Arbitrage opportunities in misspecified stochastic volatility models

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Outline

1. Motivation/ Introduction/Background
2. Set up of the problem
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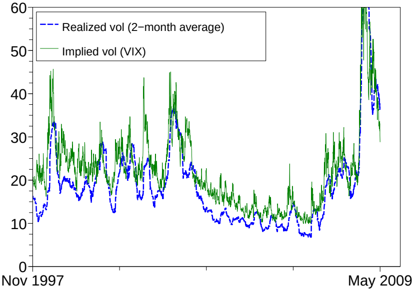
4. The Black Scholes case
5. The Stochastic Volatility case: A perturbative approach
6. Numerics/Results

Widely documented phenomenon of option mispricing.
Given set of assumptions on the real-world dynamics of an asset, the European options on this asset are not efficiently priced in options markets.
[Y-Ait Sahaliya et. al, Bakshi et. al]

Introduction

Discrepancies between the implied volatility and historical volatility levels



Substantial differences between historical and option-based measures of skewness and kurtosis [Bakshi et. al] have been documented.

Background

- Misspecification studied extensively in Black Scholes model with misspecified volatility

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We concentrate on arbitrage strategies involving

- underlying asset
- liquid European options.

Under real-world probability $\mathbb{P}$, the underlying price S follows a stochastic volatility model

$$
\begin{aligned}
dS_t/S_t &= \mu_t dt + \sigma(Y_t)\sqrt{1-\rho_t^2}\,dW_t^1 + \sigma(Y_t)\rho_t dW_t^2 \\
dY_t &= a_t dt + b_t dW_t^2,
\end{aligned}
$$

- $\sigma : \mathbb{R} \to (0, \infty)$ is a Lipschitz C^1-diffeomorphism
- $\sigma'(y) > 0$ for all $y \in \mathbb{R}$; μ, a, $b > 0$ and $\rho \in [-1, 1]$ are adapted
- (W^1, W^2) is a standard 2-dimensional Brownian motion.

Setting

Process $\tilde{\sigma}_t$, which represents the instantaneous volatility used by the option's market for all pricing purposes.
We assume that $\tilde{\sigma}_t = \tilde{\sigma}(Y_t)$

$$dY_t = a_t dt + b_t dW_t^2, \tag{1}$$

where a_t and $b_t > 0$ are adapted.
$\tilde{\sigma} : \mathbb{R} \to (0, \infty)$ is a Lipschitz C^1-diffeomorphism with $0 < \underline{\sigma} \leq \tilde{\sigma}(y) \leq \overline{\sigma} < \infty$ and $\tilde{\sigma}'(y) > 0$ for all $y \in \mathbb{R}$;

Assumptions,

- Another probability measure $\mathbb{Q}$, called market or pricing probability
- All traded assets are martingales under $\mathbb{Q}$
- The interest rate is assumed to be zero

Under $\mathbb{Q}$, the underlying asset and its volatility form a 2-dimensional Markovian diffusion:

$$
\begin{aligned}
dS_t/S_t &= \tilde{\sigma}(Y_t)\sqrt{1-\tilde{\rho}^2(Y_t,t)}dW_t^1 + \tilde{\sigma}(Y_t)\tilde{\rho}(Y_t,t)dW_t^2 \\
dY_t &= \tilde{a}(Y_t,t)dt + \tilde{b}(Y_t,t)dW_t^2,
\end{aligned}
$$

$\tilde{a}$, $\tilde{b}$ and $\tilde{\rho}$ are deterministic functions.

Setting Contd.

- Suppose that a continuum of European options for all strikes and at least one maturity, quoted in the market.
- The price of an option with maturity date T and pay-off $H(S_T)$ of S_t, Y_t and t:

$$P(S_t, Y_t, t) = E^Q[H(S_T)|\mathcal{F}_t].$$

For every such option, the pricing function P belongs to the class $C^{2,2,1}((0,\infty) \times \mathbb{R} \times [0,T))$ and satisfies the PDE

$$\tilde{a}\frac{\partial P}{\partial y} + \tilde{\mathcal{L}}P = 0,$$

where we define

$$\tilde{\mathcal{L}}f = \frac{\partial f}{\partial t} + \frac{S^2\tilde{\sigma}(y)^2}{2}\frac{\partial^2 f}{\partial S^2} + \frac{\tilde{b}^2}{2}\frac{\partial^2 f}{\partial y^2} + S\tilde{\sigma}(y)\tilde{b}\tilde{\rho}\frac{\partial^2 f}{\partial S \partial y}.$$

- Under our assumptions any such European option can be used to “complete” the $\mathbb{Q}$-market.
  (Romano, Touzi)
- And price satisfies

$$\frac{\partial P}{\partial y} > 0, \quad \forall (S, y, t) \in (0, \infty) \times \mathbb{R} \times [0, T).$$

- The real-world market may be incomplete in our setting.

