Central Limit Theorem under Model Uncertainty

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Le Master “Probabilités et Finance” de l’Université Pierre et Marie Curie et de l’Ecole Polytechnique
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BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS IN FINANCE

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We are concerned with different properties of backward stochastic differential equations and their applications to finance. These equations, first introduced by Pardoux and Peng (1990), are useful for the theory of contingent claim valuation, especially cases with constraints and for the theory of recursive utilities, introduced by Duffie and Epstein (1992a, 1992b).

Key Words: backward stochastic equation, mathematical finance, pricing, hedging portfolios, incomplete market, constrained portfolio, recursive utility, stochastic control, viscosity solution of PDE, Malliavin derivative

0. INTRODUCTION

We are concerned with backward stochastic differential equations (BSDE) and with their applications to finance. These equations were introduced by Bismut (1973) for the linear case and by Pardoux and Peng (1990) in the general case. According to these authors, the solution of a BSDE consists of a pair of adapted processes (Y, Z) satisfying

$$-dY_t = f(t, Y_t, Z_t)\,dt - Z_t^* dW_t; \qquad Y_T = \xi, \tag{0.1}$$

where f is the generator and ξ is the terminal condition.

Actually, this type of equation appears in numerous problems in finance (as pointed out in Quenez's doctorate 1993). First, the theory of contingent claim valuation in a complete market studied by Black and Scholes (1973), Merton (1973, 1991), Harrison and Kreps (1979), Harrison and Pliska (1981), Duffie (1988), and Karatzas (1989), among others, can be expressed in terms of BSDEs. Indeed, the problem is to determine the price of a contingent claim $\xi \geq 0$ of maturity T, which is a contract that pays an amount ξ at time T. In a complete market it is possible to construct a portfolio which attains as final wealth the amount ξ. Thus, the dynamics of the value of the replicating portfolio Y are given by a BSDE with linear generator f, with Z corresponding to the hedging portfolio. Then the price at time t is associated naturally with the value at time t of the hedging portfolio. However, there exists an infinite number of replicating portfolios and consequently the

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Nicole El Karoui
and Laurent Mazliak (Editors)

Backward stochastic differential equations

LONGMAN

Preface

Since the founder paper of Pardoux and Peng concerning general existence and unicity results appeared in 1990, Backward Stochastic Differential Equations have become a field of increasing activity and interest, due in particular to their connections with stochastic optimization problems where they proved to be a powerful and elegant tool to deal with state constraints. At the time of writing, the quantity of new papers on the topic is still increasing, and it is therefore not in our intentions to present an absolutely definitive theory. However, in the last three years, one has begun to understand better and better the major points of the subject and the time seems appropriate for taking stock of it.

This was the reason for us to organize a study group on the topic of Backward Stochastic Differential Equations at the Laboratoire de Probabilités of the University Paris VI during the academic year 1995-96. This group was followed by specialists and non specialists, and among them PhD students. Our aim was to make a complete basic introduction to the subject as well as to present applications and recent developments of the theory. We asked every speaker to write down a text on his conference, as pedagogical as possible. We now present the collection of these texts and we hope that it can be a useful bridge between what is already known and the most current research (the interested reader may consult the program of the Colloque du Mans sur les Equations Rétrogrades (Le Mans, June 96) to see some very new results).

The contributors were asked to make the papers as self-contained as possible. For this reason, each chapter of the book possesses its own bibliography, as well as its own notational system (though we tried to unify them when possible). We however made an exception for the numbering of sections, theorems, formulas etc... which is continuous and therefore univocal. As the different papers have very different sizes, we think this could make reading easier.

Now, last but not least, we want to thank all the contibutors for their talks and papers, Liliane Ney for her help, and Longman's Pitman Research Notes in Mathematics Collection for having accepted to publish this collection of talks.

The Editors, Paris, October 1996

2002 ICM Satellite Conference

《 Backward Stochastic Differential Equations》 Weihai

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March 9, 2006

PAGE ONE

Proving Ground

Why Students Of Prof. El Karoui Are In Demand

French Math Teacher Covers Structure Of Derivatives; Banks Clamor for 'Quants'

A Lesson on 'Smile Risk'

By CARRICK MOLLENKAMP and CHARLES FLEMING
March 9, 2006; Page A1

(See Corrections & Amplifications item below.)

When Xavier Charvet applies for a job at an investment bank next year, he thinks he'll have an advantage. The 24-year-old French student's resume begins with the phrase: "DEA d'El Karoui."

