On the robustness of Snell envelope and some approximation models

Peng HU

Centre INRIA de Bordeaux - Sud Ouest

Modeling and managing financial risks, Paris 2011

Preprints:
With P. Del Moral & N. Oudjane & B. Rémillard *On the Robustness of the Snell envelope* HAL-INRIA RR-7303 (2010).

Outline

1. Introduction
2. Broadie-Glasserman Models
3. Genealogical/Ancestral tree based model
4. Numerical examples of G.T model

1. Introduction
   - Snell envelope
   - Preliminary
   - Examples
   - Exponential concentration inequalities
2. Broadie-Glasserman Models
3. Genealogical/Ancestral tree based model
4. Numerical examples of G.T model

Snell envelope

Description

- Some Markov chain $(X_k)_{0\leq k\leq n}$, with $\eta_0 \in \mathcal{P}(E_0)$, $M_n(x_{n-1}, dx_n)$ from E_{n-1} to E_n on filtered space $(\Omega, \mathcal{F}, \mathbb{P}_{\eta_0})$, $\mathcal{F}_k$ associated natural filtration.
- For $f_k \in \mathcal{B}(E_k)$ (Payoff functions), gain process $Z_k := f_k(X_k)$ with $\mathcal{T}_k$ set of stopping times taking value in(k,k+1...n)
- **Purpose:** find $\sup_{\tau\in\mathcal{T}_k} \mathbb{E}(Z_\tau|\mathcal{F}_k)$

Snell envelope

- Y_k the **Snell envelope** of Z_k :

$$Y_n = Z_n$$
$$Y_k = Z_k \vee \mathbb{E}(Y_{k+1}|\mathcal{F}_k)$$

- Main property of the Snell envelope:

$$Y_k = \sup_{\tau \in \mathcal{T}_k} \mathbb{E}(Z_\tau|\mathcal{F}_k) = \mathbb{E}(Z_{\tau_k^*}|\mathcal{F}_k) \quad \tau_k^* = \min\{k \leq j \leq n \ : \ Y_j = Z_j\} \in \mathcal{T}_k$$

i.e.

- $Y_k = u_k(X_k)$

Snell envelope recursion:

$$u_k = f_k \vee M_{k+1}(u_{k+1}) \quad \text{with} \quad u_n = f_n$$

Numerical solution

- Replacing $(f_k, M_k)_{0\leq k\leq n}$ by some approximation model $(\widehat{f}_k, \widehat{M}_k)_{0\leq k\leq n}$ on some possibly reduced measurable subsets $\widehat{E}_k \subset E_k$.
- $\widehat{u}_k = \widehat{f}_k \vee \widehat{M}_{k+1}(\widehat{u}_{k+1})$ with terminal condition $\widehat{u}_n = \widehat{f}_n$ for $0 \leq k \leq n$

A robustness/continuity lemma

For any $0 \leq k < n$, on the state space $\widehat{E}_k$, we have that

$$|u_k - \widehat{u}_k| \quad \leq \quad \sum_{l=k}^{n} \widehat{M}_{k,l}|f_l - \widehat{f}_l| + \sum_{l=k}^{n-1} \widehat{M}_{k,l}|(M_{l+1} - \widehat{M}_{l+1})u_{l+1}|$$

Proof: By inequality $|(a \vee b) - (a' \vee b')| \leq |a \vee a'| + |b \vee b'|$ and induction.

Application examples of the lemma

Deterministic models

- Cut-off type models
- Euler approximation models
- Interpolation type models
- Quantization tree models

Monte Carlo models (stoch. N-grid approximation)

- Path space models
- ▷ Broadie-Glasserman models $[N^2]$
  ▷ BG type adapted mean-field particle model $[N^2]$
- Genealogical tree based model $[N]$

Exponential concentration inequalities

Important constants

$$\forall p \geq 0 \qquad a(2p)^{2p} = (2p)_p \; 2^{-p} \quad \text{and} \quad a(2p+1)^{2p+1} = \frac{(2p+1)_{p+1}}{\sqrt{p+1/2}} \; 2^{-(p+1/2)}$$

Proposition

If we have a Khinchine's type $\mathbb{L}_p$-mean error bounds in the following form:
$\forall$ *integer* $p \geq 1$ *and constant* c

$$\sqrt{N} \sup_{x \in E_k} ||u_k(x) - \widehat{u}_k(x)||_{\mathbb{L}_p} \leq a(p) \; c$$

then we have the following exponential concentration inequality

$$\sup_{x \in E_k} \mathbb{P}\left(|u_k(x_k) - \widehat{u}_k(x_k)| > \frac{c}{\sqrt{N}} + \epsilon \right) \leq exp\left(-N\epsilon^2/c^2\right)$$