Decay Properties of the Greeks

Lemma

Let P be the price of a call or a put option with strike K and maturity date T. Then

$$\lim_{K\to+\infty}\frac{\partial P(S,y,t)}{\partial y}=\lim_{K\to 0}\frac{\partial P(S,y,t)}{\partial y}=0,$$

$$\lim_{K\to+\infty}\frac{\partial^2 P(S,y,t)}{\partial y^2}=\lim_{K\to 0}\frac{\partial^2 P(S,y,t)}{\partial y^2}=0,$$

$$\textit{and}\quad \lim_{K\to+\infty}\frac{\partial^2 P(S,y,t)}{\partial S\partial y}=\lim_{K\to 0}\frac{\partial^2 P(S,y,t)}{\partial S\partial y}=0$$

for all $(y,t)\in\mathbb{R}\times[0,T)$. All the above derivatives are continuous in K and the limits are uniform in S,y,t on any compact subset of $(0,\infty)\times\mathbb{R}\times[0,T)$.

Sketch of the Proof

The option price satisfies,

$$\tilde{a}\frac{\partial P}{\partial y} + \tilde{\mathcal{L}}P = 0,$$

- Differentiate w.r.t. y and S,
- Use Feynman Kac representation to relate the various greeks to the fundamental solutions of pde.
- Using the classical bounds for fundamental solutions of parabolic equations.

Formulation of the Problem

The arbitrage problem is set up from the perspective of a trader,

- Who knows market is using misspecified model
- Wants to construct a strategy to benefit from this misspecification.

The first step,

- sets up a dynamic self financing delta and vega-neutral portfolio X_t with zero initial value.
  - at each date t, a stripe of European call or put options with a common time to expiry T_t.
  - $\omega_t(dK)$: quantity of options with strikes between K and $K + dK$
- $-\delta_t$ of stock
- B_t of cash.
- $\int |\omega_t(dK)| = 1$

Formulation contd.

- The value of the resulting portfolio is,
  - $$X_t = \int P_K(S_t, Y_t, t)\omega_t(dK) - \delta_t S_t + B_t,$$
  - The dynamics of this portfolio is given by,
  - $$dX_t = \int \omega_t(dK)\left(\mathcal{L}P^K dt + \frac{\partial P^K}{\partial S} dS_t + \frac{\partial P^K}{\partial y} dY_t\right) - \delta_t dS_t$$

    where,
  - $$\mathcal{L}f = \frac{\partial f}{\partial t} + \frac{S_t^2 \sigma(Y_t)^2}{2}\frac{\partial^2 f}{\partial S^2} + \frac{b_t^2}{2}\frac{\partial^2 f}{\partial y^2} + S_t \sigma(Y_t) b_t \rho_t \frac{\partial^2 f}{\partial S \partial y}$$

- choose,

$$\int \omega_t(dK)\frac{\partial P^K}{\partial y} = 0, \qquad \int \omega_t(dK)\frac{\partial P^K}{\partial S} = \delta_t$$

  to eliminate the dY_t and dS_t terms.
- The resulting portfolio is risk free.

The portfolio dynamics reduces to,

$$dX_t = \int \omega_t(dK)\mathcal{L}P^K dt,$$

Now we can write down the risk free profit from model misspecification as,

$$dX_t = \int \omega_t(dK)(\mathcal{L} - \tilde{\mathcal{L}})P^K dt.$$

At the liquidation date T^*,

$$X_{T^*} = \int_0^{T^*} \int \omega_t(dK)(\mathcal{L} - \tilde{\mathcal{L}})P^K dt,$$

where,

$$(\mathcal{L} - \tilde{\mathcal{L}})P^K = \frac{S_t^2(\sigma_t^2 - \tilde{\sigma}^2(Y_t))}{2}\frac{\partial^2 P^K}{\partial S^2} + \frac{(b_t^2 - \tilde{b}_t^2)}{2}\frac{\partial^2 P^K}{\partial y^2}$$
$$+ S_t(\sigma_t b_t \rho_t - \tilde{\sigma}(Y_t)\tilde{b}_t\tilde{\rho}_t)\frac{\partial^2 P^K}{\partial S \partial y}$$