Proving Ground
Why Students Of Prof. El Karoui Are In Demand

Nicole El Karoui

La boss des maths

PARCOURS	1944	1964	1968	1988	1990	Depuis 1997
	Naissance à Paris.	Admise à l'Ecole normale supérieure de Sèvres.	Débute dans l'enseignement comme assistante à l'université d'Orsay.	Semestre sabbatique à la Compagnie bancaire.	Cofondatrice d'une option Finance dans le DEA de probabilités de Paris-VI.	Professeur à l'Ecole polytechnique.

De Wall Street à la Bourse de Londres, cette enseignante à Polytechnique fait figure d'autorité. Mathématicienne, elle s'est spécialisée dans les produits dérivés d'actions ou d'obligations

Le 9 mars, le *Wall Street Journal* publiait à la « une » sa réponse à une énigme qui avait intrigué deux de ses journalistes. Pourquoi compte-t-on tant de Français à Wall Street ou à Londres, parmi les *quants* – les analystes quantitatifs, ces spécialistes des titres financiers sophistiqués que sont les produits dérivés d'actions ou d'obligations ? s'étaient demandé Carrick Mollenkamp et Charles Fleming ...

... faisant des mathématiques, sans être prof ni chercheur. Et même bien la gagner : le salaire annuel d'embauche d'un *quant* débutant est de 35 000 livres (50 000 euros) à Londres et de 40 000 euros à Paris. Les élèves dépassent très rapidement leur maître : arrivée au plus haut grade, et en fin de carrière, Mme El Karoui gagne 80 000 euros par an brut « *pour 60 heures par semaine* » – « *elle suit énormément ses élèves* », explique M. Vaysse.

Ses missions de conseil chez LCL (ex-Crédit lyonnais) lui permettent d'augmenter ses revenus de 50 % et de rester en contact avec le marché. Elle avoue n'avoir besoin de dormir que cinq heures par nuit, ne s'être jamais occupée du travail scolaire de ses enfants, qui ont tous fait de brillantes études, mais leur avoir fait travailler ...

... *gent ? Jamais !* » Elle n'a d'ailleurs de cesse de prévenir ses élèves : « *Je leur dis qu'ils ne valent pas le salaire qu'ils perçoivent et que si, dans cinq ans, ils en ont marre mais qu'ils ne peuvent quitter leur poste à cause de ça, ils rateront quelque chose.* »

L'argent n'est pas une préoccupation pour cette fille d'ingénieur centralien, petite-fille de pasteur par sa mère, originaire de Nancy, troisième d'une famille de huit enfants. Les études, la culture, l'ouverture sur le monde sont ses valeurs. Mariée à un Tunisien, universitaire lui aussi, elle se qualifie de spécialiste « *de la variété culturelle* » – « *entre la Tunisie et la France ; entre la banque et les mathématiques. Il faut dépasser les barrages du premier contact* ». Et elle insiste : « *dans les deux sens* ».

Bonne en maths, elle ne prend ...

... des écoles : « *En maths sup', j'ai découvert l'unité entre l'algèbre et l'analyse et ai eu du plaisir à faire des maths.* » Elle est reçue à l'Ecole normale supérieure. « *A Sèvres* [où se trouvait l'ENS de jeunes filles], *j'ai découvert les probabilités et ai très vite décidé que c'était ce que je voulais faire !* »

Mme El Karoui aime se battre contre les a priori. « *La vie, c'est subtil. C'est ce que j'ai appris en vivant avec quelqu'un d'une autre culture ; aucun de nous deux ne pouvait estimer qu'il avait raison puisque nous ne pensions pas de la même façon*, explique-t-elle. *Je n'ai rien à faire de ce que les gens disent de moi.* » Mais elle s'étonne elle-même du chemin parcouru. « *En 1967, je militais à l'UNEF* [Union nationale des étudiants de France], *au sein du cartel des ENS. Les filles faisaient le secrétariat.* ...

... *je travaillerais dans la finance, j'aurais éclaté de rire !* »

En 1988, alors que l'ENS de Fontenay, où elle enseigne, doit déménager à Lyon, elle décide de prendre un semestre sabbatique. Grâce à une amie, elle travaille quelques mois à la Compagnie bancaire (avant son intégration au Groupe BNP Paribas). Et c'est le déclic. Car « *l'échange, c'est la vie* ». Ses collègues banquiers lui font découvrir un nouvel univers – « *je ne savais pas ce qu'était une obligation !* » –, elle leur apprend le sien.

