_T) + \mu(\lambda)\Big] \end{aligned}$$

- For every fixed λ : standard stochastic control problem...

Outline

Connection with optimal transportation

- Alternatively, one may formulate the problems as :

$$\overline{U}(\mu) := \sup_{\mathbb{P}\in\mathcal{P}:B_T\sim_{\mathbb{P}}\mu} \mathbb{E}^{\mathbb{P}}[\xi] \quad \text{and} \quad \overline{L}(\mu) := \inf_{\mathbb{P}\in\mathcal{P}:B_T\sim_{\mathbb{P}}\mu} \mathbb{E}^{\mathbb{P}}[\xi]$$

Optimal transportation problem...

- If $\overline{L} = \overline{L}^{**}$, then :

$$\overline{L}(\mu) = \sup_{\phi\in\mathbb{L}^1(\mu)} \int \phi(s)\mu(ds) - \int \phi_0(s)\delta_{S_0}(ds)$$

where ϕ_0 is the value of the stochastic control problem :

$$\phi_0(s) := \overline{L}^*(\phi) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\phi(S_T) - \xi\right]$$

can be characterized by the corresponding dynamic programming equation in the Markov case, or by a backward SDE...

- Related work by Mikami and Thieullen 2004 (drift control)

Relation with Skorohod Embedding Problem

Previous literature by D. Hobson, L.C.G. Rogers, A. Cox, J. Obloj, B. Dupire, P. Carr, R. Lee

- adressed this problem by using results from the SEP :

Given μ probability measure on $\mathbb{R}$ with $\int |x|\mu(dx) < \infty$
Find a stopping time τ such that
$B_\tau \sim \mu$ and $\{B_{t\wedge\tau}, t \geq 0\}$ UI martingale

(Hall, Monroe, Azéma, Yor, Perkins, Chacon, Walsh, Rost, Root, Bass, Vallois)

- More than twenty known solutions (see Obloj for a survey)

Example : the Azéma-Yor solution

Define the barycenter function :

$$b(x) \;:=\; \frac{\int_x^\infty s\mu(ds)}{\int_x^\infty \mu(ds)}$$

Then, the Azéma-Yor solution of the SEP is :

$$\tau_{\mathrm{AY}} := \inf\{t > 0 : \; B_t^* > b(B_t)\}, \quad \text{with} \quad B_t^* := \max_{s \le t} B_s$$

i.e. $\{B_{t\wedge\tau_{\mathrm{AY}}}, t \ge 0\}$ is UI martingale and $B_{\tau_{\mathrm{AY}}} \sim \mu$.

For later use

$$\frac{c(\zeta)}{x-\zeta} \quad \text{decreases on} \quad [0, b^{-1}(x)] \quad \text{and increases on} \quad [b^{-1}(x), x)$$

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Connection with our problem

- $(M_t)_{t\geq 0}$ continuous martingale, $M_0 = 0$ and $M_T \sim \mu$,

Then $M_t = B_{\langle M\rangle_t}$ and $\langle M\rangle_T$ solution of SEP

- Let τ be a solution of SEP

Then $M_t := B_{\frac{t}{T-t}\wedge\tau}$ is a continuous martingale, $M_0 = 0$ and $M_T \sim \mu$.

Optimality of the Azéma-Yor solution

Let g be C^1 nondecreasing, and define :

$$H(m,x) \quad := \quad \int_0^m g'(r)\frac{r-x}{r-b^{-1}(r)}dr \quad \text{so that}$$

- $\{H(B_t^*, B_t), t \geq 0\}$ is a local martingale
- $g(m) - H(m,x) \leq g\big(b(x)\big) - H\big(b(x), x\big) =: G(x)$

Then

$$g\big(B_\tau^*\big) \quad \leq \quad \underbrace{H\big(B_\tau^*, B_\tau\big)}_{(\text{loc. mart.})_\tau} + G\big(B_\tau\big)$$

and

$$\sup_{\mathbb{P}\in\mathcal{P}(\mu)} \mathbb{E}^{\mathbb{P}}\left[g(S_T^*)\right] = \max_{\tau\in\mathcal{T}(\mu)} \mathbb{E}\left[g\big(B_\tau^*\big)\right] = \int G(x)\mu(dx) = \mathbb{E}\left[g\big(B_{\tau_{\mathrm{AY}}}^*\big)\right]$$

