CVaR hedging using quantization based stochastic approximation

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Outlines

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Risk hedging : Why ?

▷ Complete market, if L is measurable w.r.t. the market filtration :

$$L = \mathbb{E}[L] + \int_0^T \phi_s . dS_s, \text{ p.s.}$$

▷ Incompleteness of energy markets (electricity, gas, ...) and of financial markets

- stochastic volatility,
- spikes on *day-ahead* contracts,
- climatic (like temperature) as a source of randomness in the loss L or contingent claim.

▷ Solutions ? :

- Superhedging : N. El-Karoui, M.-C.Quenez, 1995, ...
- Maximization of expected Utility : S.D. Hodges & al., 1989, I. Karatzas & al., 1991, N. El Karoui & al., 2000,...
- Minimization of quadratic risk : H. Foellmer, D. Sonderman, 1986, M. Schweizer, 1991, ...

Framework and notations

Discrete times : $t_0 = 0 < t_1 < \cdots < t_M = T$

Source of incompleteness

L contains a source of randomness $Z := (Z_\ell)_{0\leq \ell \leq M}$ *observable* but *non-tradable* on the market.

- ▷ d assets $X = (X_\ell)_{0\leq \ell\leq M}$, X_ℓ $\mathbb{R}^d$-valued r.v.
- ▷ Filtration : $\mathbb{G} = (\mathcal{G}_\ell)_{0\leq \ell\leq M}$, $\mathcal{G}_\ell := \sigma\{X_i, Z_i; 0 \leq i \leq \ell\}$.
- ▷ Example : energy provider
  - buys a quantity C_M of gas at price S^g_T,
  - sells it to consumers at fixed price K,
  - $Z := (Z_\ell)_{0\leq \ell\leq M}$, temperature.

$$L = (S^g_M - K)\times C_M, \;\; C_M = a - bZ_M, \;\; \mathrm{d}Z_t = -\lambda(Z_t - \mu_t)\mathrm{d}t + \sigma_Z \mathrm{d}W_t.$$

▷ Risk is measured by $\mathrm{CVaR}_\alpha(L) := \mathbb{E}\left[L | L \geq \xi^*_\alpha\right]$, with $\mathrm{VaR}_\alpha(L) = \xi^*_\alpha$, such that $\mathbb{P}(L \leq \xi^*_\alpha) = \alpha$.

Outlines

Dynamic risk : $\mathcal{G}$-CVaR

Proposition : Rockafellar, Uryasev, (2000).

$\mathbb{E}[L_+] < +\infty$ and F_L (strictly) increasing.

$$\text{CVaR}_\alpha(L) = \inf_{\xi \in \mathbb{R}} \xi + \frac{1}{1-\alpha} \mathbb{E}\left[(L-\xi)_+\right] = \xi_\alpha^* + \frac{1}{1-\alpha} \mathbb{E}\left[(L-\xi_\alpha^*)_+\right]$$

and $\xi_\alpha^* = \text{VaR}_\alpha(L)$.

Definition : Bardou, Frikha, Pagès (2009).

Let L, such that $\mathbb{E}[L_+ \mid \mathcal{G}_\ell] < +\infty$ a.s.

$$\mathcal{G}_\ell\text{-CVaR}_\alpha(L) := \operatorname*{ess\,inf}_{\xi \in L^0_{\mathbb{R}}(\mathcal{G}_\ell)} \xi + \frac{1}{1-\alpha} \mathbb{E}\left[(L-\xi)_+ | \mathcal{G}_\ell\right].$$

sub-additivity, pos. homogeneity, Translation invariance and monotonicity.

CVaR hedging using self-financed strategies

Dynamic hedging vs. One step hedging

▷ $\mathcal{A}$ = self-financed strategies $\theta = (\theta_\ell)_{0 \leq \ell \leq M-1}$, $\theta_\ell \in L^0_{\mathbb{R}^d}(\mathcal{G}_\ell, \mathbb{P})$:

- Dynamic, adapted process with M trades, gain $\sum_{\ell=1}^{M} \theta_{\ell-1}.\Delta X_\ell$, with $\Delta X_\ell := X_\ell - X_{\ell-1}$
- One step, one trades ℓ_0 : $\theta_k \equiv \theta_{\ell_0}$, $k = \ell_0, \cdots, M$, gain $\theta_{\ell_0}.(X_M - X_{\ell_0})$.