« En 1967, je militais à l'UNEF. Si on m'avait dit alors que je travaillerais dans la finance, j'aurais éclaté de rire ! »

Elle modélise le mouvement des actions dans le futur, pour diminuer le risque pris par les investisseurs. Des modèles purement mathématiques, qui ne prennent pas en compte le contexte économique, grâce au calcul différentiel stochastique.

Aussi beau qu'une sonate de Brahms, ses préférées. Mme El Karoui finit son stage, mais Mme Geman, qui venait de passer quatre ans aux Etats-Unis, entre au contraire à la Banque pour y créer une structure de recherche. Elle propose à son amie, ainsi qu'à un autre mathématicien, Jean-Charles Rochet, d'y faire du conseil. « *Il y avait de vrais problèmes théoriques et pratiques à résoudre. C'était super. J'aime bien travailler à trois !* » Les deux femmes proposent alors de créer une nouvelle filière d'enseignement.

A quatre ans de la retraite, on n'imagine guère Mme El Karoui lever le pied. Elle travaille avec son collègue chinois Shige Peng à la création d'un mastère à l'université Fudan de Shanghai. Encore un nouveau passage de témoin vers une autre culture. ■

Annie Kahn

Leading Ladies

Professional women have often struggled in France, but lately more are making it to the top.

ALAIN JULIEN—AFP

MICHELE ALLIOT-MARIE
MINISTER OF DEFENSE

"It's a great victory" that no one at her macho Ministry even talks about her gender, she told *Le Figaro* last month; competence alone counts, and, by defending her budget, she's won the respect of her ground troops. If Sarkozy wins next May, she has the clout in the party to assure her a top post in his government.

PASCAL PERICH—LE MONDE

NICOLE EL KAROUI
MATH PROFESSOR

Few postgraduate degrees have the cachet of one earned under her aegis at the Ecole Polytechnique or the Université Pierre-et-Marie-Curie (Paris VI). Banks snap up her students, who have mastered the probability equations that hedge funds need to model portfolios.

关于中国复旦大学、山东大学与法国国立统计与经济管理学校、国立桥路学校及巴黎综合理工学校等3所巴黎高科成员学院联合培养数学金融硕士的合作协议

1998年2月5日，复旦大学与巴黎综合理工学校共同创建了中法应用数学研究所（ISFMA）。2004年4月23日，复旦大学与巴黎高科在上海签署合作框架协议。2005年5月23日，复旦大学、山东大学与巴黎高科在上海签署了关于数学金融硕士培养计划的框架协定。本协议是上述合作协议的继续和发展。

序言

中国加入世界贸易组织，对其金融领域带来了机遇和挑战，同时也对其在金融活动中的能力提出了更高的要求。

第 I 条　　中法数学金融硕士的培养计划

培养期为三年：

第一年的培养分别在复旦大学和山东大学进行， 教授同样的课程。
第二年，参加项目的学生共同在复旦大学继续接受培养。
第三年的培养分为两个方向，即两个特定的项目，分别在复旦大学和巴黎高科两地完成。

因此，中法数学金融硕士项目所包括的是两个不同的培养方向 :其一是完全在中国进行的 ，被称之为“上海项目”；其二被称之为“巴黎项目”。后者与前者唯一的区别在于该项目最后一年在法国进行。

“巴黎项目”第三年的培养，以巴黎综合理工学校、国立桥路学校和国立统计与经济管理学校所开设的法国国家硕士生二年级概率和金融专业课程为基础。目的 : 研究，科学与技术。方向 : 应用数学。 专业 : 概率和应用专业。

签字 ：

山东大学	展涛 校长	日期	- 6 DEC. 2005
复旦大学	陈晓漫 副校长	日期	- 6 DEC. 2005
巴黎高科	Pr. Rémi TOUSSAIN 主席	日期	- 6 DEC. 2005
	GAL Gabriel de NOMAZY 执行副主席	日期	- 6 DEC. 2005
国立统计与经济管理学校	Sylviane GASTALDO 校长	日期	- 6 DEC. 2005
国立桥路学校	Philippe COURTIER 校长	日期	- 6 DEC. 2005
巴黎综合理工学校	GAL Xavier MICHEL 总校长	日期	- 6 DEC. 2005


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Foundation of Probability Theory $(\Omega, \mathcal{F}, P)$

Probability Measure Theory (Kolmogorov,1933)