Summary

1. Introduction
2. Broadie-Glasserman Models
   - Original Broadie-Glasserman
   - BG adapted mean-field particle model
3. Genealogical/Ancestral tree based model
4. Numerical examples of G.T model

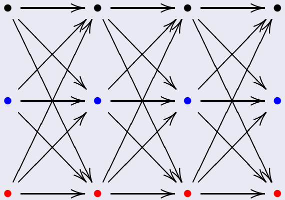
M. Broadie and P. Glasserman. A Stochastic Mesh Method for Pricing High- Dimensional American Options *Journal of Computational Finance* (04)

Original Broadie-Glasserman Models (hyp : $M'_k \ll \eta_k$)

$\eta_k \simeq \widehat{\eta}_k = \frac{1}{N}\sum_{i=1}^{N}\delta_{\xi_k^i}$ where $\xi_k := (\xi_k^i)_{1\leq i\leq N} \sim$ i.i.d. N-grid η_k on $\widehat{E}'_k = E'_k$

$$
\begin{aligned}
M'_{k+1}(x'_k, dx'_{k+1}) \simeq \widehat{M}'_{k+1}(x'_k, dx'_{k+1}) &= \widehat{\eta}_{k+1}(dx'_{k+1})\ \underbrace{R_{k+1}(x'_k, x'_{k+1})} \\
&= \widehat{\eta}_{k+1}(dx'_{k+1})\ \frac{dM'_{k+1}(x'_k, .)}{d\eta_{k+1}}(x'_{k+1})
\end{aligned}
$$

$(N = 3\ n = 3)$



$\rightsquigarrow$ N^2 *computations / time units*

By Khintchine's inequality we notice:

$$\sqrt{N}\ \left|\left|\left[M'_{l+1}-\widehat{M}'_{l+1}\right](f)(x'_l)\right|\right|_{\mathbb{L}_p}\leq 2\ a(p)\ \eta_{l+1}\left[(R_{l+1}(x'_l,\textbf{.})f)^p\right]^{\frac{1}{p}}$$

We provide the following non asymptotic convergence estimate

Theorem

For any integer $p\geq 1$, we denote by p' the smallest even integer greater than p. Then for any time horizon $0\leq k\leq n$, and any $x'_k\in E'_k$, we have

$$\begin{aligned}\sqrt{N}\,||u'_k(x'_k)-\widehat{u}'_k(x'_k)||_{\mathbb{L}_p}&\\&\leq 2a(p)\sum_{k\leq l<n}\left\{\int M'_{k,l}(x'_k,dx'_l)\eta_{l+1}\left[(R_{l+1}(x'_l,\textbf{.})u_{l+1})^{p'}\right]\right\}^{\frac{1}{p'}}\end{aligned}$$

BG adapted mean-field particle model (N^2)

algorithm with the choice $\eta_k = \mathrm{Law}(X'_k) = \eta_{k-1} M'_k$

Description (hyp. : $M'_k \ll \lambda_k$)

$\eta_k \simeq \widehat{\eta}_k = \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi_k^i}$ with i.i.d. copies ξ_k^i of X'_k

$$M'_{k+1}(x'_k, dx'_{k+1}) \simeq \widehat{M}'_{k+1}(x'_k, dx'_{k+1}) \quad = \quad \widehat{\eta}_{k+1}(dx'_{k+1}) \, \frac{H_{k+1}(x'_k, x'_{k+1})}{\widehat{\eta}_k(H_{k+1}(\,.\,, x'_{k+1}))}$$

with

$$(H)_0 \qquad H_n(x'_{n-1}, x'_n) = \frac{dM'_n(x'_{n-1}, \,.\,)}{d\lambda_n}(x'_n)$$

BG adapted mean-field particle model

Snell envelope

- Set by recursion $\widehat{u}'_k(x'_k) = f'_k(x'_k) \vee \left(\int_{\widehat{E}'_{k+1}} \widehat{M}'_{k+1}(x'_k, dx'_{k+1})\ \widehat{u}'_{k+1}(x'_{k+1}) \right)$ with terminal condition $\widehat{u}'_n = f'_n$

