Equity correlations implied by index options: estimation and model uncertainty analysis

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Multi-asset models I

An index is a weighted average of d stocks:

$$I(t) = \sum_{j=1}^{d} x_j \, S_j(t)$$

Two types of vanilla options exist:

- Single-name call options pay: $(S_j(T) - K)^+$ at maturity T.
- Index call options pay: $(I(T) - K)^+$ at maturity T.

Benchmark options: liquid options for which, prices are given by the supply/demand

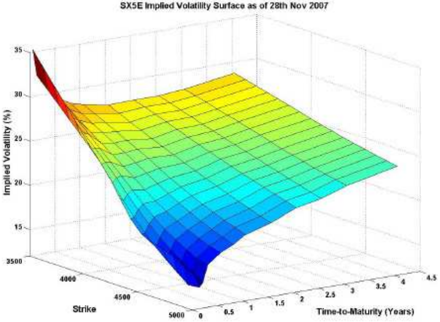

Figure: Implied Volatility (IV) surface of the Eurostoxx 50 index.

Multi-asset models II

- Merton model driven by a common Poisson process $(N_t)_{t \geq 0}$:

$$dS_j(t) = \mu_j \, S_j(t)dt + \sigma_j S_j(t) \, dB_j(t) + S_j(t^-) \left(e^{Y_j^{N(t)}} - 1 \right) dN(t),$$

$$\text{with} \qquad \mathbf{corr}(B_i(t), B_j(t)) = \rho_{i,j}^B \, t, \qquad \mathbf{corr}(Y_i, Y_j) = \rho_{i,j}^J$$

- CEV diffusion model:

$$dS_j(t) = r \, S_j(t)dt + \alpha_j S_j^{\beta_j} \, dB_j(t), \qquad \mathbf{corr}(B_i(t), B_j(t)) = \rho_{i,j}^B \, t$$

Calibration: we observe prices C_i^{bid}, C_i^{ask} of various benchmark option payoffs H_i with $i \in \mathcal{I}$ and look for $\mathbb{Q}$ (or equivalently the model parameters) such that

$$C_i^{\text{bid}} \leq \mathbb{E}^{\mathbb{Q}}[H_i] \leq C_i^{\text{ask}} \qquad \text{for } i \in \mathcal{I}$$

Goal: Joint calibration to index and single-name options

Index benchmark options $\Rightarrow (\eta, \rho_{i,j}^B, \rho_{i,j}^J)$ or $(\rho_{i,j}^B)$

- ill-posed inverse problem
- if $d = 30$ (Dow-Jones index) $\Rightarrow$ large number of parameters

Calibration approach: random mixtures of multi-asset models

- Select N multi-asset models $\mathbb{Q}_1, .., \mathbb{Q}_N \in M(S)$
- Consider random mixtures $\mathbb{Q}_{\mathbf{W}} \triangleq \frac{1}{N}\sum_{k=1}^{N} W_k \mathbb{Q}_k$ of these models, where $(W_1, \cdots, W_N) \sim \mu^N$ (prior distribution on weights)
- For any joint distribution ν of the weights, if we impose that $\frac{1}{N}\sum_{k=1}^{N} \mathbb{E}^{\nu}[W_k] = 1$ then

$$\mathbb{E}^{\nu}[\mathbb{Q}_{\mathbf{W}}] = \frac{1}{N}\sum_{k=1}^{N} \mathbb{E}^{\nu}[W_k]\mathbb{Q}_k \in M(S)$$

  defines an arbitrage-free pricing model.
- Finally, we impose that the pricing model verifies the following calibration constraints

$$\mathbb{E}^{\nu}\left[\frac{1}{N}\sum_{k=1}^{N} W_k\, \mathbb{E}_0^{\mathbb{Q}_k}[H_i]\right] \in [C_i^{\text{bid}}\,;\, C_i^{\text{ask}}]$$

Calibration approach: random mixtures of multi-asset models

- Calibration: we want the posterior ν^N, *as close as possible to* μ^N, and under which the calibration constraints are satisfied, i.e.

