Arbitrage opportunities in misspecified stochastic volatility models

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Widely documented phenomenon of option mispricing. Given set of assumptions on the real-world dynamics of an asset, the European options on this asset are not efficiently priced in options markets. [Y-Ait Sahaliya et. al, Bakshi et. al]
Discrepancies between the implied volatility and historical volatility levels

Substantial differences between historical and option-based measures of skewness and kurtosis [Bakshi et. al] have been documented.
Misspecification studied extensively in Black Scholes model with misspecified volatility

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We address the question of Misspecified stochastic volatility models.
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- misspecified volatility of volatility $\Rightarrow$ a butterfly spread
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We concentrate on arbitrage strategies involving
- underlying asset
- liquid European options.
Under real-world probability $\mathbb{P}$, the underlying price $S$ follows a stochastic volatility model

\[
\frac{dS_t}{S_t} = \mu_t dt + \sigma(Y_t) \sqrt{1 - \rho_t^2} dW^1_t + \sigma(Y_t) \rho_t dW^2_t \\
dY_t = a_t dt + b_t dW^2_t,
\]

- $\sigma : \mathbb{R} \to (0, \infty)$ is a Lipschitz $C^1$-diffeomorphism
- $\sigma'(y) > 0$ for all $y \in \mathbb{R}$; $\mu, a, b > 0$ and $\rho \in [-1, 1]$ are adapted
- $(W^1, W^2)$ is a standard 2-dimensional Brownian motion.
Process $\tilde{\sigma}_t$, which represents the instantaneous volatility used by the option’s market for all pricing purposes. We assume that $\tilde{\sigma}_t = \tilde{\sigma}(Y_t)$

$$dY_t = a_t dt + b_t dW_t^2,$$

where $a_t$ and $b_t > 0$ are adapted.
$\tilde{\sigma} : \mathbb{R} \to (0, \infty)$ is a Lipschitz $C^1$-diffeomorphism with $0 < \sigma \leq \tilde{\sigma}(y) \leq \bar{\sigma} < \infty$ and $\tilde{\sigma}'(y) > 0$ for all $y \in \mathbb{R}$;
The Market model dynamics

Assumptions,

- Another probability measure $\mathbb{Q}$, called market or pricing probability
- All traded assets are martingales under $\mathbb{Q}$
- The interest rate is assumed to be zero

Under $\mathbb{Q}$, the underlying asset and its volatility form a 2-dimensional Markovian diffusion:

$$
\frac{dS_t}{S_t} = \tilde{\sigma}(Y_t)\sqrt{1 - \tilde{\rho}^2(Y_t, t)}dW_t^1 + \tilde{\sigma}(Y_t)\tilde{\rho}(Y_t, t)dW_t^2
$$

$$
dY_t = \tilde{a}(Y_t, t)dt + \tilde{b}(Y_t, t)dW_t^2,
$$

$\tilde{a}$, $\tilde{b}$ and $\tilde{\rho}$ are deterministic functions.
Suppose that a continuum of European options for all strikes and at least one maturity, quoted in the market.

The price of an option with maturity date $T$ and pay-off $H(S_T)$ of $S_t$, $Y_t$ and $t$:

$$P(S_t, Y_t, t) = E^Q[H(S_T)|\mathcal{F}_t].$$

For every such option, the pricing function $P$ belongs to the class $C^{2,2,1}((0, \infty) \times \mathbb{R} \times [0, T))$ and satisfies the PDE

$$\tilde{a} \frac{\partial P}{\partial y} + \tilde{L} P = 0,$$

where we define

$$\tilde{L} f = \frac{\partial f}{\partial t} + \frac{S^2 \tilde{\sigma}(y)^2}{2} \frac{\partial^2 f}{\partial S^2} + \frac{\tilde{b}^2}{2} \frac{\partial^2 f}{\partial y^2} + S \tilde{\sigma}(y) \tilde{b} \rho \frac{\partial^2 f}{\partial S \partial y}. $$
Under our assumptions any such European option can be used to “complete” the $\mathbb{Q}$-market. (Romano, Touzi)