Theorem

$(H)_1$

$$\|M'_{l+1}(h^{2p}_{l+1})\| < \infty \text{ with } \sup_{x'_l, y'_l \in E'_l} \frac{H_{l+1}(x'_l, x'_{l+1})}{H_{l+1}(y'_l, x'_{l+1})} \leq h_{l+1}(x'_{l+1})$$

$$\|M'_{l+1}(u^{2p}_{l+1})\| < \infty$$

Under the conditions $(H)_0$ and $(H)_1$ stated above, for any even integer $p > 1$, any $0 \leq k \leq n$, and $x'_k \in E'_k$, we have

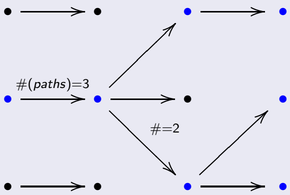
$$\sqrt{N}\ ||u'_k(x'_k) - \widehat{u}'_k(x'_k)||_{\mathbb{L}_p} \leq 2a(p) \sum_{k \leq l < n} \left(\|M'_{l+1}(h^{2p}_{l+1})\|\ \|M'_{l+1}(u^{2p}_{l+1})\| \right)^{\frac{1}{2p}}$$

Summary

1. Introduction
2. Broadie-Glasserman Models
3. Genealogical/Ancestral tree based model
4. Numerical examples of G.T model

Evolution example of genealogical tree

$(N = 3 \; n = 3)$



⊕ Snell envelope computation on the N-stochastic grid

notation

- The k-th coordinate mapping

$$\pi_k \ : \ x_n = (x'_0, \ldots, x'_n) \in E_n = (E'_0 \times \ldots \times E'_n) \mapsto \pi_k(x_n) = x'_k \in E'_k$$

- $\forall 0 \leq k < n$, $x'_k \in E'_k$ and any function $f \in \mathcal{B}(E'_{k+1})$, we have

$$\eta_n = \mathrm{Law}(X'_0, \ldots, X'_n) \quad \text{and} \quad M'_{k+1}(f)(x) := \frac{\eta_n((1_x \circ \pi_k)\ (f \circ \pi_{k+1}))}{\eta_n((1_x \circ \pi_k))}$$

- Remark $\eta_n = \eta'_0 \times M'_1 \times \cdots \times M'_n = \eta_{n-1} M_n$

Genealogical tree based model

Particle system = Neutral genetic particle algorithm

- Markov chain taking values in the product state spaces E_k^N.
- Initial system $\bar{X}_0 = (\bar{X}_0^i)_{1\leq i\leq N}$ i.i.d. random copies of X_0
- Evolution

$$\bar{X}_k \in E_k^N \xrightarrow{\text{Selection}} \widehat{X}_k := \left(\widehat{X}_k^i\right)_{1\leq i\leq N} \in E_k^N \xrightarrow{\text{Mutation}} \bar{X}_{k+1} \in E_{k+1}^N$$

Structure = Ancestral lines

$$\bar{X}_k = \begin{bmatrix} \bar{X}_k^1 \\ \vdots \\ \bar{X}_k^i \\ \vdots \\ \bar{X}_k^N \end{bmatrix} = \begin{bmatrix} (\bar{X}_{0,k}^1, \bar{X}_{1,k}^1, \ldots, \bar{X}_{k,k}^1) \\ \vdots \\ (\bar{X}_{0,k}^i, \bar{X}_{1,k}^i, \ldots, \bar{X}_{k,k}^i) \\ \vdots \\ (\bar{X}_{0,k}^N, \bar{X}_{1,k}^N, \ldots, \bar{X}_{k,k}^N) \end{bmatrix}$$

Remark

$$\begin{aligned} \bar{X}_{k+1}^i &= \left(\underbrace{(\bar{X}_{0,k+1}^i, \bar{X}_{1,k+1}^i, \ldots, \bar{X}_{k,k+1}^i)}_{||}, \bar{X}_{k+1,k+1}^i \right) \\ &= \left(\overbrace{(\widehat{X}_{0,k}^i, \widehat{X}_{1,k}^i, \ldots, \widehat{X}_{k,k}^i)}, \bar{X}_{k+1,k+1}^i \right) = \left(\widehat{X}_k^i, \bar{X}_{k+1,k+1}^i \right) \end{aligned}$$

Genealogical tree based model

Occupation measures

$$\eta_k^N := \frac{1}{N} \sum_{1 \le i \le N} \delta_{\bar{X}_k^i} \quad \text{and} \quad \widehat{\eta}_k^N := \frac{1}{N} \sum_{1 \le i \le N} \delta_{\widehat{X}_k^i}$$