▷ Dynamic hedging of static risk :

$$\inf_{\theta \in \mathcal{A}} \mathrm{CVaR}_\alpha \left(L - \sum_{\ell=1}^{M} \theta_{\ell-1}.\Delta X_\ell \right),$$

▷ One step hedging of static risk :

$$\inf_{\theta_{\ell_0} \in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell_0})} \mathrm{CVaR}_\alpha \left(L - \theta_{\ell_0}.(X_M - X_{\ell_0}) \right),$$

▷ One step hedging of forward risk :

$$\inf_{\theta_{\ell_0} \in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell_0})} \mathbb{E}\left[\mathcal{G}_{\ell_0}\text{-}\mathrm{CVaR}_\alpha \left(L - \theta_{\ell_0}.\left(X_M - X_{\ell_0}\right)\right)\right].$$

Existence of optimal strategies

The case of one step hedging

▷ Let $X = X_M - X_{\ell_0}$. Consider a regular version of the conditional law of (L, X) given $\mathcal{G}_{\ell_0}$: $\Pi(\omega, \mathrm{d}y, \mathrm{d}x)$.

Assumptions

- $L \in L^1_{\mathbb{R}}(\mathbb{P})$, $X \in \mathbb{L}^1_{\mathbb{R}^d}(\mathbb{P})$
- $\operatorname{ess\,inf}_{u \in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell_0}, \mathbb{P}),\ |u|=1} \mathbb{E}\left[(u.X)_+ \mid \mathcal{G}_{\ell_0}\right] > 0$ *a.s.*

$V : \Omega \times \mathbb{R} \times \mathbb{R}^d$ defined by

$$V(\omega, \xi, \theta) = \int \underbrace{\xi + \frac{1}{1-\alpha}(y - \theta.x - \xi)_+}_{v(\xi,\theta,y,x)} \Pi(\omega, \mathrm{d}x, \mathrm{d}y)$$

Existence of optimal strategies

There exists an optimal strategies $(\xi^*_{\alpha,2}, \theta^*_{\alpha,2}) \in \mathbb{R} \times L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell_0})$ (resp. $(\xi^*_{\alpha,3}, \theta^*_{\alpha,3}) \in L^0_{\mathbb{R}}(\mathcal{G}_{\ell_0}) \times L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell_0})$).

Why ?

Sketch of the proof

▷ Classical stochastic control result :

$$\inf_{\theta \in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell_0})} \mathrm{CVaR}_\alpha (L - \theta.X)$$

$$= \inf_{\xi \in \mathbb{R}} \mathbb{E}\left[\operatorname*{ess\,inf}_{\theta \in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell_0})} \mathbb{E}\left[\xi + \frac{1}{1-\alpha}(L - \theta.X - \xi)_+ \middle| \mathcal{G}_{\ell_0} \right]\right]$$

▷

$$\operatorname*{ess\,inf}_{\theta \in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell_0})} \mathbb{E}\left[\xi + \frac{1}{1-\alpha}(L - \theta.X - \xi)_+ \middle| \mathcal{G}_{\ell_0} \right](\omega) = \min_{\theta \in \mathbb{R}^d} V(\omega, \xi, \theta)$$

$$= V(\omega, \xi, \tilde{\theta}_\alpha) \quad p.s.$$

$V(\omega, \xi, .)$ is convex, Lipschitz continuous, $\lim_{|\theta| \to +\infty} V(\xi, \theta) = +\infty$.
▷ $\xi \mapsto \mathbb{E}\left[\min_{\theta \in \mathbb{R}^d} V(\xi, \theta)\right]$ is Lipschitz continuous, convex, and $\lim_{|\xi| \to +\infty} \mathbb{E}\left[\min_{\theta \in \mathbb{R}^d} V(\xi, \theta)\right] = +\infty$ (uniformly w.r.t. θ !). There exists $(\xi^*_{\alpha,2}, \theta^*_{\alpha,2})$ solution of the one step problem.