And price satisfies

$$\frac{\partial P}{\partial y} > 0, \quad \forall (S, y, t) \in (0, \infty) \times \mathbb{R} \times [0, T).$$

The real-world market may be incomplete in our setting.
Lemma

Let $P$ be the price of a call or a put option with strike $K$ and maturity date $T$. Then

$$
\lim_{K \to +\infty} \frac{\partial P(S, y, t)}{\partial y} = \lim_{K \to 0} \frac{\partial P(S, y, t)}{\partial y} = 0,
$$

$$
\lim_{K \to +\infty} \frac{\partial^2 P(S, y, t)}{\partial y^2} = \lim_{K \to 0} \frac{\partial^2 P(S, y, t)}{\partial y^2} = 0,
$$

and

$$
\lim_{K \to +\infty} \frac{\partial^2 P(S, y, t)}{\partial S \partial y} = \lim_{K \to 0} \frac{\partial^2 P(S, y, t)}{\partial S \partial y} = 0
$$

for all $(y, t) \in \mathbb{R} \times [0, T)$. All the above derivatives are continuous in $K$ and the limits are uniform in $S, y, t$ on any compact subset of $(0, \infty) \times \mathbb{R} \times [0, T)$. 
Sketch of the Proof

The option price satisfies,

\[ \tilde{a} \frac{\partial P}{\partial y} + \tilde{\mathcal{L}} P = 0, \]

- Differentiate w.r.t. \( y \) and \( S \),
- Use Feynman Kac representation to relate the various greeks to the fundamental solutions of pde.
- Using the classical bounds for fundamental solutions of parabolic equations.

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misspecification of SV models
The arbitrage problem is set up from the perspective of a trader,

- Who knows market is using misspecified model
- Wants to construct a strategy to benefit from this misspecification.

The first step,

- sets up a dynamic self financing delta and vega-neutral portfolio $X_t$ with zero initial value.
  - at each date $t$, a stripe of European call or put options with a common time to expiry $T_t$.
  - $\omega_t(dK)$: quantity of options with strikes between $K$ and $K + dK$
  - $-\delta_t$ of stock
  - $B_t$ of cash.
  - $\int |\omega_t(dK)| = 1$
The value of the resulting portfolio is,

\[ X_t = \int P_K(S_t, Y_t, t)\omega_t(dK) - \delta_t S_t + B_t, \]

The dynamics of this portfolio is given by,

\[ dX_t = \int \omega_t(dK) \left( \mathcal{L} P^K dt + \frac{\partial P^K}{\partial S} dS_t + \frac{\partial P^K}{\partial y} dY_t \right) - \delta_t dS_t \]

where,

\[ \mathcal{L} f = \frac{\partial f}{\partial t} + \frac{S_t^2 \sigma(Y_t)^2}{2} \frac{\partial^2 f}{\partial S^2} + \frac{b_t^2}{2} \frac{\partial^2 f}{\partial y^2} + S_t \sigma(Y_t) b_t \rho_t \frac{\partial^2 f}{\partial S \partial y} \]
choose,
\[ \int \omega_t(dK) \frac{\partial P^K}{\partial y} = 0, \quad \int \omega_t(dK) \frac{\partial P^K}{\partial S} = \delta_t \]
to eliminate the \(dY_t\) and \(dS_t\) terms.

The resulting portfolio is risk free.

The portfolio dynamics reduces to,
\[ dX_t = \int \omega_t(dK) \mathcal{L} P^K dt, \]
Now we can write down the risk free profit from model misspecification as,

\[ dX_t = \int \omega_t(dK)(\mathcal{L} - \tilde{\mathcal{L}})P^K dt. \]

At the liquidation date \( T^* \),

\[ X_{T^*} = \int_0^{T^*} \int \omega_t(dK)(\mathcal{L} - \tilde{\mathcal{L}})P^K dt, \]

where,

\[ (\mathcal{L} - \tilde{\mathcal{L}})P^K = \frac{S_t^2(\sigma_t^2 - \tilde{\sigma}^2(Y_t))}{2} \frac{\partial^2 P^K}{\partial S^2} + \frac{(b_t^2 - \tilde{b}_t^2)}{2} \frac{\partial^2 P^K}{\partial y^2} \]

\[ + S_t(\sigma_t b_t \rho_t - \tilde{\sigma}(Y_t)\tilde{b}_t \tilde{\rho}_t) \frac{\partial^2 P^K}{\partial S \partial y} \]
The problem in a Nutshell