Foundation of Probability Theory $(\Omega, \mathcal{F}, P)$

The basic rule of probability theory

$$P(\Omega) = 1, \quad P(\emptyset) = 0, \quad P(A + B) = P(A) + P(B)$$

$$P(\sum_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

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The basic idea of Kolmogorov:

Using (generalized) **Lebesgue integral** to calculate the expectation of a random variable $X(\omega)$

$$E[X] = \int_{\Omega} X(\omega) dP(\omega)$$

Markov Processes;
Itô's calculus: and pathwise stochastic analysis
Statistics: life science and medical industry; insurance, politics;
Stochastic controls;
Statistic Physics;
Economics, Finance;
Civil engineering;
Communications, internet;
Forecasting: Whether, pollution, $\cdots$

Daniell Expectation (Integral) (1918): $(\Omega, \mathcal{H}, E)$ v.s. $(\Omega, \mathcal{F}, P)$

$\mathcal{H}$: a linear space of functions on Ω (random variables) containing $c \in \mathbb{R}$, s.t. $X \in \mathcal{H} \implies |X| \in \mathcal{H}$.

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(d) $E[\lambda X] = \lambda E[X], \quad \lambda \geq 0,$

- $E[X_i] \to 0$, if $X_i \downarrow 0$.

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- $E[X_i] \to 0$, if $X_i \downarrow 0$.

Daniell-Stone Theorem

There is a unique probability measure $(\Omega, \sigma(\mathcal{H}), P)$ such that $E[X] = \int_\Omega X(\omega) dP(\omega)$ for each $X \in \mathcal{H}$.

For nonlinear expectation: No more such equivalence!!

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- Why nonlinear?

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- Why nonlinear?
- Question on $(\Omega, \mathcal{F}, P)$: Is there a probability measure P for our real world of uncertainty?

For nonlinear expectation: No more such equivalence!!

- Why nonlinear?
- Question on $(\Omega, \mathcal{F}, P)$: Is there a probability measure P for our real world of uncertainty?
- Many economists, statisticians, and risk analysts: it is more reasonable to assume that P has uncertainty (called ambiguity).

A risky position: $\{X^i(\omega)\}_{i\in I}$ a random variable(s)

—— the value of X is uncertain.

F. Knight (1921): Two types of uncertainty

1. "risk": given a probability space $(\Omega, \mathcal{F}, P)$
2. Knightian uncertainty (ambiguity):

probabability measure P is uncertain: $P \in \mathcal{P}$

Ellsberg, Ambiguity aversion (1961)

urn I: 50 red balls, 50 black bolls

urn II: 100 balls, red or black

Choquet expected utility (CEU, Schmeidler (1989))

$$U_{CEU}(X) = v_c(u(X))$$

$$= \int_0^{\infty} v(u(X) \geq t)dt + \int_{-\infty}^{0} [v(u(X) \geq t) - 1]dt$$

$$v(A) = \min_{P \in \mathcal{P}} P(A)$$

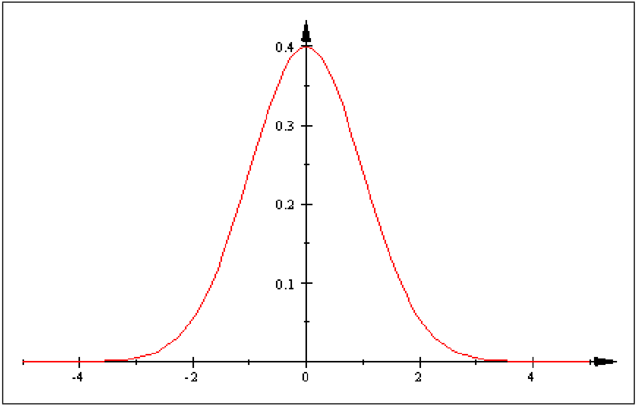
Multiple priors, Schmeidler & Gilboa (1989)

$$U^{mp}(X) = \min_{P \in \mathcal{P}} E_P[u(X)]$$

	SEU	CEU	PT	Multiple priors
Continuous state space	Savage(1954)	Gilboa(1987)		
U linear in money	de Finetti (1931,1937)	Chateauneuf (1991)		Chateauneuf (1991)
U linear in probability mixing	Anscombe & Aumann (1963)	Schmeidler (1989)		Gilboa & Schmeidler (1989)
2-stage Canonical probability	Raifa(1968)	Sarin & Wakker (1992)	Sarin & Wakker (1994)	
Continuous U tradeof consistency	Wakker(1984)	Wakker(1989)	Tversky & Kahneman (1992)	
Continuous U Multisymmetry	Nakamura (1990)	Nakamura (1990)	x	
Continuous U act independence	Cul(1992)	Chew&Karni(1994) Ghiradato et.al(2003)	x	Ghiradato et.al(2003) Casadesus-Masanell et al (2000)