Why ?
Sketch of the proof

▷ Classical stochastic control result :

$$\inf_{\theta \in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell_0})} \mathrm{CVaR}_\alpha (L - \theta.X)$$

$$= \inf_{\xi \in \mathbb{R}} \mathbb{E}\left[\operatorname*{ess\,inf}_{\theta \in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell_0})} \mathbb{E}\left[\xi + \frac{1}{1-\alpha}(L - \theta.X - \xi)_+ \middle| \mathcal{G}_{\ell_0} \right]\right]$$

▷

$$\operatorname*{ess\,inf}_{\theta \in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell_0})} \mathbb{E}\left[\xi + \frac{1}{1-\alpha}(L - \theta.X - \xi)_+ \middle| \mathcal{G}_{\ell_0} \right](\omega) = \min_{\theta \in \mathbb{R}^d} V(\omega, \xi, \theta)$$

$$= V(\omega, \xi, \tilde{\theta}_\alpha) \quad p.s.$$

$V(\omega, \xi, .)$ is convex, Lipschitz continuous, $\lim_{|\theta| \to +\infty} V(\xi, \theta) = +\infty$.

▷ $V(\omega, \xi, .)$ is differentiable so :

$$\operatorname{Arg\,min} V(\omega, \xi, .) = \left\{ \theta \in \mathbb{R}^d \middle| \nabla_\theta V(\omega, \xi, \theta) = 0 \right\} \neq \varnothing,$$

Dynamic risk hedging of static risk

$$\inf_{\theta\in\mathcal{A}} \mathrm{CVaR}_\alpha\left(L - \sum_{\ell=1}^{M}\theta_{\ell-1}.\Delta X_\ell\right)$$
$$= \inf_{\xi\in\mathbb{R}}\inf_{\theta\in\mathcal{A}} \mathbb{E}\left[\xi + \frac{1}{1-\alpha}\left(L - \sum_{\ell=1}^{M}\theta_{\ell-1}.\Delta X_\ell - \xi\right)_+\right]$$

Assumptions

- $L \in \mathbb{L}^1_{\mathbb{R}}(\mathbb{P})$, $\Delta X_\ell \in \mathbb{L}^1_{\mathbb{R}^d}(\mathbb{P})$, $\ell = 1, \cdots, M$
- $\operatorname{ess\,inf}_{u\in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell-1},\mathbb{P}),\ |u|=1} \mathbb{E}\left[(u.\Delta X_\ell)_+ \mid \mathcal{G}_{\ell-1}\right] > 0$ *p.s.*

Existence of an optimal strategies

Using the ideas of the dynamic programming principle (Bellman functions, backward induction, ...), one shows the existence of an optimal strategy.

How does the risk evolves ?

(Good) properties of $\mathcal{G}$-CVaR

▷ Risk of the holder at time t_ℓ :

$$\mathcal{G}_\ell\text{-CVaR}_\alpha\left(L-\sum_{k=1}^{M}\theta_{k-1}.\Delta X_k\right),\quad \ell=0,\cdots,M.$$

Supermartingale & Convergence

Let $M=+\infty$. Let $L\in L^1(\mathcal{G}_\infty,\mathbb{P})$ where $\mathcal{G}_\infty=\vee_\ell\mathcal{G}_\ell$.

1. $(\mathcal{G}_\ell\text{-CVaR}_\alpha(L))_{\ell\geq 0}$ is a $(\mathbb{G},\mathbb{P})$-supermartingale.
2. $\mathcal{G}_n\text{-CVaR}_\alpha(L)\xrightarrow{a.s.}L,\ n\rightarrow+\infty$.

The dynamic risk is coherent

The estimation at time 0 of dynamic risk : $(\mathbb{E}\left[\mathcal{G}_\ell\text{-CVaR}_\alpha(L)\right])_{1\leq\ell\leq M}$ decreases with time ! The random risk decreases till L.

Risk evolution of hedged portfolio

Proposition

Let X $(\mathbb{G}, \mathbb{P})$-martingale. Then

$$\mathcal{G}_k\text{-CVaR}_\alpha\left(L - \sum_{\ell=1}^{M} \theta_{\ell-1}.\Delta X_\ell\right) = \sum_{\ell=1}^{k} \theta_{\ell-1}.\Delta X_\ell + \mathcal{G}_k\text{-CVaR}_\alpha\left(L - \sum_{\ell=k+1}^{M} \theta_{\ell-1}.\Delta X_\ell\right).$$

and,

$$\mathbb{E}\left[\mathcal{G}_k\text{-CVaR}_\alpha\left(L - \sum_{\ell=1}^{M} \theta_{\ell-1}.\Delta X_\ell\right)\right] = \mathbb{E}\left[\mathcal{G}_k\text{-CVaR}_\alpha\left(L - \sum_{\ell=k+1}^{M} \theta_{\ell-1}.\Delta X_\ell\right)\right].$$