Outline

1. Super and sub-hedging under uncertain volatility
2. Connection with previous literature
   - Optimal transportation
   - The Skorohod Embedding Problem
3. Exploiting the dual formulation
   - Numerical methods
   - Lookback options with one known marginal distribution
   - Extension to many marginals

Numerical approximation

• For every fixed λ, build a numerical scheme to approximate the value function

$$u^{\lambda} \quad := \quad \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}\left[\xi - \lambda(S_T)\right]$$

this is a singular stochastic control, which can be characterized by an elliptic equation... finite differences

• Minimize over λ :

$$\inf_{\lambda} \mu(\lambda) - u^{\lambda}$$

numerical approximation by the gradient projection algorithm...

Xiaolu TAN...

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Application to Lookback derivatives

From now on :

$$\xi \; = \; g(S_T, S_T^*) \quad \text{where} \quad S_T^* := \max_{t \leq T} S_t$$

Our main results :

- recover the known explicit bounds in this context (so far, those induced by the Azéma-Yor embedding)
- extend the existing results in the case of n given marginals (Brown-Hobson-Rogers 98, Madan-Yor 02)

Dynamic Programming Equation – Fixed λ

Consider first the case :

$$\xi = g(S_T^*) \quad \text{where} \quad S_T^* := \max_{t\in[0,T]} S_t$$

Given $\lambda(.)$, the stochastic control problem is

$$u^\lambda(t,s,m) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^P\left[g(M_T^{t,s,m}) - \lambda(S_T^{t,s})\right], \quad M_T^{t,s,m} := m \vee \max_{[t,T]} S_.^{t,s}$$

$\Longrightarrow$ Optimal stopping representation and DPE characterization :

$$u^\lambda(t,s,m) = u^\lambda(s,m) = \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[g(M_\tau^{s,m}) - \lambda(S_\tau^s)\right]$$

$$\min\left\{u^\lambda - (g-\lambda), -u^\lambda_{ss}\right\} = 0 \text{ for } 0 < s < m$$

$$u^\lambda_m(m,m) = 0$$

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$$\min\left\{u^\lambda - (g-\lambda), -u_{ss}^\lambda\right\} = 0 \text{ for } 0 < s < m$$
$$u_m^\lambda(m,m) = 0$$

Lagrange Multipliers reduction

DPE implies that u^λ is a supersolution of

$$\min\left\{u^\lambda - (g - \lambda^{\text{conv}}), -u^\lambda_{ss}\right\} \;\geq\; 0$$

Then, it follows from comparison that :

$$u^\lambda(s,m) \;\geq\; u^{\lambda^{\text{conv}}}(s,m) \;:=\; \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[g(M^{s,m}_\tau) - \lambda^{\text{conv}}(S^s_\tau)\right]$$

and the converse inequality is obvious. This implies that

$$U(\mu) \;=\; \inf_{\lambda''\geq 0} \;\sup_{\tau\in\mathcal{T}} \;\mu(\lambda) + \mathbb{E}\left[g(M^{s,m}_\tau) - \lambda(S^s_\tau)\right]$$

Solving the optimal stopping problem

Given λ convex :

$$u^{\lambda}(s,m) \;=\; \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[g(M^{s,m}_{\tau}) - \lambda(S^{s}_{\tau})\right]$$

assume g is C^1 increasing, then we expect that there is a boundary ψ (continuous increasing) so that :

$$u^{\lambda}(s,m) \;=\; g(m) - \lambda(s) + \int_{\psi(m)}^{s\vee\psi(m)} (s-k)\lambda''(k)dk$$

The Neuman condition provides an equation for ψ :

$$\text{(ODE)} \qquad g'(m) - \big(m-\psi(m)\big)\lambda''\big(\psi(m)\big)\psi'(m) \;=\; 0$$

Peskir's maximality principle

Theorem Let λ be C^2 and strictly convex. Then the following are equivalent :

- (i) $u^\lambda < \infty$
- (ii) (ODE) has a maximal solution ψ which lies strictly below the diagonal ($\psi(m) < m$, $m > 0$).