Definition (Minimum entropy random mixture)

$$\inf_{\nu \in \mathcal{P}(\mathbb{R}^N)} \mathcal{E}(\nu|\mu^N) := \mathbb{E}^{\mu^N}\left[\frac{d\nu}{d\mu^N} \ln\left(\frac{d\nu}{d\mu^N}\right)\right] \qquad \text{under the constraints}$$

$$\forall i \in \mathcal{I}, \qquad C_i^{bid} \leq \mathbb{E}^{\nu}\left[\frac{1}{N}\sum_{k=1}^{N} W_k\, \mathbb{E}^{\mathbb{Q}_k}[H_i]\right] \leq C_i^{ask}$$

$$\mathbb{E}^{\nu}\left[\frac{1}{N}\sum_{k=1}^{N} W_k\right] = 1$$

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Solution to the calibration problem

Assuming (W_k) are μ^N-bounded, independent, and (Slater conditions):

(a) : $\exists \nu \in \mathcal{C}$ s.t. $\mathcal{E}(\nu|\mu^N) < \infty$

(b) : $\exists \epsilon > 0$ s.t. $]1-\varepsilon, 1+\epsilon[\subset \left\{ \mathbb{E}^{\nu}\left[\frac{1}{N}\sum_{k=1}^{N} W_k\right], \nu \in \mathcal{P}(\mathbb{R}^N) \right\}$.

(c) : $\exists \nu \in \mathcal{P}(\mathbb{R}^N)$ s.t. $\forall i \in \mathcal{I}$

$$C_i^{bid} < \mathbb{E}^{\nu}\left[\frac{1}{N}\sum_{k=1}^{N} W_k\, \mathbb{E}^{\mathbb{Q}_k}[H_i]\right] < C_i^{ask}, \quad \text{and} \quad \mathbb{E}^{\nu}\left[\frac{1}{N}\sum_{k=1}^{N} W_k\right] = 1.$$

Theorem (Solution to the calibration problem)

The primal problem has a unique solution $\nu^N \in \mathcal{P}(\mathbb{R}^N)$ given by

$$\frac{d\nu^N}{d\mu^N}(\mathbf{w}) = \frac{\exp\left[\frac{1}{N}\sum_{k=1}^{N} w_k \left(\sum_{i\in\mathcal{I}}(\lambda_i^{b*} - \lambda_i^{a*})\, \mathbb{E}^{\mathbb{Q}_k}[H_i] + \lambda_0^*\right)\right]}{Z^N(\boldsymbol{\lambda}^*)} \qquad (1)$$

where $(\lambda_0^*, \boldsymbol{\lambda}^{b*}, \boldsymbol{\lambda}^{a*}) \in \mathbb{R} \times \mathbb{R}_+^{2|\mathcal{I}|}$ is the unique maximizer of

$$\max_{(\lambda_0, \boldsymbol{\lambda}^b, \boldsymbol{\lambda}^a) \in \mathbb{R}\times\mathbb{R}_+^{2|\mathcal{I}|}} \left\{ \lambda_0 + \sum_{i\in\mathcal{I}} (\lambda_i^b C_i^{bid} - \lambda_i^a C_i^{ask}) - \ln\left(Z^N(\boldsymbol{\lambda})\right) \right\} \qquad \textbf{(D)}$$

- **(D)** is an unconstrained convex problem in finite dimension
- So, the dual (**D**) can be solved easily, and by injecting its solution $\boldsymbol{\lambda}^*$ into (1), we obtain ν^N

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Motivation for considering random mixtures I

- If the weights are chosen to be deterministic, the model uncertainty analysis which relies on the statistical approach to the problem (particularly on the posterior distribution of the weights) can no longer be carried out.
- Moreover, for deterministic weights, the choice of the objective function is less obvious.
- The use of the relative entropy as an objective function has several advantages:
  1. It leads to a convex problem
  2. The dual problem can be easily solved by gradient descent algorithm in finite dimension
  3. The dimension of the dual problem does not depend on the number of model considered, but only on the number of constraints
- If duality cannot be exploited, the dimension of the optimization problem can be high. Indeed, in order to insure the existence of a solution, one would have to increase the number of models, which in turn would increase the dimension of the optimization problem.