Outlines

Numerical methods : stochastic approximation

One step hedging of forward risk

We would like to solve the forward risk minimization problem using a one step strategy :

$$\inf_{\theta_{\ell_0}\in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell_0},\mathbb{P})} \mathbb{E}\left[\mathcal{G}_{\ell_0}\text{-CVaR}_\alpha\left(L-\theta_{\ell_0}.\left(X_M-X_{\ell_0}\right)\right)\right].$$

Assumption

The $\mathbb{R}^d\times\mathbb{R}^q$-valued process $(X_\ell, Z_\ell)_{0\leq\ell\leq M}$ is Markov under $\mathbb{G}$.

We consider that :

$$X_M - X_{\ell_0} = G(X_{\ell_0}, Z_{\ell_0}, U_{\ell_0+1}), \quad \text{and} \quad L = F(X_{\ell_0}, Z_{\ell_0}, U_{\ell_0+1}), \quad U_{\ell_0+1}\perp\mathcal{G}_{\ell_0}.$$

Markov property + vector quantization

$$V(\xi,\theta,x,z) = \mathbb{E}\left[v(\xi,\theta,U) := \xi + \tfrac{1}{1-\alpha}\left(F(x,z,U) - \theta.G(x,z,U) - \xi\right)_+\right].$$

$$\inf_{\theta_{\ell_0}\in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell_0},\mathbb{P})} \mathbb{E}\left[\mathcal{G}_{\ell_0}\text{-CVaR}_\alpha\left(L-\theta_{\ell_0}.(X_M - X_{\ell_0})\right)\right]$$

$$= \mathbb{E}\left[\underbrace{\operatorname*{ess\,inf}_{\theta_{\ell_0}\in L^0(\mathcal{G}_{\ell_0},\mathbb{P}),\xi\in L^0(\mathcal{G}_{\ell_0})} \mathbb{E}\left[\xi + \frac{1}{1-\alpha}\left(L-\theta_{\ell_0}.(X_M - X_{\ell_0}) - \xi\right)_+ \,|\, \mathcal{G}_{\ell_0}\right]}_{=\left(\min_{(\xi,\theta)\in\mathbb{R}\times\mathbb{R}^d} V(\xi,\theta,x,z)\right)_{|x=X_{\ell_0},z=Z_{\ell_0}}}\right]$$

$$\approx \sum_{j=1}^{N_{\ell_0}} \min_{(\xi,\theta)\in\mathbb{R}\times\mathbb{R}^d} V(\xi,\theta,x^j_{\ell_0},z^j_{\ell_0})\mathbb{P}\left((X_{\ell_0},Z_{\ell_0})\in C_j(x_{\ell_0},z_{\ell_0})\right)$$

The local problem and stochastic approximation

V is convex, differentiable,

$$\nabla_{(\xi,\theta)} V(\xi,\theta) = \mathbb{E}\left[\left(H_1\left(\xi,\theta,U\right), H_{2:d+1}\left(\xi,\theta,U\right)\right)\right].$$

To compute the zero of $\nabla_{(\xi,\theta)} V$, we devise a Robbins-Monro :

$$\xi_n = \xi_{n-1} - \gamma_n H_1\left(\xi_{n-1}, \theta_{n-1}, U_n\right),$$
$$\theta_n = \theta_{n-1} - \gamma_n H_{2:d+1}\left(\xi_{n-1}, \theta_{n-1}, U_n\right),$$

$\triangleright$ We compute the local CVaR using a companion procedure

$$C_n = C_{n-1} - \gamma_n H_{d+2}\left(\xi_{n-1}, \theta_{n-1}, C_{n-1}, U_n\right)$$

$$H_1\left(\xi,\theta,u\right) = 1 - \frac{1}{1-\alpha}\mathbf{1}_{\{L-\theta X \geq \xi\}}, H_{2:d+1}\left(\xi,\theta,u\right) = -\frac{X}{1-\alpha}\mathbf{1}_{\{L-\theta X \geq \xi\}}$$

and $H_{d+2}\left(\xi,\theta,C,u\right) = C - v(\xi,\theta)$.