- The trader needs to maximize this arbitrage profit.
- Taking advantage of “arbitrage opportunity” to the following optimisation problem,

\[
\text{Maximize} \quad \mathcal{P}_t = \int \omega_t(dK)(\mathcal{L} - \tilde{\mathcal{L}})P^K
\]

subject to \( \int |\omega_t(dK)| = 1 \) and \( \int \omega_t(dK) \frac{\partial P^K}{\partial y} = 0 \).

- **ANSWER**: Spread of only two options is sufficient to solve this problem.
**Proposition**

The instantaneous arbitrage profit is maximized by

$$
\omega_t(dK) = w_t^1 \delta_{K_t^1}(dK) - w_t^2 \delta_{K_t^2}(dK),
$$

where $\delta_K(dK)$ denotes the unit point mass at $K$, $(w_t^1, w_t^2)$ are time-dependent optimal weights given by

$$
w_t^1 = \frac{\partial P_{K_t^2}}{\partial y} \left( \frac{\partial P_{K_t^1}}{\partial y} + \frac{\partial P_{K_t^2}}{\partial y} \right), \quad w_t^2 = \frac{\partial P_{K_t^1}}{\partial y} \left( \frac{\partial P_{K_t^1}}{\partial y} + \frac{\partial P_{K_t^2}}{\partial y} \right),
$$

and $(K_t^1, K_t^2)$ are time-dependent optimal strikes given by

$$
(K_t^1, K_t^2) = \arg \max_{K_t^1, K_t^2} \frac{\partial P_{K_t^2}}{\partial y} (\mathcal{L} - \tilde{\mathcal{L}}) P_{K_t^1} - \frac{\partial P_{K_t^1}}{\partial y} (\mathcal{L} - \tilde{\mathcal{L}}) P_{K_t^2} \quad \frac{\partial P_{K_t^1}}{\partial y} + \frac{\partial P_{K_t^2}}{\partial y}.
$$
The proof is done in two steps,

- First show that the optimization problem is well-posed, i.e., the maximum is attained for two distinct strike values.
- show that the two-point solution suggested by this proposition is indeed the optimal one.
The misspecified model is the Black-Scholes with constant volatility $\sigma$ (but the true model is of course a stochastic volatility model).

In the Black-Scholes model ($r = 0$):

$$\frac{\partial P}{\partial \sigma} = Sn(d_1)\sqrt{T} = Kn(d_2)\sqrt{T},$$

$$\frac{\partial^2 P}{\partial \sigma \partial S} = -\frac{n(d_1)d_2}{\sigma},$$

$$\frac{\partial^2 P}{\partial \sigma^2} = \frac{Sn(d_1)d_1d_2\sqrt{T}}{\sigma},$$

where $d_{1,2} = \frac{m}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}$, $m = \log(S/K)$ and $n$ is the standard normal density.
Proposition

Let $\tilde{b} = \tilde{\rho} = 0$. The optimal option portfolio maximizing the instantaneous arbitrage profit is described as follows:

- The portfolio consists of a long position in an option with log-moneyness $m_1 = z_1 \sigma \sqrt{T} - \frac{\sigma^2 T}{2}$ and a short position in an option with log-moneyness $m_2 = z_2 \sigma \sqrt{T} - \frac{\sigma^2 T}{2}$, where $z_1$ and $z_2$ are maximizers of the function

$$f(z_1, z_2) = \frac{(z_1 - z_2)(z_1 + z_2 - w_0)}{e^{z_1^2/2} + e^{z_2^2/2}}$$

with $w_0 = \frac{\sigma(bT + 2\rho)}{b\sqrt{T}}$.

- The weights of the two options are chosen to make the portfolio vega-neutral.

