Weierstraß-Institut für
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ÖKONOMISCHES
RISIKO

Central limit theorems for coherent law-invariant risk measures

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Significance of coherent law-invariant risk measures

- Statistical functionals $\rho : \mathbb{F} \to \mathbb{R}$ of special type meeting requirements to serve as building blocks in quantitative risk management.
- Bringing viewpoints of insurance and finance together:

	insurance	finance
subject	premium calculation	risk assessment
object	contracts	financial risks
desirable properties	• law-invariant • safety loadings • risk aversion	• monotone w.r.t. stochastic order • minimal capital requirement • diversification • scale invariant
	derived Bäuerle/Müller (2006)	axiomatic Artzner et al. (1999)

Estimation of risk measures

risk measure $\rho : \mathbb{F} \to \mathbb{R}$

- **Practical problems to calculate $\rho(F)$:**
  - F unknown but data set $x_1, ..., x_n$ available
  - Calculation of $\rho(F)$ for complicated F very difficult in practice.

$\downarrow$

- **Canonical plug-in estimation:**

$$x_1, ..., x_n \to F \approx \hat{F}_n \to \rho(F) \approx \rho(\hat{F}_n)$$

$\hat{F}_n$ as easy as possible, canonical choice: $\hat{F}_n \widehat{=}$ empirical d.f. w.r.t. $(x_1, ..., x_n)$.

$$\boxed{\textbf{Subject of the talk: Limit distribution of } \rho(\hat{F}_n) \textbf{ ?}}$$

Contents

Axiomatic approach by Artzner et al. (1999)

- $\mathbb{F}_{\mathscr{X}} =: \{F_X \mid X \in \mathscr{X}\}$ set of distribution functions on $\mathbb{R}$ loss distribution functions
  - $(\Omega, \mathscr{F}, \mathbb{P})$ atomless probability space,
  - $\mathscr{L}^\infty(\Omega, \mathscr{F}, \mathbb{P}) \subseteq \mathscr{X} \subseteq \mathscr{L}^1(\Omega, \mathscr{F}, \mathbb{P})$ vector subspace, financial risks
  - $\mathscr{X}$ Stonean vector lattice

    $X \wedge Y,\ X \vee Y,\ (X-c)^+,\ X \wedge c \in \mathscr{X}$ for $X, Y \in \mathscr{X},\ c \in \mathbb{R}$.

- $\rho : \mathbb{F}_{\mathscr{X}} \to \mathbb{R}$ coherent law-invariant risk measure, if
  - monotonicity

    $\rho(F_{X_1}) \leq \rho(F_{X_2})$ for $F_{X_1} \leq F_{X_2}$.
  - cash invariance

    $\rho(F_{X+c}) = \rho(F_X) - c$ for $X \in \mathscr{X}$ and $c \in \mathbb{R}$.
  - sublinearity

    $\rho(F_{\lambda_1 X_1 + \lambda_2 X_2}) \leq \lambda_1 \rho(F_{X_1}) + \lambda_2 \rho(F_{X_2})$ for $X_1, X_2 \in \mathscr{X}$ and $\lambda_1, \lambda_2 \geq 0$.

Examples

- Concave distortion risk measure:

$$\rho_\psi(F_X) := \rho(F_X) = \int_{-\infty}^{0} \psi(F_X(x))\, dx - \int_0^\infty [1 - \psi(F_X(x))]\, dx = \oint (-X)\, d\psi(\mathbb{P}),$$

for some concave $\psi : [0,1] \to [0,1]$ which is a distortion function, i.e. nondecreasing with $\psi(0) = 0, \psi(1) = 1$.

- Average Value at Risk (Tail Value at Risk, Expected Shortfall): $\alpha \in \,]0,1[$

$$\rho(F_X) =: AV@R_\alpha(F_X) := \frac{1}{\alpha} \int_0^\alpha q_{F_{-X}}(1-\beta)\, d\beta = \frac{1}{\alpha} \int_0^\alpha V@R_\beta(X)\, d\beta$$

$AV@R_\alpha = \rho_\psi$ with $\psi(t) = \frac{1}{\alpha}\,(\alpha \wedge t)$

Examples (not concave distortion risk measures)

- $\rho = \sup_{\psi \in \Psi} \rho_\psi$, where Ψ some set of concave distortion functions.