Convergence

$$(\xi_n, \theta_n, C_n) \xrightarrow{a.s.} \left(\xi^*_\alpha(x^j_{\ell_0}, z^j_{\ell_0}), \theta^*_\alpha(x^j_{\ell_0}, z^j_{\ell_0}), \inf_{(\xi,\theta)\in\mathbb{R}\times\mathbb{R}^d} V(\xi,\theta,x^j_{\ell_0}, z^j_{\ell_0})\right).$$

Computation of dynamic strategies
Crude hedging (C.H.)

$$\inf_{\theta\in\mathcal{A}} \mathrm{CVaR}_\alpha\left(L - \sum_{\ell=1}^{M} \theta_{\ell-1}.\Delta X_\ell\right),$$

We consider that :

$$X_\ell - X_{\ell-1} = G_\ell(X_{\ell-1}, Z_{\ell-1}, U_\ell), \text{ and } Z_\ell = T_\ell(Z_{\ell-1}, U_\ell), \; U_\ell \perp \mathcal{G}_{\ell-1},$$

▷ Vector quantization $(X_\ell, Z_\ell)_{1\leq\ell\leq M}$.

▷ For each date and each point of the quantization grid, strategie : θ_ℓ^j, $\ell = 0, ..., M-1$, $j = 1, ..., N_\ell$.

▷ VaR_α and CVaR_α do not depend of the considered node since the risk is "static".

Curse of dimension

When the dimension of the algorithm is low (≤ 100), it is very efficient : few dates and assets and few points of the quantization grid).

Several alternative methods : 1/3

Backward hedging (B.H.)

At time $M-1$. We solve

$$\inf_{\theta\in\mathcal{A}} \mathbb{E}\left[\mathcal{G}_{M-1}\text{-CVaR}_\alpha\left(L-\sum_{\ell=1}^{M}\theta_{\ell-1}.\Delta X_\ell\right)\right] = \inf_{\theta_{M-1}\in L^0_{\mathbb{R}^d}(\mathcal{G}_{M-1})} \mathbb{E}\left[\mathcal{G}_{M-1}\text{-CVaR}_\alpha\left(L-\theta_{M-1}.\Delta X_M\right)\right].$$

Solution θ^b_{M-1}, then one step backward

$$\inf_{\theta_{M-2}\in L^0_{\mathbb{R}^d}(\mathcal{G}_{M-2})} \mathbb{E}\left[\mathcal{G}_{M-2}\text{-CVaR}_\alpha\left(L-\theta^b_{M-1}.\Delta X_M-\theta_{M-2}\Delta X_{M-1}\right)\right],$$

solution θ^b_{M-2}, etc...

Error propagation

It allows to control the risk dynamically but there is propagation of an error.

Several alternative methods : 2/3

Martingale Decomposition method (M.D.H.)

We write L as a sum of martingale increments

$$L = \mathbb{E}[L] + \sum_{\ell=1}^{M} \tilde{\Delta} L_\ell, \quad \tilde{\Delta} L_\ell = \mathbb{E}[L | \mathcal{G}_\ell] - \mathbb{E}[L | \mathcal{G}_{\ell-1}]$$

$$\inf_{\theta\in\mathcal{A}} \mathrm{CVaR}_\alpha \left(L - \sum_{\ell=1}^{M} \theta_{\ell-1} \Delta X_\ell \right) \leq \mathbb{E}[L] + \sum_{\ell=1}^{M} \inf_{\theta_{\ell-1}\in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell-1})} \mathrm{CVaR}_\alpha \left(\tilde{\Delta} L_\ell - \theta_{\ell-1} \Delta X$$

and we solve each problem using the minoration :

$$\inf_{\theta_{\ell-1}\in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell-1})} \mathbb{E}\left[\mathcal{G}_{\ell-1}\text{-}\mathrm{CVaR}_\alpha \left(\tilde{\Delta} L_\ell - \theta_{\ell-1} \Delta X_\ell \right) \right] \leq \inf_{\theta_{\ell-1}\in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell-1})} \mathrm{CVaR}_\alpha \left(\tilde{\Delta} L_\ell - \theta_{\ell-1} \Delta X_\ell \right).$$

Several alternative methods : 3/3

Classical decomposition hedging (C.D.H.)