In this case u^λ is given by

$$u^\lambda(s,m) \quad = \quad g(m) - \lambda(s) + \int_{\psi(m)}^{s \vee \psi(m)} (s-k)\lambda''(k)dk$$

and $\tau^* := \inf\{t > 0 : S_t \leq \psi(M_t)\}$ is an optimal stopping time.

The explicit upper bound

W.l.o.g. $\lambda(S_0) = 0$, then :

$$\begin{aligned}
U(\mu) &= g(S_0) + \inf_{\lambda''\geq 0} \mu(\lambda) + \int_{\psi(M_0)}^{S_0\vee\psi(M_0)} (s-k)\lambda''(k)dk \\
&\geq g(S_0) + \inf_{\lambda''\geq 0} \int c(k)\lambda''(k)dk \\
&= g(S_0) + \inf_{\lambda''\geq 0} \int c(\psi(x))\lambda''(\psi(x))\psi'(x)dx \\
&= g(S_0) + \inf_{\psi\in\ldots} \int c(\psi(x))\frac{g'(x)}{x-\psi(x)}dx \quad (ODE) \\
&\geq g(S_0) + \int \inf_{\psi<x} \frac{c(\psi)}{x-\psi} g'(x)dx \longrightarrow \psi^*(x) = b^{-1}(x)
\end{aligned}$$

Azéma-Yor. The reverse inequality is obvious...

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A recursive sequence of control problems

Given n functions $\lambda = (\lambda_i)_{1\le i\le n}$ define :

$$u^\lambda := u^0(S_0, M_0) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[g(M_{t_n}) - \sum_{i=1}^{n}\lambda_i(S_{t_n})\right]$$

Optimal upper bound given that $S_{t_i} \sim \mu_i$, $i \le n$:

$$\inf_{(\lambda_i)_{1\le i\le n}} \sum_{i=1}^{n}\mu_i(\lambda_i) + u^\lambda$$

Then, we introduce for $i = 1, \ldots, n$:

$$\begin{aligned} u^n(s,m) &:= g(m) \\ u^{i-1}(s,m) &:= \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[u^i(S_{t_n}, M_{t_n}) - \lambda_i(S_{t_n})\Big|(S,M)_{t_{n-1}} = (s,m)\right] \end{aligned}$$

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Extension of Peskir's maximality principle

Lemma Optimization can be restricted to those λ_i's such that $\lambda^i - u^i$ is strictly convex for all $i = 1, \ldots, n$

Theorem For λ^i s.t. $\lambda^i - u^i$ is strictly convex, $u^{i-1} < \infty$ iff there is a maximal solution ψ_i of the ODE

$$\psi_i' \left(\lambda_i''(\psi_i) - u_{ss}^i(\psi_i, m)\right) = u_{sm}^i(\psi_i, m) + \frac{u_m^i(\psi_i, m)}{m - \psi_i}$$

which stays strictly below the diagonal $\psi_i(m) < m$, $m \geq 0$.
In this case :

$$u^{i-1}(s,m) = u^i(s,m) - \lambda_i(s) + \int_{\psi_i(m)}^{s \vee \psi_i(m)} (s-k)\left(\lambda_i''(k) - u_{ss}^i(k,m)\right) dk$$

Explicit finite dimensional optimization problem

Proceeding as in the case of one marginal, we arrive at the optimization problem :

$$U(\mu_1, \ldots, \mu_n)$$
$$\geq \int \inf_{0<\psi_\cdot<x} \; \frac{c_n(\psi_n)}{x-\psi_n} + \sum_{i=1}^{n-1} \left(\frac{c_i(\overline{\psi}_i)}{x-\overline{\psi}_i} - \frac{c_i(\overline{\psi}_{i+1})}{x-\overline{\psi}_{i+1}} \right) \mathbb{1}_{\{\overline{\psi}_i < \overline{\psi}_{i+1}\}}$$

where

$$\overline{\psi}_i := \psi_i \wedge \ldots \wedge \psi_n \quad \text{for all} \quad i = 1, \ldots, n$$