The problem in a Nutshell

- The trader needs to maximize this aribtrage profit.
- Taking advantage of “arbitrage opportunity” to the following optimisation problem,

$$\text{Maximize} \quad \mathcal{P}_t = \int \omega_t(dK)(\mathcal{L} - \tilde{\mathcal{L}})P^K$$

$$\text{subject to} \quad \int |\omega_t(dK)| = 1 \quad \text{and} \quad \int \omega_t(dK)\frac{\partial P^K}{\partial y} = 0.$$

- ANSWER: Spread of only two options is sufficient to solve this problem.

General Result

Proposition

The instantaneous arbitrage profit is maximized by

$$\omega_t(dK) = w_t^1 \delta_{K_t^1}(dK) - w_t^2 \delta_{K_t^2}(dK),$$

where $\delta_K(dK)$ *denotes the unit point mass at* K*,* (w_t^1, w_t^2) *are time-dependent optimal weights given by*

$$w_t^1 = \frac{\frac{\partial P^{K_2}}{\partial y}}{\frac{\partial P^{K_1}}{\partial y} + \frac{\partial P^{K_2}}{\partial y}}, \qquad w_t^2 = \frac{\frac{\partial P^{K_1}}{\partial y}}{\frac{\partial P^{K_1}}{\partial y} + \frac{\partial P^{K_2}}{\partial y}},$$

and (K_t^1, K_t^2) *are time-dependent optimal strikes given by*

$$(K_t^1, K_t^2) = \arg\max_{K^1, K^2} \frac{\frac{\partial P^{K^2}}{\partial y}(\mathcal{L} - \tilde{\mathcal{L}})P^{K^1} - \frac{\partial P^{K^1}}{\partial y}(\mathcal{L} - \tilde{\mathcal{L}})P^{K^2}}{\frac{\partial P^{K^1}}{\partial y} + \frac{\partial P^{K^2}}{\partial y}}.$$

Sketch of Proof

The proof is done in two steps,

- First show that the optimization problem is well-posed, i.e., the maximum is attained for two distinct strike values.
- show that the two-point solution suggested by this proposition is indeed the optimal one.

The Black Scholes case

- The misspecified model is the Black-Scholes with constant volatility σ (but the true model is of course a stochastic volatility model).

In the Black-Scholes model ($r = 0$):

$$\frac{\partial P}{\partial \sigma} = Sn(d_1)\sqrt{T} = Kn(d_2)\sqrt{T},$$
$$\frac{\partial^2 P}{\partial \sigma \partial S} = -\frac{n(d_1)d_2}{\sigma},$$
$$\frac{\partial^2 P}{\partial \sigma^2} = \frac{Sn(d_1)d_1 d_2 \sqrt{T}}{\sigma},$$

where $d_{1,2} = \frac{m}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}$, $m = \log(S/K)$ and n is the standard normal density.

Proposition

Let $\tilde{b} = \tilde{\rho} = 0$. The optimal option portfolio maximizing the instantaneous arbitrage profit is described as follows:

- *The portfolio consists of a long position in an option with log-moneyness $m_1 = z_1\sigma\sqrt{T} - \frac{\sigma^2 T}{2}$ and a short position in an option with log-moneyness $m_2 = z_2\sigma\sqrt{T} - \frac{\sigma^2 T}{2}$, where z_1 and z_2 are maximizers of the function*

$$f(z_1, z_2) = \frac{(z_1 - z_2)(z_1 + z_2 - w_0)}{e^{z_1^2/2} + e^{z_2^2/2}}$$

  with $w_0 = \frac{\sigma(bT+2\rho)}{b\sqrt{T}}$.

- *The weights of the two options are chosen to make the portfolio vega-neutral.*

We define by P_{opt} the instantaneous arbitrage profit realized by the optimal portfolio.

Proof

Substituting the Black-Scholes values for the derivatives of option prices,
change of variable $z = \frac{m}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$,
the function to maximize w.r.t. z_1, z_2 becomes:

$$\frac{n(z_1)n(z_2)}{n(z_1)+n(z_2)}\left\{\frac{b\sqrt{T}}{2\sigma}(z_1^2 - z_2^2) - \frac{bT}{2}(z_1 - z_2) - \rho(z_1 - z_2)\right\},$$

from which the proposition follows directly.