We write L as a sum of classical increments

$$L = L_0 + \sum_{\ell=1}^{M} \Delta L_\ell, \quad \Delta L_\ell = L_\ell - L_{\ell-1}$$

$$\inf_{\theta \in \mathcal{A}} \mathrm{CVaR}_\alpha \left(L - \sum_{\ell=1}^{M} \theta_{\ell-1} \Delta X_\ell \right) \leq \mathbb{E}[L] + \sum_{\ell=1}^{M} \inf_{\theta_{\ell-1} \in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell-1})} \mathrm{CVaR}_\alpha \left(\Delta L_\ell - \theta_{\ell-1} \Delta X \right.$$

and we solve each local problem :

$$\inf_{\theta_{\ell-1} \in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell-1})} \mathbb{E}\left[\mathcal{G}_{\ell-1}\text{-}\mathrm{CVaR}_\alpha \left(\Delta L_\ell - \theta_{\ell-1} \Delta X_\ell \right)\right] \leq \inf_{\theta_{\ell-1} \in L^0_{\mathbb{R}^d}(\mathcal{G}_{\ell-1})} \mathrm{CVaR}_\alpha \left(\Delta L_\ell - \theta_{\ell-1} \Delta X_\ell \right).$$

Importance Sampling and Linear Control Variate

$\triangleright$ $\alpha \sim 1$, slow and chaotic convergence : rare events.

$\triangleright$ Two variance reduction techniques combined adaptively with $(\xi_n, \theta_n, C_n)_{n\geq 1}$:

1. unconstrained recursive importance sampling : translate the loss distribution into the tail distribution.
2. Linear Control Variate : X is a martingale so $\mathbb{E}[\Delta X_\ell] = 0$. ΔX_ℓ can be used as a control variable.

Outlines

Numerical examples

One step hedging of static risk of Spark Spread option

$$L = \left(S^e_M - h_R S^g_M - C\right)_+ ,$$

where $M = 1$ (an), $h_R = 4$BTU/kWh, C=3\$/MWh, $\sigma_g = 0.4$, $\sigma_e = 0.8$, S^e, S^g are two G.B.M. with $S^e_0 = 40\$/MWh$, $S^g_0 = 3\$/MMBTU$.

$\triangleright$ M.C. method gives $\mathbb{E}[L] = 11.86$ with a variance of 3692 using 3 000 000 samples.(VR$_{\text{CVaR}}$ (LCV)≈ 2)

	No hedging		Static hedging			
α	VaR	CVaR	VaR	θ^*_α	CVaR	VR$_{\text{CVaR}}$ (IS)
95%	65.1	114.4	63.1	7.8	98.3	16.7
99%	142.2	208.3	120.2	13.6	163.2	19.0
99.5%	183.1	257.8	146.8	16.4	190.2	20.2

Histogram of the loss with and without hedging

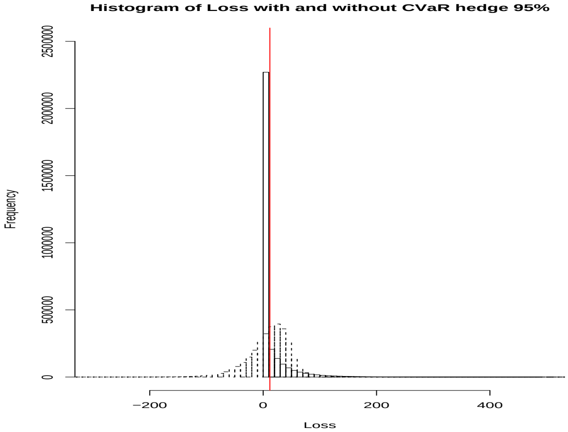

Figure: Histogram of loss distribution with (dashed lines) and without (normal lines) hedging at level $\alpha = 95\%$.

Comparison using $\alpha = 95\%$ and $\alpha = 99\%$.

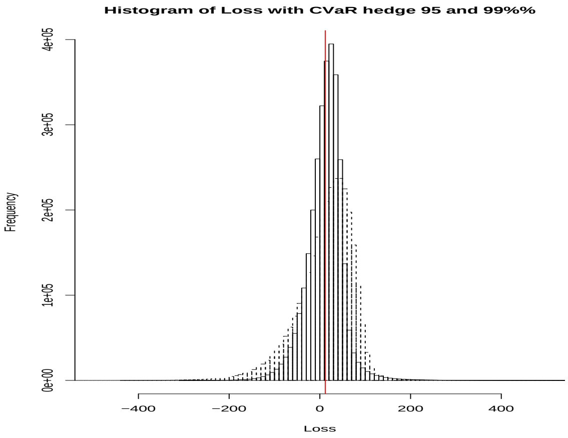

Figure: Histogram of the loss distribution with $\alpha = 95\%$ (normal lines) and $\alpha = 99\%$ (dashed lines).

