Equation for the Calibration of stochastic volatility models: theoretical and numerical study

Modeling and managing financial risks

Rémi Tachet

Paris, 10-13 January 2011

Plan

Plan

Dupire's formula

Local Volatility models

Diffusion equation for the spot :

$$dS_t = \sigma(t, S_t)S_t dW_t + rS_t dt$$

with W_t a standard brownian motion and $\sigma(t, S_t)$ the local volatility, function of the spot.

Value for the Local Volatility

We consider a given implied volatility surface $\Sigma(T, K)$.

$$\sigma^2(T, K) = 2\frac{C_T + rKC_K}{K^2 C_{KK}}$$

Limits for this model : the dynamics of this model's smile is inconsistent with the market, unstable hedges.

Local and Stochastic Volatility models (LSV)

Diffusion equations

The couple spot-volatility follows the stochastic differential equation

$$\frac{dS_t}{S_t} = a(t, S_t)b(t, y_t)dW_t^1$$
$$dy_t = \beta_t dt + \alpha(t, y_t)dW_t^2$$

- W_t^1 and W_t^2 are two ρ correlated brownian motions.
- y_t is the stochastic factor of the volatility, the function $b(t, y_t)$ transforms this factor into a proper volatility.
- $a(t, S_t)$ is the local part of the volatility, we shall use it to calibrate the vanillas of our model.

Kolmogorov equation for the LSV model

We let $p(t, S, y)$ denote the density of the couple (S_t, y_t). Such a density verifies the Kolmogorov forward equation.

Partial differential equation for p

$$\frac{\partial p}{\partial t} - \frac{\partial^2}{\partial S^2}(\frac{1}{2}a^2 b^2 S^2 p) - \frac{\partial^2}{\partial S \partial y}(\rho a b \alpha S p) - \frac{\partial^2}{\partial y^2}(\frac{1}{2}\alpha^2 p)) + \frac{\partial}{\partial y}(\beta p) = 0$$

The initial condition for p is $p(S, y, 0) = \delta(S = S_0, y = y_0)$ with (S_0, y_0) chosen according to the market.

First marginal law of the couple (S_t, y_t)

We integrate this equation with respect to y, writing $q = \int p dy$:

$$\frac{\partial q}{\partial t} - \frac{\partial^2}{\partial S^2}(\frac{1}{2}a^2 S^2 (\int b^2 p dy)) = 0$$

Calibration method

Statement : two models with the same spot marginals give the same vanillas.

Comparison with Dupire's formula

With $b = 1$ and $a = \sigma_D$, the model boils down to a local volatility model, the density q_D of S_t in this model verifies :

$$\frac{\partial q_D}{\partial t} - \frac{\partial^2}{\partial S^2}(\frac{1}{2}\sigma_D^2 S^2 q_D) = 0$$

Since the vanillas of Dupire's model are calibrated to the market, we want to match this last equation to the one on the marginal density of the LSV

$$\frac{\partial q}{\partial t} - \frac{\partial^2}{\partial S^2}(\frac{1}{2}a^2 S^2 \frac{\int b^2 p dy}{q} q) = 0$$

Equation for the calibration

Condition on the local term of the volatility

Identifying the two previous equations, the calibration of the LSV model requires that :

$$a^2(t,S) = \sigma_D^2(t,S)\frac{q}{\int b^2 p dy} = \sigma_D^2(t,S)\frac{\int p dy}{\int b^2 p dy}$$

Equation on $p(t,S,y)$, density of S_t, y_t

$$\frac{\partial p}{\partial t} - \frac{\partial^2}{\partial S^2}\left(\frac{1}{2}\sigma_D^2 b^2 S^2 \frac{\int p dy}{\int b^2 p dy} p\right) - \frac{\partial^2}{\partial S \partial y}\left(\rho \sigma_D b \alpha S \left(\frac{\int p dy}{\int b^2 p dy}\right)^{\frac{1}{2}} p\right) - \frac{\partial^2}{\partial y^2}\left(\frac{1}{2}\alpha^2 p\right)) + \frac{\partial}{\partial y}(\beta p) = 0$$

Properties of the equation

This equation is classified as a

- second order parabolic equation
- nonlinear
- integro-differential, because of the term $I(p) = \frac{\int pdy}{\int b^2 pdy}$.

The principal part of this equation contains the following term

$$\frac{\partial^2}{\partial S^2}\left(\frac{1}{2}\sigma_D^2 b^2 S^2 I(p)p\right)$$

Remarks :

- The same equation with the term $I(p)$ outside of the derivative has been studied. It is possible to prove existence and uniqueness of solutions.
- As far as our equation is concerned, we can only obtain existence under restrictive assumptions on b.