- $\widehat{\eta}_k^N$: empirical meas. of $\widehat{X}_k^i \overset{c-iid}{\sim} \eta_k^N$
- η_k^N : empirical meas. of $\bar{X}_k^i \overset{c-iid}{\sim} \widehat{\eta}_{k-1}^N M_k$

With elementary decomposition

$$[\eta_n^N - \eta_{k-1}^N M_{k,n}] = \sum_{l=k}^{n} [\eta_l^N - (\eta_{l-1}^N M_l)] M_{l,n}$$

and Khintchine's inequality, by induction we have following estimates

Lemma

For any $p \geq 1$, p' the smallest even integer greater than p. In this notation, for any $k \geq 0$ and any function f, we have the almost sure estimate

$$\sqrt{N}\, \mathbb{E}\left(\left| [\eta_n^N - \eta_{k-1}^N M_{k-1,n}](f) \right|^p \middle| \mathcal{F}_{k-1}^N \right)^{\frac{1}{p}} \leq 2a(p) \sum_{l=k}^{n} \left[\eta_{k-1}^N M_{k-1,l}(|M_{l,n}(f)|^{p'}) \right]^{\frac{1}{p'}}$$

Approximation of the Markov transitions M'_{k+1}

$$\widehat{M}'_{k+1}(f)(x) := \frac{\eta_n^N((1_x \circ \pi_k)\ (f \circ \pi_{k+1}))}{\eta_n^N((1_x \circ \pi_k))} := \frac{\sum_{1\leq i\leq N}\ 1_x(\bar{X}^i_{k,n})\ f(\bar{X}^i_{k+1,n})}{\sum_{1\leq i\leq N} 1_x(\bar{X}^i_{k,n})}$$

Construction of Model

$$\widehat{u}_k(x) = \begin{cases} f_k(x) \vee \widehat{M}'_{k+1}(\widehat{u}_{k+1})(x) & \forall x \in \widehat{E}_{k,n} \\ 0 & \text{otherwise} \end{cases}$$

In terms of the ancestors at level k, this recursion takes the following form

$$\forall 1 \leq i \leq N \qquad \widehat{u}_k\left(\bar{X}^i_{k,n}\right) = f_k\left(\bar{X}^i_{k,n}\right) \vee \widehat{M}'_{k+1}(\widehat{u}_{k+1})\left(\bar{X}^i_{k,n}\right)$$

Applying the local error given by precedent lemma and the robustness lemma, we finally get a non-asymptotic boundary for finite state space.

Theorem

For any $p \geq 1$, and $0 \leq i \leq N$ we have the following uniform estimate

$$\sup_{0 \leq k \leq n} \left\| (u_k - \widehat{u}_k)(\bar{X}^i_{k,n}) \right\|_{\mathbb{L}_p} \leq c_p(n)/\sqrt{N}$$

with some collection of finite constants $c_p(n) < \infty$ whose values only depend on the parameters p and n.

Summary

1 Introduction

2 Broadie-Glasserman Models

3 Genealogical/Ancestral tree based model

4 Numerical examples of G.T model

Numerical examples

Asset modeling

$$\frac{dX'(i)_t}{X'(i)_t} = rdt + \sigma_i dz_t^i, \quad i = 1, \ldots, d = 6.$$

- z^i independent standard Brownian motions
- r=5% annually
- $X'_0(i) = 1$ and $\sigma_i = 20\%$ annually

Bermudan options

Maturity $T = 1$ year and 11 equally distributed exercise opportunities:

- geometric average put with payoff $(K - \prod_{i=1}^{d} X'(i)_T)_+$, $K = 1$
- arithmetic average put with payoff $(K - \frac{1}{d}\sum_{i=1}^{d} X'(i)_T)_+$, $K = 1$

Benchmark

Nb. assets	1	2	3	4	5	6
Geometric	0.06033	0.07815	0.08975	0.09837	0.10511	0.11073
Arithmetic	0.0603331	0.03881	0.02945	0.02403	0.02070	0.01895

Figure: Benchmark values for the geometric and arithmetic put options (taken from B. Bouchard and X. Warin, Monte-Carlo valorisation of American options: facts and new algorithms to improve existing methods, To appear in *Numerical Methods in Finance*, ed. R. Carmona, P. Del Moral, P. Hu and N. Oudjane (2011).