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Motivation for considering random mixtures II

- Brigo and Mercurio (2002) propose log-normal mixtures with deterministic weights. They minimize the calibration error over the weights and the parameters of the models, which leads to a non-convex optimization problem!
- The Bayesian flavor of this approach relies on the fact that the weights are random with a distribution updated with market observation.
- Possible extensions of this static framework would require the weights to evolve randomly in time (e.g. Hidden Markov models).

Sensitivity to benchmark option prices

- Knowing ν^N, the price of any exotic payoff X is given by:

$$\Pi(X) = \mathbb{E}^{\nu^N}\left[\frac{1}{N}\sum_{k=1}^{N} W_k \mathbb{E}^{\mathbb{Q}_k}[X]\right] = \mathbb{E}^{\mu^N}\left[\frac{d\nu^N}{d\mu^N}\frac{1}{N}\sum_{k=1}^{N} W_k \mathbb{E}^{\mathbb{Q}_k}[X]\right]$$

- As in (Avellaneda et al, 2001), the price Π depends on the benchmark option prices $(C_i)_i \in I$ via the Lagrange multipliers $(\lambda_i^*)_{i\in I}$

Theorem (Sensitivities to input option prices)

By denoting Δ_i the sensitivity of the exotic price $\Pi(X)$ to the input price C_i, we have

$$\Delta_i = \frac{\partial \Pi(X)}{\partial C_i} = \sum_{j\in\mathcal{I}} (H^{-1})_{ij}\, \mathbf{Cov}^{\nu^N}\left(\frac{1}{N}\sum_{k=1}^{N} W_k \mathbb{E}^{\mathbb{Q}_k}[X], \frac{1}{N}\sum_{k=1}^{N} W_k \mathbb{E}^{\mathbb{Q}_k}[H_j]\right),$$

where $H_{ij} = \mathbf{Cov}^{\nu^N}\left(\frac{1}{N}\sum_{k=1}^{N} W_k \mathbb{E}^{\mathbb{Q}_k}[H_i], \frac{1}{N}\sum_{k=1}^{N} W_k \mathbb{E}^{\mathbb{Q}_k}[H_j]\right)$.

Hedging model uncertainty

The sensitivities $(\Delta_i)_{i\in I}$ correspond to the linear regression coefficients of $\frac{1}{N}\sum_{k=1}^{N} W_k \mathbb{E}^{\mathbb{Q}_k}[X]$ w.r.t. $\frac{1}{N}\sum_{k=1}^{N} W_k \mathbb{E}^{\mathbb{Q}_k}[H_i]$ under ν^N.

Therefore, the sensitivities $(\Delta_i)_{i\in I}$ solve

$$\min_{\beta}\left\{\mathbf{Var}^{\nu^N}\left(\frac{1}{N}\sum_{k=1}^{N} W_k\,\mathbb{E}^{\mathbb{Q}_k}\left[X-\beta_0-\sum_{i\in\mathcal{I}}\beta_i H_i\right]\right)\right\}.$$

- $\sum_i \Delta_i H_i$ may be viewed as a control variate to reduce the variance of the MC estimator.
- The Δ_i's represent a static hedge that minimizes the exposure to model uncertainty as measured by the variance.
- The calibration procedure provides the sensitivities with no additional computational cost.

MC algorithm for calibration/pricing/hedging

1. Generate N reference models $\mathbb{Q}_1, .., \mathbb{Q}_N$
2. Compute model prices of index vanilla options : $\mathbb{E}^{\mathbb{Q}_k}[H_i]$ for all $i \in \mathcal{I}$ and $k = 1, .., N$.
3. Solve the dual problem.
4. Generate L IID samples $W^l = (W_1^l, .., W_N^l)$, for $l = 1, .., L$ of the model weights from the prior μ^N.
5. Adjust each weight W_k with density

$$\frac{d\nu_k}{d\mu_k}(W_k) = \frac{\exp\left[\frac{1}{N} W_k \left(\sum_{i\in\mathcal{I}}(\lambda_i^{b*} - \lambda_i^{a*})\,\mathbb{E}^{\mathbb{Q}_k}[H_i] + \lambda_0^*\right)\right]}{Z_k(\boldsymbol{\lambda}^*)}.$$

6. Compute model prices of the multi-asset exotic option: $\mathbb{E}^{\mathbb{Q}_k}[X]$ for all $k = 1, .., N$.
7. An arbitrage-free price of the exotic payoff X is given by

$$\frac{1}{L}\sum_{l=1}^{L}\frac{1}{N}\sum_{k=1}^{N}\frac{d\nu_k}{d\mu_k}(W_k^l)\; W_k^l\,\pi_k \xrightarrow{L\to\infty} \Pi(X)$$

8. Without running additional Monte Carlo simulation, compute the sensitivity Δ_i of $\Pi(X)$ w.r.t. C_i, by linear regression.