Proposition

Let $\tilde{b} = \tilde{\rho} = 0$, and define by $\mathcal{P}^{opt}$ the instantaneous arbitrage profit realized by the optimal strategy.
Consider a portfolio (RR) described as follows:

- *If $bT/2 + \rho \geq 0$*
  - *buy $\frac{1}{2}$ units of options with log-moneyness $m_1 = -\sigma\sqrt{T} - \frac{\sigma^2 T}{2}$, or, equivalently, delta value $N(-1) \approx 0.16$*
  - *selling $\frac{1}{2}$ units of options with log-moneyness $m_2 = \sigma\sqrt{T} - \frac{\sigma^2 T}{2}$, or, equivalently, delta value $N(1) \approx 0.84$.*
- *if $bT/2 + \rho < 0$ buy the portfolio with weights of the opposite sign.*

Then the portfolio (RR) is the solution of the maximization problem under the additional constraint that it is Δ-antisymmetric.

Proposition

Consider a portfolio (BB) consisting in

- *buying* x_0 *units of options with log-moneyness* $m_1 = z_0\sigma\sqrt{T} - \sigma^2 T$, *or, equivalently, delta value* $N(z_0) \approx 0.055$, *where* $z_0 \approx 1.6$ *is a universal constant.*
- *buying* x_0 *units of options with log-moneyness* $m_2 = -z_0\sigma\sqrt{T} - \sigma^2 T$, *or, equivalently, delta value* $N(-z_0) \approx 0.945$
- *selling* $1 - 2x_0$ *units of options with log-moneyness* $m_3 = -\frac{\sigma^2 T}{2}$ *or, equivalently, delta value* $N(0) = \frac{1}{2}$.

The quantity x_0 *is chosen to make the portfolio vega-neutral, that is,* $x_0 \approx 0.39$.
Then, the portfolio (BB) is the solution of the maximization problem under the additional constraint that it is Δ*-symmetric.*

Proposition

Define by $\mathcal{P}^{RR}$ the instantaneous arbitrage profit realized by the portfolio of part 1 and by $\mathcal{P}^{BB}$ that of part 2. Let

$$\alpha = \frac{\sigma|bT + 2\rho|}{\sigma|bT + 2\rho| + 2bK_0\sqrt{T}}$$

where K_0 is a universal constant, defined below in the proof, and approximately equal to 0.459*. Then*

$$\mathcal{P}^{RR} \geq \alpha\mathcal{P}^{opt} \quad \text{and} \quad \mathcal{P}^{BB} \geq (1-\alpha)\mathcal{P}^{opt}.$$

Sketch of Proof

The maximization problem can be reduced to,

$$\max \frac{Sb^2\sqrt{T}}{2\sigma}\int z^2 n(z)\bar{\omega}_t(dz) - Sb(bT/2+\rho)\int zn(z)\bar{\omega}_t(dz)$$

$$\text{subject to} \quad \int n(z)\bar{\omega}_t(dz) = 0, \quad \int |\bar{\omega}_t(dz)| = 1.$$

Observe that the contract (BB) maximizes the first term while the contract (RR) maximizes the second term. The values for the contract (BB) and (RR) are given by

$$\mathcal{P}^{BB} = \frac{Sb^2\sqrt{T}}{\sigma\sqrt{2\pi}}e^{-z_0^2/2}, \qquad \mathcal{P}^{RR} = \frac{Sb|bT/2+\rho|}{\sqrt{2\pi}}e^{-\frac{1}{2}}.$$

therefore

$$\frac{\mathcal{P}^{RR}}{\mathcal{P}^{BB}+\mathcal{P}^{RR}} = \frac{\sigma|bT+2\rho|}{\sigma|bT+2\rho|+2bK_0\sqrt{T}} \quad \text{with} \quad K_0 = e^{\frac{1}{2}-\frac{z_0^2}{2}}.$$

Since the maximum of a sum is always no greater than the sum of maxima, $\mathcal{P}^{opt} \leq \mathcal{P}^{BB} + \mathcal{P}^{RR}$

Remarks

- Risk reversals are never optimal and butterflies are not optimal unless $\rho = -\frac{bT}{2}$.
- Nevertheless, risk reversals and butterflies are relatively close to being optimal, and have the additional advantage of being independent from the model parameters, whereas the optimal claim depends on the parameters.
- This near-optimality is realized by a special universal risk reversal (16-delta risk reversal in the language of foreign exchange markets) and a special universal butterfly (5.5-delta vega weighted buttefly).
- When $b \to 0$, $\alpha \to 1$, In this case RR is nearly optimal.