A well-known puzzle: why normal distribution is so widely used and abused?

$$N(\mu, \sigma^2): \quad p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$



A key explanation:

A key explanation:

Theorem (Central Limit Theorem (CLT))

$\{X_i\}_{i=1}^{\infty}$ is assumed to be i.i.d., with

$$\mu = E[X_1], \quad \sigma^2 = E[(X_1 - \mu)^2].$$

Then for each bounded and continuous function $\varphi \in C(\mathbb{R})$, we have

$$\lim_{i \to \infty} \mathbb{E}[\varphi(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu))] = E[\varphi(X)], \quad X \overset{d}{=} N(0, \sigma^2).$$

- Practically, normal distributions are so frequently and widely used. But often it is far from true that the above $\{X_i\}_{i=1}^{\infty}$ is i.i.d.

Dirty work?

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- Good dirty workers?

- We don't assume one probability measure P;

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- We don't assume $X_i \overset{d}{=} X_j$.

- We don't assume one probability measure P;
- We don't assume $X_i \overset{d}{=} X_j$.
- Instead,

$$X_i \in \{F_\theta(x) : \theta \in \Theta\} : \text{ subset of uncertain distributions .}$$

$$\hat{\mathbb{E}}[X] = \sup_{\theta\in\Theta} E_\theta[X] = \sup_{\theta\in\Theta} \int_\Omega X dP_\theta$$

$\{P_\theta,\ \theta \in \Theta\}$: the subset of uncertain probabilities.

Sublinear expectation on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$

$$\hat{\mathbb{E}}[X] = \sup_{\theta \in \Theta} E_\theta[X]$$

$\hat{\mathbb{E}}$ has the properties:

(a) $X \geq Y$ then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$

(b) $\hat{\mathbb{E}}[c] = c$

(c) $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$

(d) $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X], \lambda \geq 0.$

Artzner, Delbean, Eber & Heath [ADEH1999]

$$\rho(X) := \hat{\mathbb{E}}[-X]$$

- Huber Robust Statistics (1981).
- (ADEH, 1999)
- Föllmer & Schied (2004)

Robust representation of sublinear expectations

- Huber Robust Statistics (1981).
- (ADEH, 1999)
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Theorem

$\hat{\mathbb{E}}[\cdot]$ is a sublinear expectation on $\mathcal{H}$ if and only if there exists a subset of linear expectation $\{E_\theta, \theta \in \Theta\}$ such that

$$\hat{\mathbb{E}}[X] = \sup_{\theta \in \Theta} E_\theta[X], \quad \forall X \in \mathcal{H}.$$

Meaning of the robust representation: Statistic model uncertainty

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X], \quad \forall X \in \mathcal{H}.$$

The size of the subset $\mathcal{P}$ represents the degree of model uncertainty: The stronger the $\hat{\mathbb{E}}$ the more the uncertainty

$$\hat{\mathbb{E}}_1[X] \geq \hat{\mathbb{E}}_2[X], \quad \forall X \in \mathcal{H} \iff \mathcal{P}_1 \supset \mathcal{P}_2$$

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Advantage:

The information of uncertainty of probabilities is very well kept in $\hat{\mathbb{E}}$.

Definition

X, Y are said to be identically distributed, denoted by $X \overset{d}{=} Y$, if they have same distributions:

$$\hat{\mathbb{E}}[\varphi(X)] = \hat{\mathbb{E}}[\varphi(Y)], \quad \forall \varphi \in C_b(\mathbb{R}^n).$$

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Distribution of of X under uncertainty

$$\mathbb{F}_X[\varphi] := \hat{\mathbb{E}}[\varphi(X)], \varphi \in C_b(\mathbb{R}^n)$$

$(\mathbb{R}^n, C_b(\mathbb{R}^n), \mathbb{F}_X)$ forms a sublinear expectation.

Independence under $\hat{\mathbb{E}}$

Definition

$Y(\omega)$ is said to be independent from $X(\omega)$, denoted by $Y \perp\!\!\!\perp X$, if we have:

$$\hat{\mathbb{E}}[\varphi(X,Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x,Y)]_{x=X}], \ \forall \varphi(\cdot).$$

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The realization of X does not change the distribution uncertainty of Y.