Numerical results

Calibration procedure:

- We consider 4 classes of multi-asset models:
  (1) Merton model with intensity $\eta = 1$
  (2) Merton model with intensity $\eta = 2$
  (3) Merton model with intensity $\eta = 3$
  (4) CEV model
- We calibrate these models to single-name vanilla options available on the market
- We use the random mixture approach to calibrate the remaining parameters: $\theta = \{\rho^B_{i,j}\}$ for the CEV, and $\theta = \{\rho^B_{i,j}, \rho^J_{i,j}\}$ for the Merton (using index vanilla options):
  (1) Choose a correlation structures $\theta = (\rho^B_{i,j}, \rho^J_{i,j})$ for each multi-asset model $\mathbb{Q}_1, \ldots, \mathbb{Q}_N$
  (2) Choose a prior distribution μ^N for the weights: IID uniform, IID truncated exponential, ...
  (3) Solve the dual problem $(\mathbf{D})$ with a gradient descent algorithm $\Rightarrow \boldsymbol{\lambda}^*$
  (4) From $\boldsymbol{\lambda}^*$ we get ν^N

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Simulated data with Merton model (intensity $\eta = 1$)

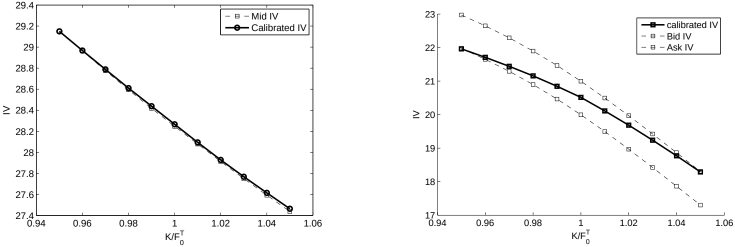

Figure: Calibrated IV of the single-names (left) and the index (right).

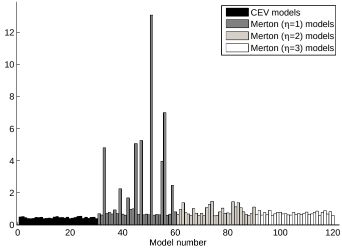

Figure: Means of the weights under the posterior distribution. Data simulated with Merton ($\eta = 1$)

Dow Jones market data

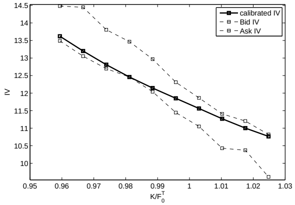

Figure: Calibrated IV of the DJ index.

η	ρ^B	ρ^J	Terminal correlation
1	27 %	93%	30%
5	14 %	95%	32%
5	22 %	77%	30%
10	1.3%	84%	29%
10	-1.5%	85%	28%
10	14 %	70%	28%
10	3.8%	81%	28%

Table: 7 models appear to give DJ index vanilla prices well within the bid/ask spread

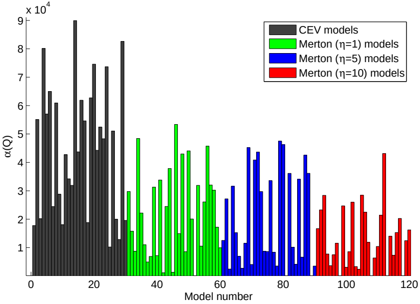

Figure: Biggest calibration error $\alpha(\mathbb{Q})$ on the DJ index vanilla (nominal $= 10^5$)

Quantifying model uncertainty (Cont, 2006)