Consumption hedging

▷ Gas provider :

- buys on spot market a quantity C_M of gas at price S_M^g
- sells it to consumers at price $K = 11$€/MWh.

▷ $C_M = a - b \times T_M$ is the consumption at time $M = 1$ year with $a = 100$ Mwh and $b = 3$ MWh/C^o.

▷ The temperature is modelized using an O-U process,

$$\mathrm{d}T_t = -\lambda(T_t - m)\mathrm{d}t + \sigma_T \mathrm{d}W_t,$$

The two B.M. are correlated with $\rho = -0.8$. The loss writes

$$L = (S_M^g - K)C_M.$$

Results

	No hedging		Static hedging		
α	VaR	CVaR	VaR	θ^*_α	CVaR
95%	784.6	1226.3	259.6	81.6	366.5
99%	1452.4	2012.3	437.1	89.9	537.3
99.5%	1769.9	2382.8	505.7	92.3	608.6

Table: Static risk hedging of the consumption using a one step strategy

Histograms of losses

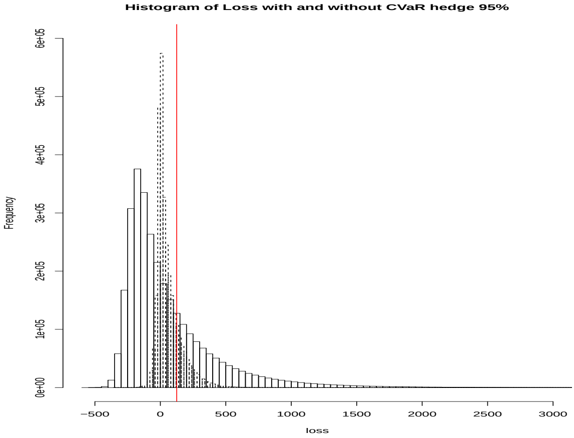

Figure: Histograms of losses with (dashed lines) and without (normal lines) CVaR hedging with $\alpha = 95\%$.

Dynamic hedging

Three number of trades : $M = 4$ (each quarter), $M = 12$ (each month), $M = 52$ (each week) and the confidence level $\alpha = 95\%$ is fixed. 10 points for the quantization grids.

	C.H.		B.H.		M.D.H.		C.D.H.	
M	VaR	CVaR	VaR	CVaR	VaR	CVaR	VaR	CVaR
4	178.3	240.9	175.9	252.5	177.8	252.9	178.9	259.2
12	163.2	214.1	160.7	233.8	158.7	221.7	161.9	232.9
52	272.6	395.1	158	233.2	148.7	210.1	153.1	223.7

Table: Dynamic CVaR hedging at level 95% using 4 methods.

Comparison one step and dynamic hedging

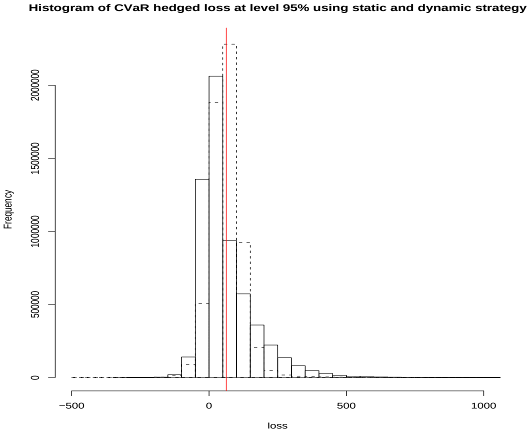

Figure: Histograms of CVaR hedged losses with $\alpha = 95\%$ using a one step (normal lines) and dynamic (dashed lines) strategies (M.D.H.) with 52 dates of trading.

Conclusion

$\triangleright$ One can solve other control problems like utility maximization problems using similar ideas.

$\triangleright$ $L^p(\mathbb{P})$-risk measure ρ

$$\inf_{\theta \in \mathcal{A}} \rho \left(L - \sum_{\ell=1}^{M} \theta_{\ell-1} . \Delta X_\ell \right)$$