Stochastic Volatility Model

A simple stochastic volatility model, which captures all the desired effects, the SABR $\beta = 1$.
The dynamics of the underlying asset under $\mathbb{Q}$ is

$$dS_t = \tilde{\sigma}_t S_t(\sqrt{1-\tilde{\rho}^2}dW_t^1 + \tilde{\rho}dW_t^2) \tag{2}$$

$$d\tilde{\sigma}_t = \tilde{b}\tilde{\sigma}_t dW_t^2 \tag{3}$$

The true dynamics of the instantaneous implied volatility is

$$d\tilde{\sigma}_t = b\tilde{\sigma}_t dW_t^2, \tag{4}$$

and the dynamics of the underlying under the real-world measure is

$$dS_t = \sigma_t S_t(\sqrt{1-\rho^2}dW_t^1 + \rho dW_t^2). \tag{5}$$

First order correction

Call option price C satisifies the following pricing equation,

$$\frac{\partial C}{\partial t} + S^2\sigma^2\frac{\partial^2 C}{\partial S^2} + \frac{b^2}{2}\frac{\partial^2 C}{\partial \sigma^2} + S\sigma b\rho\frac{\partial^2 C}{\partial S\partial\sigma} = 0$$

- stochastic volatility is introduced as a perturbation $b = \epsilon\sigma$.
- Look for asymptotic solutions of the form,
  - $$C = C_0 + \epsilon C_1 + \epsilon^2 C_2 + O(\epsilon^3)$$
  - Here C_0 corresponds to the leading Black Scholes solution.
  - $$\frac{\partial C_0}{\partial t} + S^2\sigma^2\frac{\partial^2 C_0}{\partial S^2} = 0$$

The first leading order to ϵ satisfies the following equation neglecting the higher order terms $O(\epsilon^2)$,

$$C_1 = \frac{\tilde{\sigma}^2\tilde{\rho}(T-t)}{2}S\frac{\partial^2 C_0}{\partial S\partial\sigma}$$

Perturbation Results

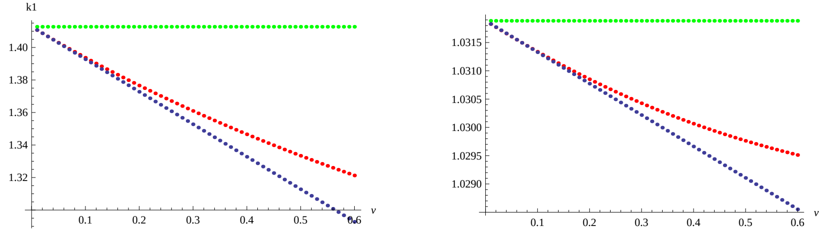

Figure: Optimal Strikes for the set of parameters $\sigma = .2$, $S = 1, b = .3, \rho = -.3, \tilde{\rho} = -.5, t = 1$, as a function of the misspecified $\tilde{b} \in [.01, .4]$.

- The trader is aware about the misspecification.
- Stock price $= 100$ and volatility $\sigma = 0.1$
- Real world parameters: $b = .8, \rho = -.5$
- Market or pricing parameters: $\tilde{b} = .3, \tilde{\rho} = -.7$
- Demonstration for only one month options.
- Results are shown for 100 trajectories of the stock and volatility.

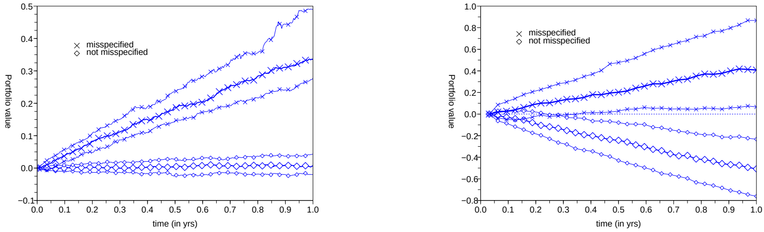

Figure: The evolution of portfolios using options with 1 month

- Left: The true parameters are $\rho = -.2, b = .1$. The misspecified or the market parameters are $\tilde{\rho} = -.3, \tilde{b} = .9$.
- include a bid ask-fork of 0.45% in implied volatility terms for every option transaction.The evolution of the portfolio performance with 32 rebalancing dates.

Thank You

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