- We define a set $\mathcal{Q}$ of martingale measures on Ω (not necessary calibrated)
- Then, we penalize each element $\mathbb{Q} \in \mathcal{Q}$ by the biggest pricing error on the benchmark options $H_1, \ldots, H_{|\mathcal{I}|}$ using $\mathbb{Q}$:

$$\alpha(\mathbb{Q}) = \max_{i \in \mathcal{I}} \max\{(C_i^{bid} - \mathbb{E}^{\mathbb{Q}}[H_i])^+, (\mathbb{E}^{\mathbb{Q}}[H_i] - C_i^{ask})^+\}$$

- Note: The nominal of H_i is determined by the (maximal) quantity of the i-th option available to the investor.
- For a given payoff X, we compute the convex risk measures:

$$\begin{aligned} \pi^*(X) &= \sup_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{E}^{\mathbb{Q}}[X] - \alpha(\mathbb{Q}) \right\} \\ \pi_*(X) = -\pi^*(-X) &= \inf_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{E}^{\mathbb{Q}}[X] + \alpha(\mathbb{Q}) \right\} \end{aligned}$$

  and build a model uncertainty measure ε as:

$$\varepsilon(X) = \pi^*(X) - \pi_*(X).$$

Model uncertainty analysis

Exotic Payoff X	Strike K	Confidence interval for $\widehat{\Pi_0(X)}$	$\epsilon(X)/\widehat{\Pi_0(X)}$
$(I(T)-K)^+$	138	[2.71082 ; 2.7113]	3.28%
$(I(T)-K)^+$	145	[0.4591 ; 0.4596]	12.35%
$(I(T)-K)^+ \ 1_{\max_{0\leq t\leq T} I(t)<B}$	138	[1.2661 ; 1.2677]	11.85%
$F_0^T \left(\min_{1\leq j\leq d} \frac{S_j(T)}{S_j(0)} - K\right)^+$	0.8	[9.6993 ; 9.7106]	5.59%
$F_0^T \left(\max_{1\leq j\leq d} \frac{S_j(T)}{S_j(0)} - K\right)^+$	1.1	[11.4348 ; 11.4491]	14.87%

Table: Model uncertainty measures and 95% confidence intervals for model prices of different multi-asset exotic options. The maturity of the options is $T = 11$ weeks. The barrier for the knock-out option is $B = 145$.

Conclusion

- We propose a method for constructing an arbitrage-free multi-asset pricing model which is consistent with a set of observed single- and multi-asset derivative prices.
- CEV models cannot be simultaneously calibrated to index and single-name vanilla options
- Our results are consistent with previous findings, (Branger and Schlag, 2004), which point to common jumps as an explanation for the steepness of the index smile.
- Among the Merton models which can be perfectly calibrated, the common jump intensity and the Brownian correlations can be very different!
- Nonetheless, all calibrated models exhibit the same terminal correlation:

$$\rho_{i,j}^{T'} = \frac{\sigma_i \sigma_j \rho_{i,j}^B + (m_i\, m_j + \rho_{i,j}^J \sqrt{v_i\, v_j})\eta}{\sqrt{\sigma_i^2 + (m_i^2 + v_i)\eta}\, \sqrt{\sigma_j^2 + (m_j^2 + v_j)\eta}}$$

- Low model uncertainty for ATM and worst-of call option.
- High model uncertainty for best-of, barrier, deep OTM call options.

Comparison with the Weighted Monte Carlo Algorithm (Avellaneda et al, 2001)

- Our approach can match the Weighted Monte Carlo setting if:
  1. The weights are deterministic
  2. The reference probabilities $\mathbb{Q}_k$ are chosen to be Dirac masses δ_{ω_k}, at specific market scenarios $\omega_k \in \Omega$.
- It is immediate to see that δ_{ω_k} is no longer a martingale probability since it corresponds to a specific path. Therefore, many constraints need to be added to restore the martingality of the weighted average $\frac{1}{N}\sum_{k=1}^{N} w_k\,\delta_{\omega_k}$:
  1. $\frac{N(N-1)}{2}$ constraints in discrete time (one for each pair of time)
  2. Infinitely many in continuous time.
- The duality approach would therefore no longer be useful to transform the calibration procedure into a finite dimensional optimization problem.
- Hence, our approach can be seen as an arbitrage-free version of the Weighted Monte Carlo method.