A sequence $\{X_i\}_{i=1}^{\infty}$ is called i.i.d. in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ if

$X_i \overset{d}{=} X_1$ and $X_{i+1} \perp\!\!\!\perp (X_1, X_2, \cdots, X_i), \ \ i = 1, 2, \cdots.$

Central Limit Theorem (CLT) [Peng 2007, 2010]

Theorem

Let $\{X_i\}_{i=1}^{\infty}$ *be i.i.d. in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *and* $\hat{\mathbb{E}}[X_1] = -\hat{\mathbb{E}}[-X_1] = 0$. *Let* $S_n = X_1 + \cdots + X_n$. *Then*

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\varphi\left(S_n/\sqrt{n}\right)] = \hat{\mathbb{E}}[\varphi(X)], \quad X \overset{d}{=} N(0, [\underline{\sigma}^2, \overline{\sigma}^2]).$$

where $\overline{\sigma}^2 = \hat{\mathbb{E}}[X_1^2]$, $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-X_1^2]$.

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where $\overline{\sigma}^2 = \hat{\mathbb{E}}[X_1^2]$, $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-X_1^2]$.

In particular

$$\hat{\mathbb{E}}[\varphi(X)] = \begin{cases} \frac{1}{\sqrt{2\pi\overline{\sigma}^2}} \int_{-\infty}^{\infty} \varphi(x) e^{-\frac{x^2}{2\overline{\sigma}^2}} dx, & \varphi \text{ convex}, \\ \frac{1}{\sqrt{2\pi\underline{\sigma}^2}} \int_{-\infty}^{\infty} \varphi(x) e^{-\frac{x^2}{2\underline{\sigma}^2}} dx & \varphi \text{ concave}. \end{cases}$$

Normal distribution under Sublinear expectation

A fundamentally important sublinear distribution

Definition

A random variable X in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called normally distributed, denoted by

$$X \overset{d}{=} N(0, [\underline{\sigma}^2, \overline{\sigma}^2]) \quad \text{if} \quad aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2} X, \quad \forall a, b \geq 0.$$

where $\bar{X}$ is an independent copy of X.

Theorem

$X \overset{d}{=} N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ iff, for each $\varphi \in C_b(\mathbb{R})$, the function

$$u(t,x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)], \quad x \in \mathbb{R}, \quad t \geq 0$$

is the solution of the PDE

$$u_t = G(u_{xx}), \quad u|_{t=0} = \varphi.$$

where $G(a) = \hat{\mathbb{E}}[\frac{a}{2}X^2]$: G-normal distribution.

Theorem

Let $\{Y_i\}_{i=1}^{\infty}$ *be a i.i.d. in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. *Then*

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\varphi\left(\frac{Y_1 + \cdots + Y_n}{n}\right)] = \hat{\mathbb{E}}[\varphi(Y)]$$

Y *is maximally distributed*: $aY + b\bar{Y} \overset{d}{=} (a+b)Y$

Law of Large Numbers (LLN)

Theorem

Let $\{Y_i\}_{i=1}^{\infty}$ *be a i.i.d. in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. *Then*

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Y is maximally distributed: $aY + b\bar{Y} \overset{d}{=} (a+b)Y$

Maximal distribution

$$\hat{\mathbb{E}}[\varphi(Y)] = \max_{v\in[\underline{\mu},\bar{\mu}]} \varphi(v), \quad \bar{\mu} = \hat{\mathbb{E}}[X_1], \ \underline{\mu} = -\hat{\mathbb{E}}[-X_1].$$

Theorem (Peng2007, 2010)

Let $\{X_i, Y_i\}_{i=1}^{\infty}$ *be i.i.d. s.t.*

$$\hat{\mathbb{E}}[|X_1|^{2+\alpha}] + \hat{\mathbb{E}}[|Y_1|^{1+\alpha}] < \infty \text{ and } \hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[-X_1] = 0.$$

Then

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\varphi(\frac{X_1 + \cdots + X_n}{\sqrt{n}} + \frac{Y_1 + \cdots + Y_n)}{n}] = \hat{\mathbb{E}}[\varphi(X + Y)], \, ,$$

where (X, Y) *is G-distributed with*

$$G(p, a) := \hat{\mathbb{E}}[pY_1 + \frac{1}{2}aX_1^2]$$

and $u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X + tY)]$ *solves the PDE*

$$\partial_t u = G(\partial_x u, \partial_{xx}^2 u), \quad u|_{t=0} = \varphi$$

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- Stochastic analysis, stochastic processes?