- **Risk measures based on one-sided moments**: [Fischer (2003)]

$$\rho(F_X) =: \rho_{p,a}(F_X) := -\mathbb{E}[X] + a\, \|(X - \mathbb{E}[X])^-\|_p \quad (a \in [0,1],\ p \in [1,\infty[).$$

- Expectiles: [Newey/Powell (1987), Müller (2010)]

$$\rho_\alpha(F_X) := \operatorname*{argmin}_{x \in \mathbb{R}} \left[(1-\alpha)\, \left\|\left((-X)-x\right)^-\right\|_2^2 + \alpha \left\|\left((-X)-x\right)^+\right\|_2^2 \right] \quad (\alpha \in [1/2, 1[)$$

Regular coherent law-invariant risk measures

$\rho : \mathbb{F}_{\mathscr{X}} \to \mathbb{R}$ coherent law-invariant risk measure

- associated distortion function: [K./Zähle (2010)]

$$\psi_\rho : [0,1] \to \mathbb{R},\ t \mapsto \rho(F_{-X(t)}),\ X(t) \sim B(1,t),$$ is a distortion function

- ρ regular if $\lim_{k\to\infty} \rho(F_{-(X-k)^+}) = 0$ for nonnegative $X \in \mathscr{X}$.

  ρ regular $\Rightarrow \rho(F_X) = \sup_{m\in\mathbb{N}} \inf_{k\in\mathbb{N}} \rho\left(F_{[X^+\wedge k]-[X^-\wedge m]} \right)$ for $X \in \mathscr{X}$ [K. (2007)]

- ρ strictly regular if ρ regular, and $\lim_{t\to 0+} \psi_\rho(t) = 0$.

- **Prominent examples:** Exists Riesz-seminorm $\|\cdot\|$ on $\mathscr{X}$
  - $\|\cdot\|$ complete $\Rightarrow \rho$ regular.
  - $\|\cdot\|$ complete and $\sigma-$order continuous $\Rightarrow \rho$ strictly regular.
  - $\|\cdot\|$ Luxemburg seminorm on Orlicz space or Orlicz heart w.r.t. **continuous** Young function $\Rightarrow \rho$ is strictly regular.

Representation by concave distortion risk measures

- **2.1 Proposition:** [Belomestny/K. (2010)]
  - $\rho : F_{\mathscr{X}} \to \mathbb{R}$ regular coherent law-invariant risk measure $\Rightarrow$ $\rho = \sup_{\psi \in \Psi} \rho_\psi$,

    where Ψ some set of concave distortion function.
  - Representing Ψ compact w.r.t. uniform metric $\Leftarrow$ ρ strictly regular.

  [Based on Kusuoka (2001) with Jouini/Schachermayer/Touzi (2006)]

- **Representation typically unknown!** Partial information by ψ_ρ

  $\psi_\rho = \sup_{\psi \in \Psi} \psi$ and $\sup_{\psi \in \Psi} |\psi(t) - \psi(s)| \leq \psi_\rho(|t-s|)$ if $\rho = \sup_{\psi \in \Psi} \rho_\psi$

The set up

- $\rho : \mathbb{F}_{\mathscr{X}} \to \mathbb{R}$ strictly regular coherent law-invariant risk measure.
- $(X_i)_i$ strictly stationary sequence of random variables with common distribution function $F \in \mathbb{F}_{\mathscr{X}}$.
- $F_n \widehat{=}$ empirical distribution function based on $(X_1, \ldots, X_n)$
- **Estimation:**

$$\rho_n(F) := \rho(F_n).$$

Examples

- **Concave distortion risk measure:** $\rho = \rho_\psi$

$$\rho_n(F) = -\int_0^1 q_{F_n}(t)\ \psi'(t)\, dt. \qquad L\text{–Statistic}.$$

- **Average Value at Risk:** $\rho = AV@R_\alpha$

$$\rho_n(F) = \frac{1}{\alpha}\frac{1}{n}\sum_{i=1}(q_{F_n}(\alpha) - X_i)^+ - q_{F_n}(\alpha).$$

- **Risk measures based on one-sided moments:** $\rho = \rho_{p,a}$

$$\rho_n(F) = -\frac{1}{n}\sum_{j=1}^{n} X_j + a\left(\sum_{i=1}^{n}\left(\left[\frac{1}{n}\sum_{j=1}^{n} X_j - X_i\right]^+\right)^p \frac{\sharp\{k \mid X_k = X_i\}}{n}\right)^{1/p}.$$

- **Expectiles:** $\rho = \rho_\alpha$

$$\rho_n(F) = \operatorname*{argmin}_{x\in\mathbb{R}} \sum_{i=1}^{n}\left[(1-\alpha)\Big(\big((-X_i)-x\big)^-\Big)^2 + \alpha\Big(\big((-X_i)-x\big)^+\Big)^2\right].$$

Basic assumptions

(M) Data

$(X_i)_{i\in\mathbb{N}}$ strongly mixing with mixing coefficients $\alpha(i) \leq \bar{\alpha}_0 \exp(-\bar{\alpha}_1 i)$ for some $\bar{\alpha}_0, \bar{\alpha}_1 > 0$ (e.g. i.i.d. data, also general ARMA- or GARCH-processes).

(J) Jumps of distribution

F has a finite set $D(F)$ s.t. $F|]q_F(0), q_F(1)[\backslash D(F)$ is continuously differentiable with strictly positive derivative.