Existence Theorem

The idea of the proof is based upon the fact that, if b does not "move" too much, we have

$$||I(p) - \frac{1}{b^2}|| \leq \epsilon ||p||$$

Replacing $I(p)$ by this approximation, the equation becomes classic and we have existence and uniqueness results. Using those results, the appropriate spaces (Holder) and a fixed point method, it is possible to prove the

Theorem

If $||b - \bar{b}|| \leq \epsilon$ for ϵ small enough and $\bar{b}$ a constant, then there exists a function p solution of our calibration problem.

Plan

Calibrated model

Let us consider a simple mean reverting model for the volatility.

Lognormal model

We let σ_t denote the stochastic volatility.

$$\begin{aligned} \sigma_t &= b(y_t) = e^{y_t} \\ dy_t &= \kappa(\alpha - y_t)dt + \gamma dW_t^2 \end{aligned}$$

The parameters of the model are given for instance by a historical calibration. Here, for the sake of simplicity, we choose

$$\begin{aligned} \kappa = \alpha = \gamma = \sigma_0 &= 0.2 \\ \rho &= -0.8 \end{aligned}$$

Algorithm for the resolution

We discretize the equation on a $(t, x = log(S), y)$ grid, it becomes

$$\frac{\partial p}{\partial t} - \frac{\partial^2}{\partial x^2}(\frac{1}{2}\sigma_D^2 b^2 I(p)p) - \frac{\partial^2}{\partial x \partial y}(\rho\sigma_D b\gamma\sqrt{I(p)}p) - \frac{\partial^2}{\partial y^2}(\frac{1}{2}\gamma^2 p))$$
$$-\frac{\partial}{\partial x}(\frac{1}{2}\sigma_D^2 b^2 I(p)p) + \frac{\partial}{\partial y}(\kappa(\alpha - y)p) = 0$$

Algorithm

- Two-step predictor-corrector Alternate Direction Implicit (ADI) scheme, convergence in dt^2, dx^2, dy^2.
- At time step i, we use the coefficient $I(p)(S_j, t_{i-1})$ instead of the unknown $I(p)(S_j, t_i)$.

Results of the calibration

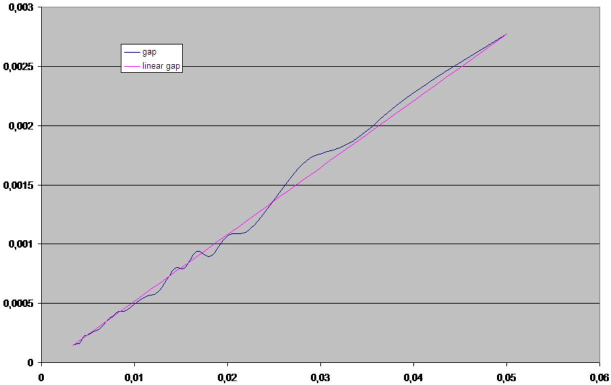
We use a realistic implied volatility surface as input, compute Dupire's volatility σ_D, solve the pde with our algorithm and compare the vanilla prices we get

T	K	Call BS	Call	Gap	Put BS	Put	Gap
1.0	0.5	51.29	51.32	-0.03	1.29	1.31	-0.02
1.0	0.6	42.56	42.60	-0.04	2.56	2.58	-0.02
1.0	0.7	34.43	34.48	-0.05	4.43	4.46	-0.03
1.0	0.8	26.99	27.04	-0.05	6.99	7.02	-0.03
1.0	0.9	20.33	20.38	-0.05	10.33	10.36	-0.04
1.0	1.0	14.54	14.59	-0.05	14.54	14.58	-0.04
1.0	1.1	9.85	9.90	-0.04	19.85	19.88	-0.03
1.0	1.2	6.30	6.35	-0.04	26.30	26.33	-0.03
1.0	1.3	3.81	3.85	-0.04	33.81	33.83	-0.02
1.0	1.4	2.18	2.21	-0.03	42.18	42.20	-0.02
1.0	1.5	1.20	1.22	-0.02	51.20	51.21	-0.01

Maturity = 1 year

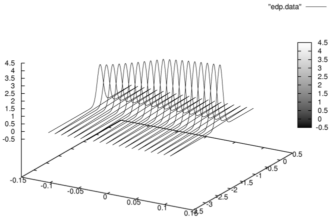
Time convergence rate

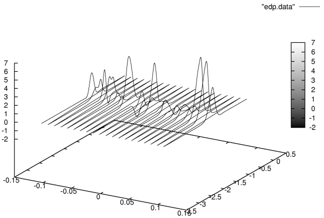
The method used to deal with the term $I(p)$ obviously changes the rate dt^2. We plot the error for at-the-money 1-year vanillas against the inverse of the number of time steps (from 20 to 300 per year).