Definition

A process $\{B_t(\omega)\}_{t\geq 0}$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a **Brownian motion** under $\hat{\mathbb{E}}$ if:

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- B is called a symmetric Brownian motion if $\hat{\mathbb{E}}[B_t] = -\hat{\mathbb{E}}[-B_t] = 0$.

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Theorem (Peng 2007)

For a symmetric Brownian motion B, ***(iii)*** $\Longleftrightarrow$
(iii') $B_t \overset{d}{=} N(0, [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$, *with* $\overline{\sigma}^2 = \hat{\mathbb{E}}[B_1^2]$, $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-B_1^2]$.

Remark.

G-Brownian motion $B_t(\omega) = \omega_t$, $t \geq 0$, can in fact strongly correlated under the unknown 'objective probability', it can even be have very long memory. But it's increments are independent. By which we can have many advantages in analysis, calculus and computation, compare with, e.g. fractal B.M. $\square$

Main idea.

Following Kolmogorov's approach of consistent finite dimensional distributions. $\square$

Construction of B.M in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$

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- Define

$$\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[\varphi(B_{t_1}, \cdots, B_{t_{n-1}} - B_{t_{n-2}}, B_{t_n} - B_{t_{n-1}})]$$

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- Consistency: automatically holds;

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- Consistency: automatically holds;
- Use $\|\cdot\|^p := \left(\hat{\mathbb{E}}[|X|^p]\right)^{1/p}$ to get the completion $L^p(\mathcal{H}_t)$ of $\mathcal{H}_t$.

□

Itô's integral of G–Brownian motion

For each process $(\eta_t)_{t\geq 0} \in$ of the form

$$M^{2,0}(0,T) := \{\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbf{I}_{[t_j,t_{j+1})}(t), \;\; \xi_j \in Ł^2(\mathcal{H}_{t_j})\}$$

we define

$$I(\eta) = \int_0^T \eta(s)dB_s := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}).$$

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Lemma

We have

$$\hat{\mathbb{E}}[\int_0^T \eta(s)dB_s] = 0$$

and

$$\hat{\mathbb{E}}[(\int_0^T \eta(s)dB_s)^2] \leq \int_0^T \hat{\mathbb{E}}[(\eta(t))^2]dt.$$

Definition

Under the Banach norm $\|\eta\|^2 := \int_0^T \hat{\mathbb{E}}[(\eta(t))^2]dt$,

$$I(\eta) : M^{2,0}(0, T) \mapsto L^2(\mathcal{F}_T) \text{ is a contract mapping}$$

We then extend $I(\eta)$ to $M^2(0, T)$ and define, the stochastic integral

$$\int_0^T \eta(s)dB_s := I(\eta), \quad \eta \in M^2(0, T).$$

Lemma

We have

(i) $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$.

(ii) $\int_s^t (\alpha\eta_u + \theta_u) dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u, \ \ \alpha \in L^1(\mathcal{F}_s)$

(iii) $\hat{\mathbb{E}}[X + \int_r^T \eta_u dB_u | \mathcal{H}_s] = \hat{\mathbb{E}}[X], \forall X \in L^1(\mathcal{F}_s)$.

Quadratic variation process of G–BM

We denote:

$$\langle B\rangle_t = B_t^2 - 2\int_0^t B_s dB_s = \lim_{\max(t_{k+1}-t_k)\to 0} \sum_{k=0}^{N-1} (B_{t_{k+1}^N} - B_{t_k})^2$$

$\langle B\rangle$ is an increasing process called **quadratic variation process** of B.

$$\hat{\mathbb{E}}[\langle B\rangle_t] = \overline{\sigma}^2 t \ \text{ but } \ \hat{\mathbb{E}}[-\langle B\rangle_t] = -\underline{\sigma}^2 t$$

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Lemma

$B_t^s := B_{t+s} - B_s,\ t \geq 0$ is still a G-Brownian motion. We also have

$$\langle B\rangle_{t+s} - \langle B\rangle_s \equiv \langle B^s\rangle_t \,.$$

We have the following isometry

$$\hat{\mathbb{E}}[(\int_0^T \eta(s)dB_s)^2] = \hat{\mathbb{E}}[\int_0^T \eta^2(s)d\langle B\rangle_s],$$
$$\eta \in M_G^2(0,T)$$

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \eta_s d\langle B\rangle_s + \int_0^t \beta_s dB_s$$

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Theorem.