State space discretization

Methods : random tree, stochastic mesh, Binomial tree, quantization approach or quantization-like approach:

State space partitioning

1. Simulate N i.i.d. paths according to asset dynamic
2. At each time step, partition the particles into M subsets
3. For each subset, compute the representative state $(S_k^j)_{1 \leq j \leq M, 1 \leq k \leq n}$ as average of particles

Finite state space Markov chain

1. Define $\tilde{E}_k = \{S_k^1, \ldots, S_k^M\}$ as new finite state space.
2. The dynamic of new Markov chain $\tilde{X}_k$:

$$\mathbb{P}\left(\tilde{X}_k = S_k^j \,|\, \tilde{X}_{k-1} = S_{k-1}^i\right) = \mathbb{P}\left(X_k \in V_k^j \,|\, X_{k-1} = S_{k-1}^i\right)$$

Complexity and errors

Complexity : Forward step $O(MN)$, Backward step $O(N)$

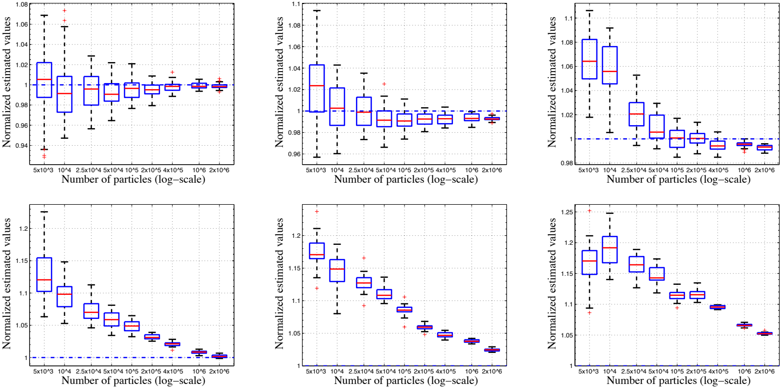
- State discretization error bounded by $\frac{c}{M^{\frac{1}{d}}}$
- G.T algorithm error bounded by $c\frac{M^{\beta}}{N}$, for $\beta > 0$

Optimization

Set $M = O(N^{\frac{d}{2\beta d+2}})$

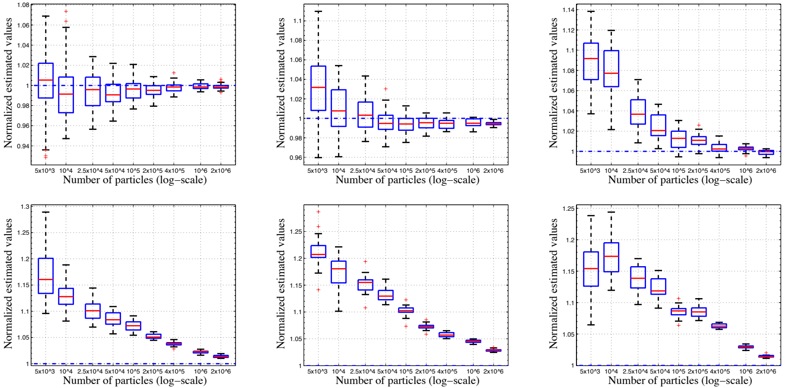
- Global complexity of order $N^{\frac{(1+2\beta)d+2}{2\beta d+2}}$
- Approximation error bounded by $\frac{c}{N^{\frac{1}{2\beta d+2}}}$

In following example, we set $\beta = 1/2$ so that the complexity grows with the dimension from $N^{4/3}, N^{3/2}, N^{8/5}, \cdots, N^{2}$ for dimensions $d = 1, 2, 3, \cdots, \infty$.



Boxplots for estimated option values (divided by the benchmark values) as a function of the number of particles for the **geometric** put-payoff. The box stretches from the 25th percentile to the 75th percentile, the median is shown as a line across the box, the whiskers extend from the box out to the most extreme data value within 1.5 IQR (Interquartile Range) and red crosses indicates outliers.

Numerical examples



Boxplots for estimated option values (divided by the benchmark values) as a function of the number of particles for the **arithmetic** put-payoff.

- Bermudan options with path dependent payoff
  see P. Del Moral & N. Oudjane *Snell envelope with path dependent multiplicative optimality criteria* `HAL-INRIA RR-7360 (2010)`
- Extend to the more general case of reflected Backward Stochastic Differential Equations (BSDE) with non zero driver that does not depend on the z variable and which satisfies suitable regularity conditions.
- Extend for the computation of price sensitivities for hedging purposes.

Thank you!