(I) Tails of distribution

F fulfills the following integrability condition:

$$\int_{\mathbb{R}} F(x)^{-1/2-2\delta}(1-F(x))^{1/2-\delta}\ \psi_\rho(\lambda F(x)^{1+\delta})\ dx < \infty$$

for some $\lambda \in]0,1/2[, \delta \in [0,1/2[$, where $\delta = 0$ iff i.i.d. data **"loss" tail matters!**

Basic idea

$\rho : \mathbb{F}_{\mathscr{X}} \to \mathbb{R}$ strictly regular coherent law-invariant risk measure

- $\rho = \sup_{\psi \in \Psi} \rho_\psi$, Ψ compact w.r.t. uniform metric (**repres. possibly unknown!**).
- **Reduction to convergence of stochastic processes:**
  - (T) $\to$ stochastic processes $\big(\rho_\psi(F_n)\big)_{\psi\in\Psi}, \big(\rho_\psi(F)\big)_{\psi\in\Psi}$ Borel random elements in path space

    $UCB(\Psi) := \{f : \Psi \to \mathbb{R} \mid f$ uniformly continuous w.r.t. uniform metric$\}$

    endowed with sup norm

  - Under (M), (J), (I) convergence in law in $UCB(\Psi)$

$$\big(\sqrt{n}[\rho_\psi(F_n) - \rho_\psi(F)]\big)_{\psi\in\Psi} \xrightarrow{L} G$$

$\Rightarrow$ application of functional Delta method for sup functionals

$$\sqrt{n}[\rho_n(F) - \rho(F)] = \sqrt{n}\,[\sup_\psi \rho_\psi(F_n) - \sup_\psi \rho_\psi(F)] \xrightarrow{L} \sup_{\psi,\rho(F)=\rho_\psi(F)} G(\psi)$$

Ingredients of proof

- Approximation lemma for Borel random elements in separable metric spaces
  $\leftarrow$ compactness of Ψ, (T) [Billingsley (1968)]
- Donsker's functional theorem $\leftarrow$ (M) [Ben Hariz (2005)]
- Bounds for empirical distribution function $\leftarrow$ (M) [Puri/Tran (1980)]
- Functional Delta method for quantile transforms $\leftarrow$ (J) [van der Vaart (1998)]

- **4.1 Theorem:** [Belomestny/K. (2010)]

  Under (M), (J), (I), we may find

  - a compact set $S(\rho(F))$ of continuous, concave distortion functions,
  - some centered Gaussian process $(G(\psi))_{\psi\in S(\rho(F))}$ with continuous paths w.r.t. uniform metric, and

$$\begin{aligned}\mathbb{E}\big[G(\psi_1)G(\psi_2)\big] &= \int_{\mathbb{R}^2} \psi_1'(F(x))\psi_2'(F(y))\big\{F(x\wedge y)-F(x)F(y)\\ &\quad +2\sum_{k=1}^{\infty}\left[\mathbb{P}(\{X_1\leq x, X_{k+1}\leq y\})-F(x)F(y)\right]\big\}\,dx\,dy\end{aligned}$$

  such that $\left(\sqrt{n}[\rho_n(F)-\rho(F)]\right)_{n\in\mathbb{N}}$ converges in law to $\max\limits_{\psi\in S(\rho(F))} G(\psi)$.

- Moreover, $\sup\limits_{\psi\in S(\rho(F))} G(\psi)=G(Z)$ for some Borel-random element Z of $S(\rho(F))$ if $\mathbb{E}[G(\psi_1)-G(\psi_2)]^2\neq 0$ for any two different $\psi_1,\psi_2\in S(\rho(F))$.

  In particular $\sup\limits_{\psi\in S(\rho(F))} G(\psi)=G(\psi_\rho)$ if ρ concave distortion risk measure.

5 Final remarks

Limit random variable $\sup_{\psi \in S(\rho(F)} G(\psi)$ in main result with $\sigma := \sup_{\psi \in S(\rho(F))} \mathbb{E}\left[G(\psi)^2\right]$

How to utilize for confidence intervals?

- **Large deviation principle for suprema of Gaussian processes:**

$$\mathbb{P}(\{\sup_{\psi \in S(\rho(F))} G(\psi) < z\}) \geq \Phi(z/\sigma)$$

- **Idea for one-sided confidence interval:**

Find estimator $\widehat{\sigma}_n \xrightarrow{p} \overline{\sigma} \geq \sigma$, and choose suitable quantile λ of $N(0,1)$.

$$\text{confidence interval: } \left] -\infty,\ \rho_n(F) - \frac{\lambda \widehat{\sigma}_n}{\sqrt{n}} \right[$$

- In case of i.i.d. data simple ad hoc methods for $\widehat{\sigma}_n$ available, but in general not!

$\rightarrow$ **future work (subsampling? use of associated distortion function might help!)**