Let α, β and η be process in $L_G^2(0,T)$. Then for each $t \geq 0$ and in $L_G^2(\mathcal{H}_t)$ we have

$$\Phi(X_t) = \Phi(X_0) + \int_0^t \Phi_x(X_u)\beta_u dB_u + \int_0^t \Phi_x(X_u)\alpha_u du + \int_0^t [\Phi_x(X_u)\eta_u + \frac{1}{2}\Phi_{xx}(X_u)\beta_u^2] d\langle B\rangle_u$$

Problem

We consider the following SDE:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t h(X_s)d\langle B\rangle_s + \int_0^t \sigma(X_s)dB_s,\ t > 0.$$

where $X_0 \in \mathbb{R}^n$ *is given*

$$b, h, \sigma : \mathbb{R}^n \mapsto \mathbb{R}^n \ \text{ are given Lip. functions.}$$

The solution: a process $X \in M_G^2(0, T; \mathbb{R}^n)$ *satisfying the above SDE.*

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Theorem

There exists a unique solution $X \in M_G^2(0, T; \mathbb{R}^n)$ *of the stochastic differential equation.*

- Risk measures and pricing under dynamic volatility uncertainties ([A-L-P1995], [Lyons1995]) —for path dependent options;
- Stochastic (trajectory) analysis of sublinear and/or nonlinear Markov process.

Prospectives

- Risk measures and pricing under dynamic volatility uncertainties ([A-L-P1995], [Lyons1995]) —for path dependent options;
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- New Feynman-Kac formula for fully nonlinear PDE: path-interpretation.

$$u(t,x)=\hat{\mathbb{E}}_x[(B_t)\exp(\int_0^t c(B_s)ds)]$$

$$\partial_t u=G(D^2u)+c(x)u,\ \ u|_{t=0}=\varphi(x).$$

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- Fully nonlinear Monte-Carlo simulation.
- BSDE driven by G-Brownian motion: a challenge.

Probability Space $(\Omega, \mathcal{F}, P)$	Nonlinear Expectation Space $(\Omega, \mathcal{H}, \mathbb{E})$
Distribution $X \overset{d}{=} Y$	Distribution $X \overset{d}{=} Y$
Independence $Y \perp\!\!\!\perp X$	Independence $Y \perp\!\!\!\perp X$
Normal distribution $N(\mu, \sigma^2)$	Maximal distr. $M([\underline{\mu}, \overline{\mu}])$ Normal distr. $N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$
Law of Large Numbers Central Limit Theorem	LLN CLT
Brownian Motion $B_t(\omega) = \omega_t$	G-Brownian Motion $B_t(\omega) = \omega_t$

Quadratic variation process	$\langle B\rangle_t = t$ $\langle B\rangle_t$: a non-symmetric Brownian
I Itô's formula	$dX_t = \beta_t dB_t + \alpha_t dt$ $d\Phi(X_t) = \partial_x\Phi(X_t)dX_t + \frac{1}{2}\partial_{xx}^2\Phi(X_t)\beta_t^2 dt$
G-Itô's formula	$dX_t = \beta_t dB_t + \eta_t d\langle B\rangle_t + \alpha_t dt$ $d\Phi(X_t) = \partial_x\Phi(X_t)dX_t + \frac{1}{2}\partial_{xx}^2\Phi(X_t)\beta_t^2 d\langle B\rangle_t$

SDE $(\Omega, \mathcal{H}, P)$ $(\Omega, \mathcal{H}, \mathbb{E})$	$dx_t = b(x_t)dt + \sigma(x_t)dB_t$ $dx_t = b(x_t)dt + \sigma(x_t)dB_t + \beta(x_t)d\langle B\rangle_t$
Heat equation G-Heat equation	$\partial_t u = \Delta u$ $\partial_t u = G(Du, D^2u)$
Martingale Representaion	$\mathbb{E}[X\|\mathcal{F}_t] = \mathbb{E}[X] + \int_0^t z_s dB_s$ $\mathbb{E}[X\|\mathcal{F}_t] = \mathbb{E}[X] + \int_0^t z_s dB_s + K_t$ $K_t \overset{?}{=} \int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s)ds$

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- [Hu, Denis, Hu 2009], [Hu, 2009], [Hu, Peng2009], [Xu, Zhang 2010], [Li, Hu2009],
- [Soner, Touzi, Zhang, 2009-2010a,b,c,d], [Hu, Y.-Peng2010], [Song, 2010a] 2010b].

谢谢

Mercie

Thank you