We define by $P_{opt}$ the instantaneous arbitrage profit realized by the optimal portfolio.
Proof

Substituting the Black-Scholes values for the derivatives of option prices,
change of variable \( z = \frac{m}{\sigma \sqrt{T}} + \frac{\sigma \sqrt{T}}{2} \),
the function to maximize w.r.t. \( z_1, z_2 \) becomes:

\[
\frac{n(z_1)n(z_2)}{n(z_1) + n(z_2)} \left\{ \frac{b\sqrt{T}}{2\sigma} (z_1^2 - z_2^2) - \frac{bT}{2} (z_1 - z_2) - \rho (z_1 - z_2) \right\},
\]

from which the proposition follows directly.
Proposition

Let $\tilde{b} = \tilde{\rho} = 0$, and define by $P^{\text{opt}}$ the instantaneous arbitrage profit realized by the optimal strategy. Consider a portfolio (RR) described as follows:

- If $bT/2 + \rho \geq 0$
  - buy $\frac{1}{2}$ units of options with log-moneyness $m_1 = -\sigma \sqrt{T} - \frac{\sigma^2 T}{2}$, or, equivalently, delta value $N(-1) \approx 0.16$
  - selling $\frac{1}{2}$ units of options with log-moneyness $m_2 = \sigma \sqrt{T} - \frac{\sigma^2 T}{2}$, or, equivalently, delta value $N(1) \approx 0.84$.
- If $bT/2 + \rho < 0$ buy the portfolio with weights of the opposite sign.

Then the portfolio (RR) is the solution of the maximization problem under the additional constraint that it is $\Delta$-antisymmetric.
Proposition

Consider a portfolio (BB) consisting in

- **buying** $x_0$ units of options with log-moneyness $m_1 = z_0 \sigma \sqrt{T} - \sigma^2 T$, or, equivalently, delta value $N(z_0) \approx 0.055$, where $z_0 \approx 1.6$ is a universal constant.
- **buying** $x_0$ units of options with log-moneyness $m_2 = -z_0 \sigma \sqrt{T} - \sigma^2 T$, or, equivalently, delta value $N(-z_0) \approx 0.945$
- **selling** $1 - 2x_0$ units of options with log-moneyness $m_3 = -\frac{\sigma^2 T}{2}$ or, equivalently, delta value $N(0) = \frac{1}{2}$.

The quantity $x_0$ is chosen to make the portfolio vega-neutral, that is, $x_0 \approx 0.39$.

Then, the portfolio (BB) is the solution of the maximization problem under the additional constraint that it is $\Delta$-symmetric.
Proposition

Define by $P^{RR}$ the instantaneous arbitrage profit realized by the portfolio of part 1 and by $P^{BB}$ that of part 2. Let

$$\alpha = \frac{\sigma |bT + 2\rho|}{\sigma |bT + 2\rho| + 2bK_0 \sqrt{T}}$$

where $K_0$ is a universal constant, defined below in the proof, and approximately equal to 0.459. Then

$$P^{RR} \geq \alpha P^{opt} \quad \text{and} \quad P^{BB} \geq (1 - \alpha) P^{opt}.$$
The maximization problem can be reduced to,

\[
\max \frac{Sb^2 \sqrt{T}}{2\sigma} \int z^2 n(z) \bar{\omega}_t(dz) - Sb(bT/2 + \rho) \int zn(z) \bar{\omega}_t(dz)
\]

subject to \(\int n(z) \bar{\omega}_t(dz) = 0, \int |\bar{\omega}_t(dz)| = 1\).