Stability tests

The existence proof requires that the function b does not vary too much. In order to test if this condition is indeed necessary, we plot the density $p(1, x, y)$ for $b(y) = exp(y)$ and $b(y) = exp(10 * y)$





Plan

Hadamard instability

Definition

We say that a linear operator L is Hadamard unstable when for all $\epsilon > 0$, there exists two functions $\phi(x)$ and $p(t,x)$ verifying

$$\frac{\partial p}{\partial t} - Lp = 0 \qquad ||p(1,.)|| \geq 1$$

$$p_{|t=0} = \phi \qquad ||\phi|| \leq \epsilon$$

Example : backward heat equation

$L = -\Delta$ is Hadamard unstable. Indeed, with $(\lambda_n)_{n\in\mathbb{N}}$ the eigenvalues of L and $(p_n)_{n\in\mathbb{N}}$ corresponding eigenvectors, of norm 1, we see that $p = \epsilon exp(\lambda_n t)p_n$ is solution of $\frac{\partial p}{\partial t} - (-\Delta p) = 0$. Since the λ_n grow to infinity, with n big enough, $||p(1,.)||$ is bigger than 1.

Linearized equation

The proof of the first theorem was equivalent to an implicit functions theorem. We see the equation as an operator on the couple (p, b) and prove that the p- differential of the operator is an isomorphism.
Linearized equation around a given function p, with a correlation equal to 0

$$\frac{\partial h}{\partial t} - \frac{\partial^2}{\partial S^2}(\frac{1}{2}f^2(\frac{\int pdy}{\int b^2 pdy}h + \frac{\int hdy}{\int b^2 pdy}p - \frac{\int pdy \int b^2 hdy}{(\int b^2 pdy)^2}p)) - \frac{\partial^2}{\partial y^2}(\frac{1}{2}\alpha^2 h)) + \frac{\partial}{\partial y}(\beta h) = 0$$

An explanation for the numerical instability can be that this linearized equation is Hadamard unstable.

Ill-posedness of a generalized problem (1)

After suitable changes of variables, we write the linearized equation under the following form

$$\frac{\partial h}{\partial t} - \frac{\partial^2}{\partial x^2}(h + \nu \int \rho h dy) - \frac{\partial^2}{\partial y^2}(h) = 0$$

We choose $\nu = \sum_{k=0}^{n} a_k(t,x) y^k$ and search h under the form $h = \sum_{k=0}^{n} h_k(t,x) y^k$. By projecting the equation on the powers of y, we get coupled differential equations

$$\frac{\partial h_1}{\partial t} - \frac{\partial^2}{\partial x^2}(h_1(1 + a_1 \int y \rho dy) + \sum_{k \neq 1} h_k a_1 \int y^k \rho dy) = 0$$

that can be written as

$$(\frac{\partial h^i}{\partial t}) = M(\frac{\partial^2 h^i}{\partial x^2}) + \ldots$$

Ill-posedness of a generalized problem (2)

In the case n = 2, the matrix M is

$$\begin{pmatrix} 1 + a_0 \int \rho dy & a_0 \int y\rho dy & a_0 \int y^2 \rho dy \\ a_1 \int \rho dy & 1 + a_1 \int y\rho dy & a_1 \int y^2 \rho dy \\ a_2 \int \rho dy & a_2 \int y\rho dy & 1 + a_2 \int y^2 \rho dy \end{pmatrix}$$

A necessary and sufficient condition for the system of pde's to be well-posed is that the matrix M is definite positive. Here, its eigenvalues are 1, 1 and $1 + a_0 \int \rho dy + a_1 \int y\rho dy + a_2 \int y^2 \rho dy$. The condition is then

$$1 + a_0 \int \rho dy + a_1 \int y\rho dy + a_2 \int y^2 \rho dy = 1 + \int \nu\rho > 0$$

Since this is true for all polynoms, it is true for all functions. The operator is Hadamard unstable when $1 + \int \nu\rho < 0$.

"Orthogonality" of the LSV linearized equation

This computation does not work in the case of the LSV model.

$$\nu = \frac{p}{\int p} \qquad \rho = 1 - \frac{b^2}{\int b^2 p}$$

gives $1 + \int \nu\rho = 1$. The function ν is orthogonal to the function ρ.
Axis of research

- explain the instability when this orthogonality property is verified
- generalize the result when the correlation is different than 0

Plan

Conclusion

- The equation steming from the calibration of a Local and Stochastic Volatility model is a challenging mathematical problem. It is possible to prove existence of solutions under certain assumptions but the general case is still an open problem.
- From a numerical point of view, the calibration is efficient when applied to realistic data. However, instabilities seem to occur in more "extreme" cases.
- The study of those instabilities brought us towards Hadamard-type instabilities for the linearized problem. Unfortunately, that method was not conclusive. New angles have to be explored.

References

1. B. Dupire, Pricing and Hedging with Smiles, 1993
2. A. Friedman, Partial differential equations of parabolic type, 1964
3. N. Alibaud, Existence, uniqueness and regularity for nonlinear parabolic equations with nonlocal terms, 2007
4. D. Joseph and J. C. Saut, Short-wave instabilities and Ill-posed initial-value problems, 1990