Optimal upper bound given two marginals

The case $n = 2$ reduces to :

$$\inf_{\psi_i < x} \left\{ c_2(\psi_2) - \mathbb{1}_{\{\psi_2 > \psi_1\}} \left(\frac{c_1(\psi_2)}{x - \psi_2} - \frac{c_1(\psi_1)}{x - \psi_1} \right) \right\}$$

which recovers Hobson and Rogers 1998 $\Longrightarrow$

- $\psi_1(x) = b_1^{-1}(x)$ (Azéma-Yor)
- $\psi_2(x)$ defined by

$$\inf_{\psi_2 < x} \left\{ c_2(\psi_2) - \mathbb{1}_{\{\psi_2 > \psi_1(x)\}} \left(\frac{c_1(\psi_2)}{x - \psi_2} - \frac{c_1(\psi_1(x))}{x - \psi_1(x)} \right) \right\}$$

The $n-$marginals problem reduces to 2$-$marginals problems

- First, minimize wrt $\overline{\psi}_1 \le \overline{\psi}_2$, given $\underline{\psi}_2, \ldots, \underline{\psi}_n < x$:

$$\min_{\overline{\psi}_1 \le \overline{\psi}_2} \left(\frac{c_1\left(\overline{\psi}_1\right)}{x - \overline{\psi}_1} - \frac{c_1\left(\overline{\psi}_2\right)}{x - \overline{\psi}_2} \right) \mathbb{1}_{\{\overline{\psi}_1 < \overline{\psi}_2\}}$$

- For $i \le n$, assume $\overline{\psi}^*_{i-1}(x)$ does not depend on $\overline{\psi}_i$ on $\{\overline{\psi}^*_{i-1}(x) < \overline{\psi}_i\}$ for all $x \ge 0$. Then with $\psi_{n+1}(x) = x$:

$$\min_{\overline{\psi}_i \le \overline{\psi}_{i+1}} \frac{c_i\left(\overline{\psi}_i\right)}{x - \overline{\psi}_i} - \left(\frac{c_{i-1}\left(\overline{\psi}_i\right)}{x - \overline{\psi}_i} - \frac{c_{i-1}\left(\overline{\psi}^*_{i-1}(x)\right)}{x - \overline{\psi}^*_{i-1}(x)} \right) \mathbb{1}_{\{\overline{\psi}^*_{i-1}(x) < \overline{\psi}_i\}}$$

These steps are similar to the 2$-$marginals case of Brown, Hobson and Rogers...

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Solving the $n-$marginals problem

Step 1 : $\psi_1^* = b_1^{-1}$

Step i : minimization over $\overline{\psi}_i \leq \overline{\psi}_{i+1}$, given $\overline{\psi}_{i-1}^*$:

• If $b_i \leq b_{i+1}$, we find $\psi_i^* = b_i^{-1}$ (this recovers Madan and Yor)

• In the general case (not covered in the literature), we rely on :

Lemma Let $i = 2, \ldots, n$ be fixed, assume that $c_i \geq c_{i-1}$, and let $\overline{\psi_i}^*(x)$ be any minimizer of the Step i problem. Then, the function $x \longmapsto \overline{\psi_i}^*(x)$ is nondecreasing

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Step 1 : $\psi_1^* = b_1^{-1}$

Step i : minimization over $\overline{\psi}_i \leq \overline{\psi}_{i+1}$, given $\overline{\psi}_{i-1}^*$:

- If $b_i \leq b_{i+1}$, we find $\psi_i^* = b_i^{-1}$ (this recovers Madan and Yor)

- In the general case (not covered in the literature), we rely on :

Lemma Let $i = 2, \ldots, n$ be fixed, assume that $c_i \geq c_{i-1}$, and let $\overline{\psi_i}^*(x)$ be any minimizer of the Step i problem. Then, the function $x \longmapsto \overline{\psi_i}^*(x)$ is nondecreasing

Future projects

- Continuous-time limit (extension of Madan-Yor 02)
- Construct martingale processes corresponding to the bound
- Lower/upper bound on Variance calls given 1 (and more generally n) marginals,
- General theory for the treatment of stochastic control problems given marginals, i.e. optimal transportation along controlled stochastic dynamics

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