Observe that the contract (BB) maximizes the first term while the contract (RR) maximizes the second term. The values for the contract (BB) and (RR) are given by

\[
\mathcal{P}^{BB} = \frac{Sb^2 \sqrt{T}}{\sigma \sqrt{2\pi}} e^{-z_0^2/2}, \quad \mathcal{P}^{RR} = \frac{Sb|bT/2 + \rho|}{\sqrt{2\pi}} e^{-1/2}.
\]
therefore

\[
\frac{\mathcal{P}^{RR}}{\mathcal{P}^{BB} + \mathcal{P}^{RR}} = \frac{\sigma |bT + 2\rho|}{\sigma |bT + 2\rho| + 2bK_0 \sqrt{T}} \quad \text{with} \quad K_0 = e^{1 - \frac{z_0^2}{2}}.
\]

Since the maximum of a sum is always no greater than the sum of maxima, \( \mathcal{P}^{opt} \leq \mathcal{P}^{BB} + \mathcal{P}^{RR} \)
Risk reversals are never optimal and butterflies are not optimal unless $\rho = -\frac{bT}{2}$.

Nevertheless, risk reversals and butterflies are relatively close to being optimal, and have the additional advantage of being independent from the model parameters, whereas the optimal claim depends on the parameters.

This near-optimality is realized by a special universal risk reversal (16-delta risk reversal in the language of foreign exchange markets) and a special universal butterfly (5.5-delta vega weighted butterfly).

When $b \to 0$, $\alpha \to 1$, In this case RR is nearly optimal.
A simple stochastic volatility model, which captures all the desired effects, the SABR $\beta = 1$.

The dynamics of the underlying asset under $\mathbb{Q}$ is

$$dS_t = \tilde{\sigma}_t S_t (\sqrt{1 - \tilde{\rho}^2} dW_t^1 + \tilde{\rho} dW_t^2)$$  \hspace{1cm} (2)

$$d\tilde{\sigma}_t = \tilde{b} \tilde{\sigma}_t dW_t^2$$ \hspace{1cm} (3)

The true dynamics of the instantaneous implied volatility is

$$d\tilde{\sigma}_t = b \tilde{\sigma}_t dW_t^2,$$ \hspace{1cm} (4)

and the dynamics of the underlying under the real-world measure is

$$dS_t = \sigma_t S_t (\sqrt{1 - \rho^2} dW_t^1 + \rho dW_t^2).$$ \hspace{1cm} (5)
Call option price $C$ satisfies the following pricing equation,

$$\frac{\partial C}{\partial t} + S^2 \sigma^2 \frac{\partial^2 C}{\partial S^2} + \frac{b^2}{2} \frac{\partial^2 C}{\partial \sigma^2} + S \sigma b \rho \frac{\partial^2 C}{\partial S \partial \sigma} = 0$$

- Stochastic volatility is introduced as a perturbation $b = \epsilon \sigma$.
- Look for asymptotic solutions of the form,

$$C = C_0 + \epsilon C_1 + \epsilon^2 C_2 + O(\epsilon^3)$$

Here $C_0$ corresponds to the leading Black Scholes solution.

$$\frac{\partial C_0}{\partial t} + S^2 \sigma^2 \frac{\partial^2 C_0}{\partial S^2} = 0$$

The first leading order to $\epsilon$ satisfies the following equation neglecting the higher order terms $O(\epsilon^2)$,

$$C_1 = \frac{\tilde{\sigma}^2 \tilde{\rho} (T - t)}{2} S \frac{\partial^2 C_0}{\partial S \partial \sigma}$$
Figure: Optimal Strikes for the set of parameters $\sigma = .2$, $S = 1$, $b = .3$, $\rho = -.3$, $\tilde{\rho} = -.5$, $t = 1$, as a function of the missspecified $\tilde{b} \in [.01, .4]$. 

misspecification of SV models
The trader is aware about the misspecification.

- Stock price = 100 and volatility $\sigma = 0.1$
- Real world parameters: $b = 0.8$, $\rho = -0.5$
- Market or pricing parameters: $\tilde{b} = 0.3$, $\tilde{\rho} = -0.7$
- Demonstration for only one month options.
- Results are shown for 100 trajectories of the stock and volatility.
Figure: The evolution of portfolios using options with 1 month

- Left: The true parameters are $\rho = -0.2$, $b = 0.1$. The misspecified or the market parameters are $\tilde{\rho} = -0.3$, $\tilde{b} = 0.9$.
- include a bid ask-fork of 0.45% in implied volatility terms for every option transaction. The evolution of the portfolio performance with 32 rebalancing dates.
Thank You

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